

Research Article

Solution of Singular Integral Equations via Riemann–Liouville Fractional Integrals

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In this attempt, we introduce a new technique to solve main generalized Abel's integral equations and generalized weakly singular Volterra integral equations analytically. This technique is based on the Adomian decomposition method, Laplace transform method, and Ψ -Riemann–Liouville fractional integrals. Finally, some examples are proposed and they illustrate the rapidness of our new technical method.

1. Introduction

Fractional dynamic varying systems with singular kernels either in the Riemann–Liouville sense or in the Caputo sense have been investigated in the literature [1–3]. To solve a fractional dynamic equation, we always apply a corresponding fractional integral operator. The action of this integral operator will transform the fractional dynamic equation into its corresponding integral equation whose singularity is reflected in the kernel. Motivated by this fact, in this article, we introduce a new technique to solve main generalized Abel's integral equations and generalized weakly singular Volterra integral equations analytically.

Let $\bar{\delta} \in (0, 1)$ and $0 < \gamma < b$. Consider the main generalized Abel's integral equation [4, 5]:

$$u(\gamma) = \int_0^\gamma \frac{u(t)dt}{[g(\gamma) - g(t)]^{\bar{\delta}}}, \quad (1)$$

and the generalized weakly singular Volterra type integral equation of the second kind [4, 5]:

$$u(\gamma) = g(\gamma) + \int_0^\gamma \frac{\bar{\beta}u(t)dt}{[g(\gamma) - g(t)]^{\bar{\delta}}}, \quad (2)$$

where g is a strictly monotonically increasing and differentiable function in the interval $(0, b)$ with $g(\gamma) \neq 0$ for every γ in $(0, b)$ and $\bar{\beta}$ is a constant.

Particularly, if $\bar{\delta} = (1/2)$ and $g(\gamma) = \gamma$, then integral equation (1) reduces to the classical Abel's integral equation in which Abel, in 1823, investigated the motion of a particle that slides down along a smooth unknown curve under the influence of the gravity in a vertical plane. The particle takes the time $g(\gamma)$ to move from the highest point of vertical height γ to the lowest point 0 on the curve. This problem is derived to find the equation of that curve. Indeed, Abel's integral equation is one of the most famous equations that frequently appear in many engineering problems and physical properties such as heat conduction, semiconductors, chemical reactions, and metallurgy (see, e.g., [6, 7]). Besides, over the past few years, many numerical methods for solving Abel's integral equation have been developed, such as collocation methods [8], product integration methods [9, 10], fractional multistep methods [11–13],

methods based on wavelets [14–16], backward Euler methods [9], Adomian decomposition method [17], and Tau approximation method [18].

Recently, Wazwaz [4, 5] solved integral equations (1) and (2). Unlike [4, 5], in the present paper, we are going to introduce a new technique, which is based on the Adomian decomposition method, Laplace transform method, and Ψ -Riemann–Liouville fractional integrals, for solving main generalized Abel’s integral equations and generalized weakly singular Volterra integral equations. For related works on generalized fractional derivatives in the Riemann–Liouville and Caputo senses and their Laplace transforms, we refer to [19]. Moreover, there are several methods used in obtaining approximate solutions to linear and nonlinear fractional ordinary differential equations (FODEs) and fractional partial differential equations (FPDEs) and their real-world applications. For this reason, we advise the readers to visit [20–28].

The paper is organized as follows. In Section 2, we recall the definitions of Riemann–Liouville fractional integrals, Ψ -Riemann–Liouville fractional integrals, and some essential properties. Section 3 is devoted to deliver the main results for the generalized Abel’s integral equations and generalized weakly singular Volterra integral equations. In Section 4, several examples are considered to illustrate the applicability of our main results.

2. Preliminaries

Here, we give the definitions of Riemann–Liouville fractional integrals, Ψ -Riemann–Liouville fractional integrals, and some essential properties.

Definition 1. Let $g: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and let $(a, b) \subseteq (-\infty, \infty)$ be a finite interval on the real-axis \mathbb{R} . The left and right-sided Riemann–Liouville fractional integrals of order $\bar{\delta} > 0$ are, respectively, defined by [29]

$$\begin{aligned} \mathfrak{I}_{a^+}^{\bar{\delta}} g(\gamma) &= \frac{1}{\Gamma(\bar{\delta})} \int_a^\gamma (\gamma - t)^{\bar{\delta}-1} g(t) dt, \quad \gamma > a, \\ \mathfrak{I}_{b^-}^{\bar{\delta}} g(\gamma) &= \frac{1}{\Gamma(\bar{\delta})} \int_\gamma^b (t - \gamma)^{\bar{\delta}-1} g(t) dt, \quad \gamma < b. \end{aligned} \tag{3}$$

Definition 2. Let $(a, b) \subseteq (-\infty, \infty)$ be a finite or infinite interval of the real-axis \mathbb{R} and $\bar{\delta} > 0$. Let $\psi(\gamma)$ be an increasing and positive function on the interval (a, b) with a continuous derivative $\psi'(\gamma)$ on the interval (a, b) . Then, the left and right-sided ψ -Riemann–Liouville fractional integrals of a function f with respect to another function $\psi(\gamma)$ on $[a, b]$ are defined by [29, 30]

$$\begin{aligned} \mathfrak{I}_{a^+}^{\bar{\delta}; \psi} g(\gamma) &= \frac{1}{\Gamma(\bar{\delta})} \int_a^\gamma \psi'(t) (\psi(\gamma) - \psi(t))^{\bar{\delta}-1} g(t) dt, \\ \mathfrak{I}_{b^-}^{\bar{\delta}; \psi} g(\gamma) &= \frac{1}{\Gamma(\bar{\delta})} \int_\gamma^b \psi'(t) (\psi(t) - \psi(\gamma))^{\bar{\delta}-1} g(t) dt. \end{aligned} \tag{4}$$

If we set $\psi(\gamma) = \gamma$ in (4), then Definition 2 reduces to Definition 1.

The following lemmas hold in [29, 30].

Lemma 1. Let $\bar{\delta} > 0, \bar{\mu} > 0$, and $g(\gamma) = \gamma^{\bar{\mu}}$; then,

$$\mathfrak{I}_{a^+}^{\bar{\delta}} g(\gamma) = \frac{\Gamma(\bar{\mu} + 1)}{\Gamma(\bar{\delta} + \bar{\mu} + 1)} \gamma^{\bar{\delta} + \bar{\mu}}. \tag{5}$$

Lemma 2. Let $\bar{\delta} > 0, \bar{\mu} > 0$, and $g(\gamma) = (\psi(\gamma) - \psi(a))^{\bar{\mu}}$; then,

$$\mathfrak{I}_{a^+}^{\bar{\delta}; \psi} g(\gamma) = \frac{\Gamma(\bar{\mu} + 1)}{\Gamma(\bar{\delta} + \bar{\mu} + 1)} (\psi(\gamma) - \psi(a))^{\bar{\delta} + \bar{\mu}}. \tag{6}$$

Remark 1. In this context, $\mathfrak{I}^{\bar{\delta}} g(\gamma)$ and $\mathfrak{I}^{\bar{\delta}; \psi} g(\gamma)$ stand for $\mathfrak{I}_{0^+}^{\bar{\delta}} g(\gamma)$ and $\mathfrak{I}_{0^+}^{\bar{\delta}; \psi} g(\gamma)$, respectively.

In this paper, using the Adomian decomposition method and Laplace transform method combined with Lemma 1, we produce a new powerful technique. By using this technique, we obtain exact solution for main generalized Abel’s integral equations and generalized weakly singular Volterra integral equations.

3. Main Results

Now, we give our main results.

Lemma 3 (see [4, 5]). If $g(\gamma)$ is bounded on $0 < \gamma < b$, g is strictly monotonically increasing and differentiable function in some interval $(0, b)$ and $g(\gamma) \neq 0$ for every γ in $(0, b)$. Then, Abel’s integral equation (1) has the following solution:

$$u(\gamma) = \frac{\sin(\bar{\delta}\pi)}{\pi} \frac{d}{d\gamma} \int_0^\gamma \frac{g'(t)f(t)}{[g(\gamma) - g(t)]^{1-\bar{\delta}}} dt, \quad 0 < \bar{\delta} < 1. \tag{7}$$

Theorem 1. If $f(t)$ is bounded on $0 < \gamma < b$, g is strictly monotonically increasing and differentiable function in some interval $(0, b)$ and $g(\gamma) \neq 0$ for every γ in $(0, b)$. Abel’s integral equation (1) has the following solution:

$$u(\gamma) = \frac{\Gamma(\bar{\delta})\sin(\bar{\delta}\pi)}{\pi} \frac{d}{d\gamma} \left[\mathfrak{I}^{\bar{\delta}; \psi} g(\gamma) \right]. \tag{8}$$

Proof. From Lemma 3 for $\bar{\delta} \in (0, 1)$, we have

$$u(\gamma) = \frac{\Gamma(\bar{\delta})\sin(\bar{\delta}\pi)}{\pi} \frac{d}{d\gamma} \left(\frac{1}{\Gamma(\bar{\delta})} \int_0^\gamma g'(t) (g(\gamma) - g(t))^{\bar{\delta}-1} f(t) dt \right). \tag{9}$$

From this and Definition 2, we get (8). This completes the proof. \square

Corollary 1. Under the similar assumptions of Theorem 1, if $g(\gamma) = (g(\gamma) - g(0))^{\bar{\mu}}$, then the solution of Abel’s integral equation (1) is

$$u(\gamma) = \frac{\sin(\bar{\delta}\pi)}{\pi} \frac{\Gamma(\bar{\delta})\Gamma(\bar{\mu} + 1)}{\Gamma(\bar{\delta} + \bar{\mu} + 1)} \frac{d}{d\gamma} \left[(g(\gamma) - g(0))^{\bar{\delta} + \bar{\mu}} \right]. \tag{10}$$

Now, the solution of (2) can be obtained in the following theorem.

Theorem 2. If $g(\gamma) = \bar{\theta}(g(\gamma) - g(0))^{\bar{\mu}}$, g is strictly monotonically increasing and differentiable function in some interval $(0, b)$ and $g(\gamma) \neq 0$ for every γ in $(0, b)$. The generalized weakly singular Volterra type integral equation (2) has the following solution:

$$u(\gamma) = \bar{\theta}\Gamma(\bar{\mu} + 1)(g(\gamma) - g(0))^{\bar{\mu}} E_{1-\bar{\delta}, \bar{\mu}+1} \left(\bar{\beta}\Gamma(1 - \bar{\delta})(g(\gamma) - g(0))^{(1-\bar{\delta})} \right), \tag{11}$$

where $E_{\bar{\delta}, \bar{\mu}}$ is the 2-parameter Mittag-Leffler function which is defined by [31, 32]

$$E_{\bar{\delta}, \bar{\mu}}(\gamma) = \sum_{n=0}^{\infty} \frac{\gamma^n}{\Gamma(\bar{\delta}n + \bar{\mu})}. \tag{12}$$

Proof. From the Adomian decomposition method, we substitute the decomposition series

$$u(\gamma) = \sum_{n=0}^{\infty} u_n(\gamma), \tag{13}$$

into both sides of (2) to obtain

$$\sum_{n=0}^{\infty} u_n(\gamma) = g(\gamma) + \int_0^\gamma \frac{\bar{\beta}}{(g(\gamma) - g(t))^{\bar{\delta}}} \left(\sum_{n=0}^{\infty} u_n(t) \right) dt. \tag{14}$$

The components $u_0(\gamma), u_1(\gamma), u_2(\gamma), \dots$ are determined by using (3) in the Adomian recurrence relation:

$$\begin{aligned} u_0(\gamma) &= g(\gamma) = \bar{\theta}(g(\gamma) - g(0))^{\bar{\mu}}, \\ u_1(\gamma) &= \bar{\beta} \int_0^\gamma \frac{(g(t) - g(0))^{\bar{\mu}}}{(g(\gamma) - g(t))^{\bar{\delta}}} dt = \bar{\theta}\bar{\beta} \frac{\Gamma(1 - \bar{\delta})\Gamma(\bar{\mu} + 1)}{\Gamma(2 - \bar{\delta} + \bar{\mu})} (g(\gamma) - g(0))^{1-\bar{\delta}+\bar{\mu}}, \\ u_2(\gamma) &= \bar{\theta}\bar{\beta}^2 \int_0^\gamma \frac{u_1(t)}{(g(\gamma) - g(t))^{\bar{\delta}}} dt = \bar{\theta}\bar{\beta}^2 \frac{(\Gamma(1 - \bar{\delta}))^2\Gamma(\bar{\mu} + 1)}{\Gamma(3 - 2\bar{\delta} + \bar{\mu})} (g(\gamma) - g(0))^{2(1-\bar{\delta})+\bar{\mu}}, \\ u_3(\gamma) &= \bar{\theta}\bar{\beta}^3 \int_0^\gamma \frac{u_2(t)}{(g(\gamma) - g(t))^{\bar{\delta}}} dt = \bar{\theta}\bar{\beta}^3 \frac{(\Gamma(1 - \bar{\delta}))^3\Gamma(\bar{\mu} + 1)}{\Gamma(4 - 3\bar{\delta} + \bar{\mu})} (g(\gamma) - g(0))^{3(1-\bar{\delta})+\bar{\mu}}, \\ &\vdots \end{aligned} \tag{15}$$

Thus, the exact solution is

$$\begin{aligned} u(\gamma) &= \bar{\theta} \left((g(\gamma) - g(0))^{\bar{\mu}} + \bar{\beta} \frac{\Gamma(1 - \bar{\delta})\Gamma(\bar{\mu} + 1)}{\Gamma((1 - \bar{\delta}) + \bar{\mu} + 1)} (g(\gamma) - g(0))^{(1-\bar{\delta})+\bar{\mu}} + \bar{\beta}^2 \frac{(\Gamma(1 - \bar{\delta}))^2\Gamma(\bar{\mu} + 1)}{\Gamma(2(1 - \bar{\delta}) + \bar{\mu} + 1)} (g(\gamma) - g(0))^{2(1-\bar{\delta})+\bar{\mu}} + \dots \right) \\ &= \bar{\theta}\Gamma(\bar{\mu} + 1)(g(\gamma) - g(0))^{\bar{\mu}} \sum_{n=0}^{\infty} \frac{\left(\bar{\beta}\Gamma(1 - \bar{\delta})(g(\gamma) - g(0))^{(1-\bar{\delta})} \right)^n}{\Gamma((1 - \bar{\delta})n + \bar{\mu} + 1)} \\ &= \bar{\theta}\Gamma(\bar{\mu} + 1)(g(\gamma) - g(0))^{\bar{\mu}} E_{1-\bar{\delta}, \bar{\mu}+1} \left(\bar{\beta}\Gamma(1 - \bar{\delta})(g(\gamma) - g(0))^{(1-\bar{\delta})} \right), \end{aligned} \tag{16}$$

which completes the proof. \square

Corollary 2. Under the similar assumptions of Theorem 2, if $g(\gamma) = \gamma$, then the solution of the generalized weakly singular Volterra type integral equation (2) is

$$u(\gamma) = \bar{\theta}\Gamma(\bar{\mu} + 1)\gamma^{\bar{\mu}}E_{1-\bar{\delta},\bar{\mu}+1}\left(\bar{\beta}\Gamma(1-\bar{\delta})\gamma^{(1-\bar{\delta})}\right). \quad (17)$$

The noise terms may appear between components of $u_0(\gamma)$ and $u_1(\gamma)$ of (15) with opposite signs. Hence, by canceling these noise terms between these components, we may give the exact solution that should be justified through substitution and thus minimize the size of the calculations. In this situation, we use the following corollary.

Corollary 3. Under the similar assumptions of Theorem 2, the solution of the generalized weakly singular Volterra type integral equation (2) can be obtained as

$$u(\gamma) = \frac{\bar{\theta}\Gamma(\bar{\mu} + 1)}{\bar{\beta}\Gamma(1-\bar{\delta})}\gamma^{\bar{\delta}+\bar{\mu}-1}\left(E_{1-\bar{\delta},\bar{\delta}+\bar{\mu}}\left(\bar{\beta}\Gamma(1-\bar{\delta})\gamma^{(1-\bar{\delta})}\right) - \frac{1}{\Gamma(\bar{\delta} + \bar{\mu})}\right). \quad (18)$$

Proof. To prove this, we use one of the Mittag-Leffler function's properties, that is [31, 32],

$$E_{\bar{\delta},\bar{\mu}}(\gamma) = \frac{1}{\gamma}E_{\bar{\delta},\bar{\mu}-\bar{\delta}}(\gamma) - \frac{1}{x\Gamma(\bar{\mu} - \bar{\delta})}. \quad (19)$$

From this, we can write (17) as

$$u(\gamma) = \frac{\bar{\theta}\Gamma(\bar{\mu} + 1)\gamma^{\bar{\mu}}}{\bar{\beta}\Gamma(1-\bar{\delta})\gamma^{1-\bar{\delta}}}\left(E_{1-\bar{\delta},\bar{\delta}+\bar{\mu}}\left(\bar{\beta}\Gamma(1-\bar{\delta})\gamma^{(1-\bar{\delta})}\right) - \frac{1}{\Gamma(\bar{\delta} + \bar{\mu})}\right). \quad (20)$$

This completes the proof. \square

Theorem 3. Under the similar assumptions of Theorem 2, if $g(\gamma) = \sum_{j=1}^{\infty} \bar{\theta}_j \gamma^{\bar{\mu}_j}$, then the exact of the generalized weakly singular Volterra type integral equation (2) is given by

$$u(\gamma) = \sum_{j=1}^{\infty} \bar{\theta}_j \Gamma(\bar{\mu}_j + 1) \gamma^{\bar{\mu}_j} E_{1-\bar{\delta},\bar{\mu}_j+1}\left(\bar{\beta}\Gamma(1-\bar{\delta})\gamma^{(1-\bar{\delta})}\right), \quad (21)$$

or equivalently

$$u(\gamma) = \sum_{j=1}^{\infty} \frac{\bar{\theta}_j \Gamma(\bar{\mu}_j + 1)}{\bar{\beta}\Gamma(1-\bar{\delta})} \gamma^{\bar{\delta}+\bar{\mu}_j-1} \left(E_{1-\bar{\delta},\bar{\delta}+\bar{\mu}_j} \left(\bar{\beta}\Gamma(1-\bar{\delta})\gamma^{(1-\bar{\delta})} \right) - \frac{1}{\Gamma(\bar{\delta} + \bar{\mu}_j)} \right). \quad (22)$$

Proof. By the same manner of Theorem 2, we get

$$\begin{aligned} u(\gamma) &= \bar{\theta}_1 \Gamma(\bar{\mu}_1 + 1) \gamma^{\bar{\mu}_1} E_{1-\bar{\delta},\bar{\mu}_1+1} \left(\bar{\beta}\Gamma(1-\bar{\delta})\gamma^{(1-\bar{\delta})} \right) \\ &+ \bar{\theta}_2 \Gamma(\bar{\mu}_2 + 1) \gamma^{\bar{\mu}_2} E_{1-\bar{\delta},\bar{\mu}_2+1} \left(\bar{\beta}\Gamma(1-\bar{\delta})\gamma^{(1-\bar{\delta})} \right) \\ &+ \bar{\theta}_3 \Gamma(\bar{\mu}_3 + 1) \gamma^{\bar{\mu}_3} E_{1-\bar{\delta},\bar{\mu}_3+1} \left(\bar{\beta}\Gamma(1-\bar{\delta})\gamma^{(1-\bar{\delta})} \right) + \dots \end{aligned} \quad (23)$$

This completes the proof of the first part of this theorem.

By using property (19), we easily obtain the proof of the second part of this theorem. Thus, the proof of Theorem 3 is completed. \square

Remark 2. Due to the occurrence of the noise terms between $u_0(\gamma)$ and $u_1(\gamma)$, we can write $u(\gamma)$ as

$$\begin{aligned} u(\gamma) &= \bar{\theta}_1 \Gamma(\bar{\mu}_1 + 1) \gamma^{\bar{\mu}_1} E_{1-\bar{\delta},\bar{\mu}_1+1} \left(\bar{\beta}\Gamma(1-\bar{\delta})\gamma^{(1-\bar{\delta})} \right) \\ &+ \frac{\bar{\theta}_2 \Gamma(\bar{\mu}_2 + 1)}{\bar{\beta}\Gamma(1-\bar{\delta})} \gamma^{\bar{\delta}+\bar{\mu}_2-1} \left(E_{1-\bar{\delta},\bar{\delta}+\bar{\mu}_2} \left(\bar{\beta}\Gamma(1-\bar{\delta})\gamma^{(1-\bar{\delta})} \right) \right. \\ &\left. - \frac{1}{\Gamma(\bar{\delta} + \bar{\mu}_2)} \right) + \dots, \end{aligned} \quad (24)$$

which is obtained from (21) and (22).

Now, we introduce some spaces of the continuous functions in order to obtain the boundness of the above solution.

Definition 3. Let $0 < \bar{\mu} \leq 1$ and $J = [a, b]$ be a finite interval on the half-axis \mathbb{R}^+ with $0 \leq a < b < \infty$. Then,

(i) We denote by $C[a, b]$ the space of continuous functions g on J with the norm

$$\|g\|_C := \max\{|g(\gamma)|; \gamma \in [a, b]\}. \quad (25)$$

(ii) We define the weighted space $C_{\bar{\mu}}^{\psi}[a, b]$ of functions g with respect to an increasing function ψ on $(a, b]$ by

$$\begin{aligned} C_{\bar{\mu}}^{\psi}[a, b] &:= \max\{g: [a, b] \longrightarrow \mathbb{R}, (\psi(\gamma) - \psi(a))^{\bar{\mu}} g \\ &\cdot (\gamma) \in C[a, b]\}, \end{aligned} \quad (26)$$

with the norm

$$\|g\|_{C_{\bar{\mu}}^{\psi}} := \max\left\{ |(\psi(\gamma) - \psi(a))^{\bar{\mu}} g(\gamma)|; \gamma \in [a, b] \right\}. \quad (27)$$

Note that $C_0^{\psi}[a, b] = C[a, b]$.

Remark 3. Let $\bar{\delta}, \bar{\beta}, \tau > 0$; then, we have

$$\int_{\tau}^t (g(t) - g(s))^{\bar{\delta}} (g(s) - g(\tau))^{\bar{\beta}} g'(s) ds = (g(t) - g(\tau))^{\bar{\delta} + \bar{\beta} + 1} \int_0^1 (1 - z)^{\bar{\delta}} z^{\bar{\beta}} dz$$

$$= \frac{\Gamma(\bar{\delta} + 1)\Gamma(\bar{\beta} + 1)}{\Gamma(\bar{\delta} + \bar{\beta} + 2)} (g(t) - g(\tau))^{\bar{\delta} + \bar{\beta} + 1}. \tag{28}$$

Proof. The proof follows directly from the substitution $g(s) = g(\tau) + z(g(t) - g(\tau))$ and the definition of the beta function. \square

Theorem 4. Let $0 \leq a < b < \infty$, $\bar{\delta} \in (0, 1)$, and $\bar{\mu} > 0$. Then, If $\bar{\mu} \leq \bar{\delta}$, the fractional operator $\mathfrak{S}_{a^+}^{\bar{\delta}; \psi}$ is bounded from $C_{\bar{\mu}}^{\psi}[a, b]$ into $C[a, b]$ with

$$\left\| \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g \right\|_C \leq \|g\|_{C_{\bar{\mu}}^{\psi}} \frac{\Gamma(1 - \bar{\mu})}{\Gamma(\bar{\delta} - \bar{\mu} + 1)} (\psi(b) - \psi(a))^{\bar{\delta} - \bar{\mu}}. \tag{29}$$

(i) Moreover, if $\bar{\mu} \leq \bar{\delta} - 1$, the solution $u(x)$ in Theorem 1 is bounded from $C_{\bar{\mu}}^{\psi}[a, b]$ into $C[a, b]$ with

$$\|u\|_C \leq \|g\|_{C_{\bar{\mu}}^{\psi}} \frac{\sin(\bar{\delta}\pi)\Gamma(\bar{\delta})\Gamma(1 - \bar{\mu})}{\pi\Gamma(\bar{\delta} - \bar{\mu})} (\psi(b) - \psi(0))^{\bar{\delta} - \bar{\mu} - 1} \psi'(b). \tag{30}$$

(ii) For any $g \in C[a, b]$, the fractional operator $\mathfrak{S}_{a^+}^{\bar{\delta}; \psi}$ is a mapping from $C[a, b]$ into $C[a, b]$ with

$$\left\| \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g \right\|_C \leq \|g\|_C \frac{1}{\Gamma(\bar{\delta} + 1)} (\psi(b) - \psi(a))^{\bar{\delta}}. \tag{31}$$

Moreover, for any $g \in C[a, b]$, the solution $u(x)$ in Theorem 1 is bounded from $C[a, b]$ into $C[a, b]$ with

$$\|u\|_C \leq \|g\|_C \frac{\sin(\bar{\delta}\pi)}{\pi} (\psi(b)\psi(0))^{\bar{\delta} - 1} \psi'(b). \tag{32}$$

Proof

(i) From Definition 3, we get

$$\left\| \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g \right\|_C = \max \left\{ \left| \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g(x) \right|; x \in [a, b] \right\}$$

$$= \max_{\gamma \in (a, b)} \left\{ \left| \frac{1}{\Gamma(\bar{\delta})} \int_a^{\gamma} \psi'(s) (\psi(\gamma) - \psi(s))^{\bar{\delta} - 1} g(s) ds \right| \right\}$$

$$= \max_{\gamma \in (a, b)} \left\{ \left| \frac{1}{\Gamma(\bar{\delta})} \int_a^{\gamma} \psi'(s) (\psi(\gamma) - \psi(s))^{\bar{\delta} - 1} (\psi(s) - \psi(a))^{\bar{\mu}} (\psi(s) - \psi(a))^{-\bar{\mu}} g(s) ds \right| \right\}$$

$$\leq \|g\|_{C_{\bar{\mu}}^{\psi}} \left| \frac{1}{\Gamma(\bar{\delta})} \int_a^{\gamma} \psi'(s) (\psi(\gamma) - \psi(s))^{\bar{\delta} - 1} (\psi(s) - \psi(a))^{-\bar{\mu}} ds \right|.$$

Then, by making use of Remark 3, it follows that

$$\left\| \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g \right\|_C \leq \|g\|_{C_{\bar{\mu}}^{\psi}} \left| \frac{\Gamma(1 - \bar{\mu})}{\Gamma(\bar{\delta} - \bar{\mu} + 1)} (\psi(\gamma) - \psi(a))^{\bar{\delta} - \bar{\mu}} \right|. \tag{34}$$

Since ψ is an increasing function, it follows that

$$\left\| \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g \right\|_C \leq \|g\|_{C_{\bar{\mu}}^{\psi}} \frac{\Gamma(1 - \bar{\mu})}{\Gamma(\bar{\delta} - \bar{\mu} + 1)} (\psi(b) - \psi(a))^{\bar{\delta} - \bar{\mu}}. \tag{35}$$

This completes the proof of the first part inequality in (i). By making use of Theorem 1 and inequality (34) with $a = 0$, we obtain

$$\|u\|_C \leq \frac{\sin(\bar{\delta}\pi)\Gamma(\bar{\delta})}{\pi} \|g\|_{C_{\bar{\mu}}^{\psi}} \left| \frac{\Gamma(1 - \bar{\mu})}{\Gamma(\bar{\delta} - \bar{\mu} + 1)} \frac{d}{d\gamma} (\psi(\gamma) - \psi(0))^{\bar{\delta} - \bar{\mu}} \right|$$

$$= \|g\|_{C_{\bar{\mu}}^{\psi}} \left| \frac{\sin(\bar{\delta}\pi)\Gamma(\bar{\delta})\Gamma(1 - \bar{\mu})}{\pi\Gamma(\bar{\delta} - \bar{\mu})} (\psi(\gamma) - \psi(0))^{\bar{\delta} - \bar{\mu} - 1} \psi'(x) \right|$$

$$\leq \|g\|_{C_{\bar{\mu}}^{\psi}} \frac{\sin(\bar{\delta}\pi)\Gamma(\bar{\delta})\Gamma(1 - \bar{\mu})}{\pi\Gamma(\bar{\delta} - \bar{\mu})} (\psi(b) - \psi(0))^{\bar{\delta} - \bar{\mu} - 1} \psi'(b). \tag{36}$$

Let $g \in C[a, b]$; then, by using Definition 3 and Lemma 2 and since ψ is an increasing function, we have

$$\begin{aligned} \left\| \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g \right\|_C &= \max_{\gamma \in (a, b)} \left| \frac{1}{\Gamma(\bar{\delta})} \int_a^\gamma \psi'(s) (\psi(\gamma) - \psi(s))^{\bar{\delta}-1} g(s) ds \right| \\ &= \|g\|_C \left| \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} 1 \right| \\ &= \|g\|_C \left| \frac{1}{\Gamma(\bar{\delta} + 1)} (\psi(\gamma) - \psi(a))^{\bar{\delta}} \right| \\ &\leq \|g\|_C \frac{1}{\Gamma(\bar{\delta} + 1)} (\psi(b) - \psi(a))^{\bar{\delta}}. \end{aligned} \quad (37)$$

This completes the proof of the first inequality of (ii). By making use of Theorem 1 and inequality (37) with $a = 0$, we can deduce the second part inequality in (ii). \square

Theorem 5. For any $\bar{\delta} \in (0, 1]$, $\bar{\mu}, a \geq 0$, we have

$$\left(\mathfrak{S}_{a^+}^{\bar{\delta}; \psi} \right) (\psi(s)\psi(a))^{\bar{\mu}}(\gamma) = \frac{\Gamma(\bar{\mu} + 1)}{\Gamma(\bar{\delta} + \bar{\mu} + 1)} (\psi(\gamma) - \psi(a))^{\bar{\delta} + \bar{\mu}}. \quad (38)$$

$$\begin{aligned} \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g(\gamma) &= \frac{1}{\Gamma(\bar{\delta})} \int_a^\gamma \psi'(t) (\psi(\gamma) - \psi(t))^{\bar{\delta}-1} g(t) dt \\ &= \frac{1}{\Gamma(\bar{\delta})} \int_a^\gamma \psi'(s) (\psi(\gamma) - \psi(s))^{\bar{\delta}-1} (\psi(s) - \psi(\gamma))^{-\bar{\mu}} (\psi(s) - \psi(\gamma))^{\bar{\mu}} g(s) ds. \end{aligned} \quad (41)$$

Then, by making use of Remark 3 and Theorem 5, it follows that

$$\begin{aligned} \left\| \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g \right\| &\leq \frac{M}{\Gamma(\bar{\delta})} \int_a^\gamma \psi'(s) (\psi(\gamma) - \psi(s))^{\bar{\delta}-1} (\psi(s) - \psi(\gamma))^{-\bar{\mu}} ds \\ &= M \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} (\psi(s) - \psi(\gamma))^{-\bar{\mu}} \\ &= \frac{\Gamma(1 - \bar{\mu})}{\Gamma(\bar{\delta} - \bar{\mu} + 1)} (\psi(\gamma) - \psi(a))^{\bar{\delta} - \bar{\mu}}. \end{aligned} \quad (42)$$

Taking the limit on both sides, it follows that

$$\lim_{\gamma \rightarrow a} \left\| \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g \right\| = 0, \quad (43)$$

which completes the proof of the first part. By making use of Theorem 1 and formula (42) with $a = 0$, we can deduce the second part of theorem. \square

4. Test Examples

In this section, we consider several test problems corresponding to the equations (1) and (2) to demonstrate the efficiency of our new mechanism. \square

Proof. The proof is similar to Lemma 2, so it is omitted. \square

Theorem 6. Let $0 \leq a < b < \infty$, $\bar{\delta} \in (0, 1)$, $\bar{\mu} > 0$ and $g \in C_{\bar{\mu}}^{\psi}[a, b]$. If $\bar{\mu} < \bar{\delta}$, then we have

$$\mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g(a) = \lim_{x \rightarrow a} \mathfrak{S}_{a^+}^{\bar{\delta}; \psi} g(\gamma) = 0. \quad (39)$$

Moreover, the solution $u(\gamma)$ in Theorem 1 vanishes at $\gamma = 0$.

Proof. Let $g \in C_{\bar{\mu}}^{\psi}[a, b]$; then, $(\psi(\gamma)\psi(a))^{\bar{\mu}}g(\gamma) \in C[a, b]$ and there exists some $M > 0$ such that

$$\left| (\psi(\gamma) - \psi(a))^{\bar{\mu}} g(\gamma) \right| \leq M, \quad \gamma \in [a, b], \quad (40)$$

and

Example 1. Consider the generalized Abel's integral equation [4]:

$$\frac{4}{3} (\sin \gamma)^{(3/4)} = \int_0^\gamma \frac{u(t) dt}{(\sin \gamma - \sin t)^{(1/4)}, \quad 0 < \gamma < \frac{\pi}{2}, \quad (44)$$

where $\bar{\delta} = (1/4)$, $g(\gamma) = (4/3) (\sin \gamma)^{(3/4)}$ and $g(\gamma) = \sin \gamma$. It is clear that $g(\gamma)$ is strictly monotonically increasing in $0 < \gamma < (\pi/2)$ and $g'(\gamma) \neq 0$ for each γ in $0 < \gamma < (\pi/2)$. Using Corollary 1 with $\bar{\mu} = (3/4)$, we get

$$u(\gamma) = \frac{1}{\sqrt{2\pi}} \frac{4}{3} \frac{(3/4)\Gamma(1/4)\Gamma(3/4)}{\Gamma(2)} \frac{d}{d\gamma} [\sin \gamma] \quad (45)$$

$$= \cos \gamma,$$

which is the exact solution, where we have used the fact that $\Gamma(1/4)\Gamma(3/4) = \sqrt{2\pi}$.

Example 2. Consider the generalized Abel's integral equation [4]:

$$\frac{6}{25} \gamma^{(25/6)} = \int_0^\gamma \frac{u(t) dt}{(\gamma^5 - t^5)^{(1/6)}, \quad 0 < \gamma < 2. \quad (46)$$

Here, equation (7) takes the following form:

$$\begin{aligned}
 u(\gamma) &= \frac{\sin(\bar{\delta}\pi)}{\pi} \frac{d}{d\gamma} \int_0^x \frac{5t^4 \cdot (6/25)t^{(25/6)}}{(\gamma^5 - t^5)^{(1/6)}} dt \\
 &= \frac{1}{2\pi} \frac{6}{25} \frac{d}{d\gamma} \int_0^x \frac{5t^4 (t^5)^{(5/6)}}{(\gamma^5 - t^5)^{(1/6)}} dt.
 \end{aligned}
 \tag{47}$$

From this, we have $\bar{\delta} = (1/6)$, $g(\gamma) = (6/25)\gamma^{(25/6)}$, $g(\gamma) = \gamma^5$, and $\bar{\mu} = (5/6)$. Thus, Corollary 1 gives the exact solution:

$$u(\gamma) = \frac{1}{2\pi} \frac{d}{d\gamma} \left(\frac{2}{5} \pi \gamma^5 \right) = \gamma^4,
 \tag{48}$$

where we have used the fact that $\Gamma(1/6)\Gamma(5/6) = 2\pi$.

Example 3. Consider the weakly singular second kind Volterra integral equation [4, 33, 34]:

$$u(\gamma) = 2\sqrt{\gamma} - \int_0^\gamma \frac{u(t)dt}{\sqrt{\gamma-t}}, \quad x \in [0, 2].
 \tag{49}$$

Using Corollary 3 with $\bar{\theta} = 2$, $\bar{\mu} = (1/2)$, $\bar{\beta} = -1$, and $\bar{\delta} = (1/2)$, we can easily obtain

$$\begin{aligned}
 u(\gamma) &= 1 - E_{(1/2),1}(-\sqrt{\pi\gamma}) \\
 &= 1 - e^{-\pi x} \operatorname{erfc}(\sqrt{\pi\gamma}),
 \end{aligned}
 \tag{50}$$

where erfc is the complementary error function which is defined as

$$\operatorname{erfc}(w) = \frac{2}{\sqrt{\pi}} \int_w^\infty e^{-z^2} dz.
 \tag{51}$$

Example 4. Consider the weakly singular second kind Volterra integral equation [4, 33, 34]:

$$u(\gamma) = \gamma^2 + \frac{16}{15} \gamma^{(52/15)} - \int_0^\gamma \frac{u(t)dt}{\sqrt{\gamma-t}}, \quad x \in [0, 1],
 \tag{52}$$

where $\bar{\theta}_1 = 1$, $\bar{\theta}_2 = (16/15)$, $\bar{\mu}_1 = 2$, $\bar{\mu}_2 = (5/2)$, $\bar{\beta} = -1$, and $\bar{\delta} = (1/2)$. Thus, by formula (24), we get

$$\begin{aligned}
 u(\gamma) &= \Gamma(2+1)\gamma^2 E_{(1/2),3}(-\sqrt{\pi\gamma}) \\
 &\quad - \frac{16}{15} \frac{\Gamma((5/2)+1)\gamma^{(5/2)}}{\Gamma(1/2)\gamma^{(1/2)}} \left(E_{(1/2),3}(-\sqrt{\pi\gamma}) - \frac{1}{\Gamma(3)} \right) \\
 &= 2\gamma^2 E_{(1/2),3}(-\sqrt{\pi\gamma}) - 2\gamma^2 E_{(1/2),3}(-\sqrt{\pi\gamma}) + \gamma^2 = \gamma^2.
 \end{aligned}
 \tag{53}$$

Hence, the exact solution is $u(\gamma) = \gamma^2$.

Example 5. Consider the weakly singular second kind Volterra integral equation [9]:

$$u(\gamma) = \sqrt{\gamma} - \pi\gamma + 2 \int_0^\gamma \frac{u(t)dt}{\sqrt{\gamma-t}}
 \tag{54}$$

In this example, $\bar{\theta}_1 = 1$, $\bar{\theta}_2 = -\pi$, $\bar{\mu}_1 = (1/2)$, $\bar{\mu}_2 = 1$, $\bar{\beta} = 2$, and $\bar{\delta} = (1/2)$. Thus, formula (24) gives

$$\begin{aligned}
 u(\gamma) &= \frac{1}{2} \sqrt{\pi\gamma} E_{(1/2),(3/2)}(2\sqrt{\pi\gamma}) \\
 &\quad - \frac{1}{2} \sqrt{\pi\gamma} E_{(1/2),(3/2)}(2\sqrt{\pi\gamma}) + \sqrt{\gamma} = \sqrt{\gamma},
 \end{aligned}
 \tag{55}$$

which is the exact solution of integral equation (54).

Example 6. Consider the weakly singular second kind Volterra integral equation [9]:

$$u(\gamma) = \frac{1}{2} - \sqrt{\gamma} + \int_0^\gamma \frac{u(t)dt}{\sqrt{\gamma-t}}
 \tag{56}$$

From formula (24) with $\bar{\theta}_1 = (1/2)$, $\bar{\theta}_2 = -1$, $\bar{\mu}_1 = 0$, $\bar{\mu}_2 = (1/2)$, $\bar{\beta} = 1$, and $\bar{\delta} = (1/2)$, we get

$$u(\gamma) = \frac{1}{2} E_{(1/2),1}(\sqrt{\pi\gamma}) - \frac{1}{2} E_{(1/2),1}(\sqrt{\pi\gamma}) + \frac{1}{2} = \frac{1}{2},
 \tag{57}$$

which is the exact solution of integral equation (56).

Example 7. Consider the Abel integral equation of the second kind [33]:

$$u(\gamma) = \gamma + \frac{4}{3} \gamma^{(3/2)} - \int_0^\gamma \frac{u(t)dt}{\sqrt{\gamma-t}}
 \tag{58}$$

From formula (24) with $\bar{\theta}_1 = 1$, $\bar{\theta}_2 = (4/3)$, $\bar{\mu}_1 = 1$, $\bar{\mu}_2 = (3/2)$, $\bar{\beta} = -1$, and $\bar{\delta} = (1/2)$, we get

$$u(\gamma) = \gamma E_{(1/2),2}(-\sqrt{\pi\gamma}) - \gamma E_{(1/2),2}(-\sqrt{\pi\gamma}) + \gamma = \gamma,
 \tag{59}$$

which is the exact solution of (58).

5. Conclusion

In the present article, a new technique involving Riemann–Liouville fractional integrals has been used to solve main generalized Abel’s integral equations and generalized weakly singular Volterra integral equations. Also, we have solved several examples with our proposed technique. We can observe that our developed technique is easy and straightforward to apply.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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