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Research Article

Turing Instability and Amplitude Equation of Reaction-Diffusion System with Multivariable

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In this paper, we investigate pattern dynamics with multivariable by using the method of matrix analysis and obtain a condition under which the system loses stability and Turing bifurcation occurs. In addition, we also derive the amplitude equation with multivariable. This is an effective tool to investigate multivariate pattern dynamics. The example and simulation used in this paper validate our theoretical results. The method presented is a novel approach to the investigation of specific real systems based on the model developed in this paper.

1. Introduction

The pattern formation was first investigated and interpreted by Alan Turing 60 years ago [1]. In general, Turing model contains two reactants: activator and inhibitor, which engage in diffusion. Recently, the study of Turing bifurcation, amplitude equation, and secondary bifurcation have paid more attention on the pattern formation [2–4], and Lee and Cho found that dynamical parameters and external periodic forcing play an important role in the shape and type of pattern formation [5]. And the robustness problem is also investigated [6]. The effects of cross-diffusion, the phenomenon in which a gradient in the concentration of one species induces other species, on pattern formation in reaction-diffusion systems have been discussed in many theoretical papers [7–14]. Regarding noise, noise is a ubiquitous phenomenon in nature and always deemed to play a very important role in the natural synthetic system [15]. Viney and Reece [16] treated noise as adaptive and suggested that applying evolutionary rigour to the study of noise is necessary to fully understand organismal phenotypes, and Shen considered the Lévy noise in the gene network [17, 18].

Recently, the pattern formation with three or four variables has been investigated, and it obtains promising results [19, 20], and Xu et al. made a concrete analysis with three variables [21]. As we all know that amplitude equation is a promising tool to investigate the pattern dynamics of the reaction-diffusion system [2, 22], however, the amplitude equation is a complex process [3], and the researcher often chose the amplitude equation [23–26] to investigate the reaction-diffusion system. In conclusion, spatial patterns in reaction-diffusion systems have attracted the interest of experimentalists and theorists during the last few decades. But, these previous works did not give a general method to define Turing instability and derive the amplitude equation with n variables.

Besides, the study of patterns can offer useful information on the underlying processes causing possible changes in the system. In order to better understand the reaction-diffusion model, first, we propose to study the Turing instability with n variables by matrix theory [27]. In addition, we also derive the amplitude equation by using the standard multiple scale analysis [28, 29] which provides a way to investigate the mechanism of pattern formation.

The paper is organized as follows. In Section 2, we give the general reaction diffusion with multivariable and derive

the condition of Turing instability. In Section 3, we derive the amplitude equation from the reaction-diffusion system with multivariable. In Section 4, we utilize an example to illustrate the application of these ideas. The simulation validates our theoretical results. Finally, we summarize our results and conclusion.

2. Turing Instability with Multivariate

For a multivariate reaction-diffusion system, we have

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}(\mathbf{u}) + \mathbf{D}\nabla^2\mathbf{u}, \quad \mathbf{u} = (u_{11}, \ldots, u_{nn}), \tag{1}$$

where the function $\mathbf{f}(\mathbf{u})$ specifies dynamics of the interaction of the species and $\mathbf{D}$ is the diffusion parameter diagonal matrix.

Also, we can obtain the following linear system at equilibrium $\mathbf{u} = \mathbf{0}$ from (1):

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{A}\mathbf{u} + \mathbf{D}\nabla^2\mathbf{u}, \tag{2}$$

where A is the Jacobian matrix.

As we all know that the stability of a system depends on the sign of the real part of eigenvalue [23], for coefficient matrix $\mathbf{A}$, there is a nonsingular matrix $\mathbf{P}$ subject to $\mathbf{A} = \mathbf{P}^{-1}\mathbf{J}\mathbf{P}$, and $\mathbf{J}$ is a Jordan form [27]. Also, we have

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{A}| = |\lambda\mathbf{I} - \mathbf{J}| &= \begin{vmatrix} \lambda - \mathbf{J}_1 & 0 & 0 & 0 \\ 0 & \lambda - \mathbf{J}_2 & 0 & \vdots \\ 0 & 0 & \lambda - \mathbf{J}_3 & 0 \\ 0 & \cdots & \cdots & \lambda - \mathbf{J}_n \end{vmatrix} \\ &= (\lambda + \lambda_1)(\lambda + \lambda_2)\cdots(\lambda + \lambda_n) = 0, \end{aligned} \tag{3}$$

where

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & 0 & \vdots \\ 0 & \mathbf{J}_2 & \vdots \\ \cdots & \cdots & \mathbf{J}_i \end{pmatrix}, \tag{4}$$

where

$$\mathbf{J}_k = \begin{pmatrix} \lambda_k & 1 & 0 & 0 \\ 0 & \lambda_k & 1 & \vdots \\ 0 & 0 & \lambda_k & 1 \\ 0 & \cdots & \cdots & \lambda_k \end{pmatrix}, \tag{5}$$

where $-\lambda_i$ is the eigenvalue and has a negative real part which means stable without diffusion. In addition, we can get the condition of stability by Routh–Hurwitz criteria.

In the standard way, we assume that $\mathbf{u}$ takes the form

$$\mathbf{u} = \mathbf{c}e^{\lambda_k t + ikr}, \tag{6}$$

and obtains the characteristic equation from system (1.2) as follows:

$$\left|\lambda\mathbf{I} - \mathbf{A} + \mathbf{D}k^2\right| = \left|\lambda\mathbf{I} - \mathbf{J} - \mathbf{D}k^2\right| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0, \tag{7}$$

and there is at least $\lambda > 0$ which exists when $a_n < 0$ based on Routh–Hurwitz criteria.

Now, we obtain the definition of Turing instability with n variables.

Theorem 1. *Turing instability occurs when $a_n < 0$.*

In addition, we obtain the critical condition of Turing instability. Assume $x = k^2$ and $p(x) = x^n + p_1x^{n-1} + \cdots + p_n$, $p'(x) = 0$ has $n-1$ roots, and $x_i, i = 1, \ldots, n-1$ and can get the minimum value $p(x_c > 0)$ [30]. We can obtain the critical condition of Turing instability from $a_n(k_c^2) = 0$.

3. Amplitude Equation with Multivariable

In this paper, we continue to study the system with n variables. In the following, we use multiple scale analysis to derive the amplitude equations.

The solutions of systems can be expanded as

$$c = c_0 + Z_i e^{ik_i r} + c.c., \quad i = 1, 2, 3. \tag{8}$$

And system (1) can be written as

$$\frac{\partial \mathbf{c}}{\partial t} = \mathbf{L}\mathbf{c} + N(\mathbf{c}, \mathbf{c}), \tag{9}$$

where $\mathbf{c} = \mathbf{u}$ is the variable, $\mathbf{L} = \mathbf{A} + \mathbf{D}\nabla^2$ is the linear operator, $N = N_1\mathbf{u}^2 + N_2\mathbf{u}^3$ is the nonlinear term, $N_1\mathbf{u}^2$ is all the twice term, and $N_2\mathbf{u}^3$ is all the triple term.

We need to investigate the dynamical behavior when γ is close to γ_c, and then we expand γ as

$$\gamma_c - \gamma = \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \cdots, \tag{10}$$

where γ_c is the critical value and ε is a small enough parameter.

We expand $\mathbf{c}$ and N as the series form of ε:

$$\begin{aligned} \mathbf{c} &= \mathbf{u}_1\varepsilon + \mathbf{u}_2\varepsilon^2 + \cdots, \\ N &= N_1\mathbf{u}_1^2\varepsilon^2 + \left(N_1\mathbf{u}_1\mathbf{u}_2 + N_2\mathbf{u}_1^3\right)\varepsilon^3 + o\left(\varepsilon^4\right). \end{aligned} \tag{11}$$

Linear operator L can be expanded as

$$\mathbf{L} = \mathbf{L}_c + (\gamma_c - \gamma)M. \tag{12}$$

Let

$$\begin{aligned} T_0 &= t, \\ T_1 &= \varepsilon t, \\ T_2 &= \varepsilon^2 t, \ldots. \end{aligned} \tag{13}$$

T_i is a dependent variable, and amplitude is a slow variable. For the derivation of time, we have that

$$\frac{\partial W}{\partial t} = \varepsilon\frac{\partial W}{\partial T_1} + \varepsilon^2\frac{\partial W}{\partial T_2} + \cdots. \tag{14}$$

Substituting the above equations into (1) and expanding (1) according to different orders of ε, we can obtain three equations as follows:

$$\begin{aligned}
&\varepsilon : \mathbf{L}_c\mathbf{u}_1 = 0,\\
&\varepsilon^2 : \mathbf{L}_c\mathbf{u}_2 = \frac{\partial}{\partial T_1}\mathbf{u}_1 - \gamma_1 M\mathbf{u}_1 - N_1\mathbf{u}_1^2,\\
&\varepsilon^3 : \mathbf{L}_c\mathbf{u}_3 = \frac{\partial}{\partial T_1}\mathbf{u}_2 + \frac{\partial}{\partial T_2}\mathbf{u}_1 - \gamma_1 M\mathbf{u}_2 - \gamma_2 M\mathbf{u}_1 - N_1\mathbf{u}_1\mathbf{u}_2 - N_2\mathbf{u}_1^3.
\end{aligned} \tag{15}$$

We first consider the case of the first order of ε. Since L_c is the linear operator of the system close to the onset, $\mathbf{u}_1$ is the linear combination of the eigenvectors that corresponds to the eigenvalue zero. Since that

$$\mathbf{u}_1 = \mathbf{b}W_i e^{ik_i r} + c.c, \tag{16}$$

where $\mathbf{L}_c\mathbf{b} = 0$, nonzero root exists [27].

Now, we consider the case of ε^2 and the zero eigenvectors of operator $\mathbf{L}_c^+$, $\exists\mathbf{b}^+$ s.t.:

$$\mathbf{L}_c^+\mathbf{b}^+ = 0. \tag{17}$$

By investigating $e^{ik_1 r}$ only in the following, another case we can get is by changing subscript which is not described in detail here. It can be obtained from the orthogonality condition that

$$\mathbf{b}^{+T}\mathbf{b}\frac{\partial}{\partial T_1}W_1 = \mathbf{b}^{+T}M\mathbf{b}W_1 + \left(\mathbf{b}^{+T}\mathbf{b}^2\right)N_1\overline{W_2W_3}. \tag{18}$$

By using the same methods, one has

$$\begin{aligned}
\mathbf{u}_2 = {} & a_0 + a_i Z_i e^{ik_i r} + a_{ii}Z_{ii}e^{i2k_i r} + a_{12}Z_{12}e^{i(k_1-k_2)r}\\
& + a_{23}Z_{23}e^{i(k_2-k_3)r} + a_{31}Z_{31}e^{i(k_3-k_1)r} + c.c.
\end{aligned} \tag{19}$$

For the case of ε^3, we have that

$$\mathbf{L}_c\mathbf{u}_2 = 0. \tag{20}$$

We can obtain all the coefficients and $a_i = \mathbf{b}$.

Using the Fredholm solubility condition again, we can obtain

$$\begin{aligned}
\left(\mathbf{b}^{+T}\mathbf{b}\right)\frac{\partial Z}{\partial T_1} + \left(\mathbf{b}^{+T}\mathbf{b}\right)\frac{\partial Z}{\partial T_2} = {} & \gamma_1\mathbf{b}^{+T}M\mathbf{b}Z_1 + \gamma_2\mathbf{b}^{+T}M\mathbf{b}W_1\\
& + \mathbf{b}^{+T}\mathbf{b}^2N_1\left(\overline{Z_2W_3} + \overline{Z_3W_2}\right)\\
& + \mathbf{b}^{+T}\mathbf{b}^3N_2\left(|W|_1^2 + |W_2|^2 + |W_3|^2\right)W_1,
\end{aligned} \tag{21}$$

and then we substitute systems (18) and (21) into (14) for simplification, in which we can obtain

$$\begin{aligned}
\left(\mathbf{b}^{+T}\mathbf{b}\right)\frac{\partial W_1}{\partial t} = {} & (\gamma_c - \gamma)W_1 + \mathbf{b}^{+T}\mathbf{b}^2N_1\overline{W_2W_3}\\
& + \mathbf{b}^{+T}\mathbf{b}^3N_2\left(|W|_1^2 + |W_2|^2 + |W_3|^2\right)W_1.
\end{aligned} \tag{22}$$

4. Method

In the following, we consider the Turing instability of a system with 3 variables:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{k_0 u}{1 + k_0 u} - \mu u + d_1\nabla^2 u,\\
\frac{\partial v}{\partial t} &= \frac{k_1 u + k_2 v}{1 + k_1 u + k_2 v} - \mu v + d_2\nabla^2 v,\\
\frac{\partial w}{\partial t} &= \frac{k_3 u + k_4 v + k_5 w}{1 + k_3 u + k_4 v + k_5 w} - \mu w + d_3\nabla^2 w,
\end{aligned} \tag{23}$$

and we obtain the Jacobian matrix at equilibrium $(u_0, v_0, w_0) = (0, 0, 0)$:

$$A = \begin{pmatrix} k_0 - \mu & 0 & 0\\ k_1 & k_2 - \mu & 0\\ k_3 & k_4 & k_5 - \mu \end{pmatrix}. \tag{24}$$

The characteristic equation is

$$|\lambda I - A| = (\lambda - k_0 + \mu)(\lambda - k_2 + \mu)(\lambda - k_5 + \mu) = 0. \tag{25}$$

The system is stable without diffusion when $\mu - k_0 < 0, \mu - k_2 < 0$ and $\mu - k_5 < 0$, that is to say it is stable when $\mu < min(k_0, k_2, k_5)$.

Then, the Jacobian matrix with diffusion is given as follows:

$$B = \begin{pmatrix} k_0 - \mu - d_1k^2 & 0 & 0\\ k_1 & k_2 - \mu - d_2k^2 & 0\\ k_3 & k_4 & k_5 - \mu - d_3k^2 \end{pmatrix}. \tag{26}$$

The characteristic equation is

$$|\lambda I - B| = \lambda^3 + a\lambda^2 + b\lambda + c = 0, \tag{27}$$

where

$$\begin{aligned}
a &= (d_3 + d_1 + d_2)k^2 - k_0 - k_5 + 3\mu - k_2,\\
b &= (d_2d_3 + d_1d_3 + d_1d_2)k^4 + (-d_1k_2 - k_0d_3 + 2\mu d_3 + 2\mu d_2\\
&\quad - k_0d_2 - k_2d_3 - d_2k_5 - d_1k_5 + 2d_1\mu)k^2\\
&\quad - 2\mu k_2 + 3\mu^2 - 2\mu k_5 - 2k_0\mu + k_0k_2 + k_2k_5 + k_0k_5,\\
c &= a_5k^6 + b_5k^4 + c_5k^2 + d_5,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
a_5 &= d_1d_2d_3,\\
b_5 &= \mu d_2d_3 - k_0d_2d_3 - d_1k_2d_3 + d_1\mu d_3 - d_1d_2k_5 + d_1d_2\mu,\\
c_5 &= -\mu d_2k_5 - d_1\mu k_5 + \mu^2d_2 - k_0d_2\mu - d_1k_2\mu - \mu k_2d_3\\
&\quad + k_0d_2k_5 + d_1\mu^2 + d_1k_2k_5 + \mu^2d_3 + k_0k_2d_3 - k_0\mu d_3,\\
d_5 &= -\mu^2k_5 + k_0\mu k_5 + \mu k_2k_5 + \mu^3 + k_0k_2\mu - k_0k_2k_5 - \mu^2k_2 - k_0\mu^2.
\end{aligned} \tag{29}$$

For convenience, we assume $x = k^2$, and then $f(x) = a_5x^3 + b_5x^2 + c_5x + d_5$ and $f'(x) = 3a_5x^2 + 2b_5x + c_5$ which have two roots $x_{12} = -b_5 \pm \sqrt{b_5^2 - 3a_5c_5}$. We know $f(x) \geq f(x_1)$, $k_c^2 = x_1 = -b_5 + \sqrt{b_5^2 - 3a_5c_5}$, and the critical value is $f(x_1) = 0$.

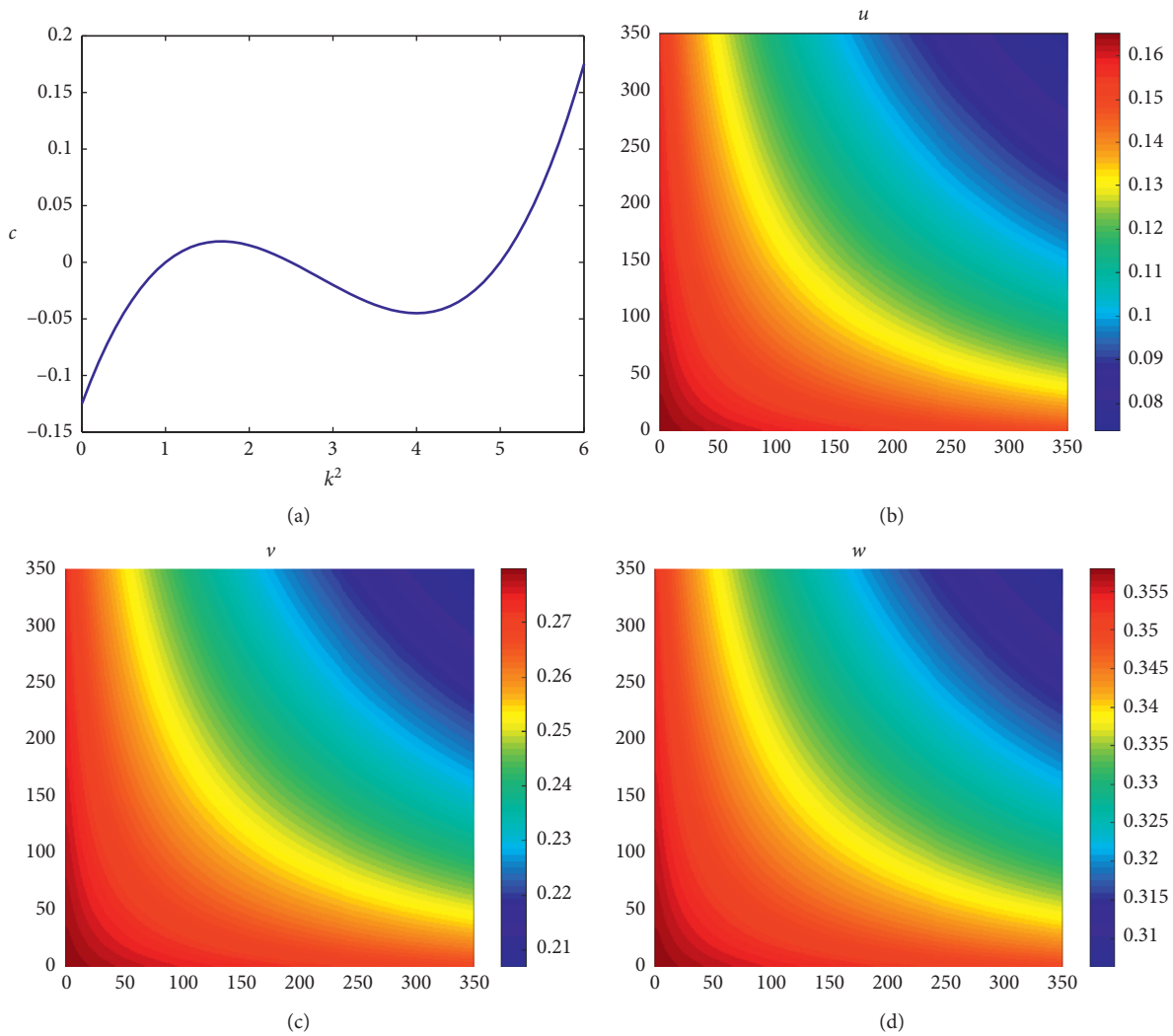


Figure 1: Dispersion curve and rainbow pattern.

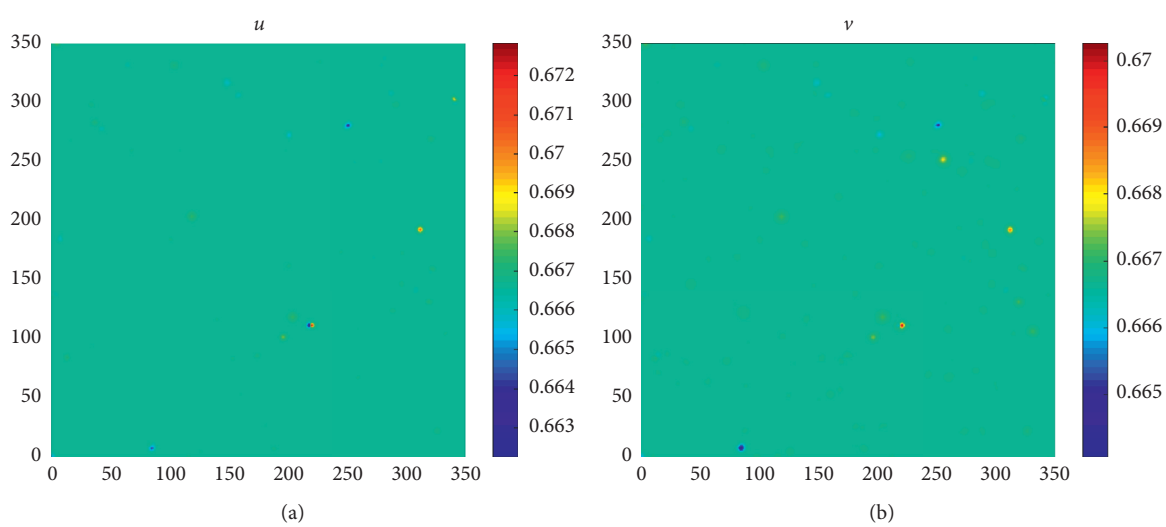
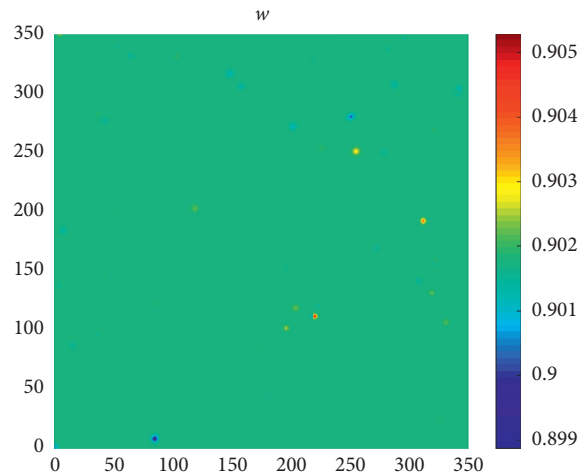
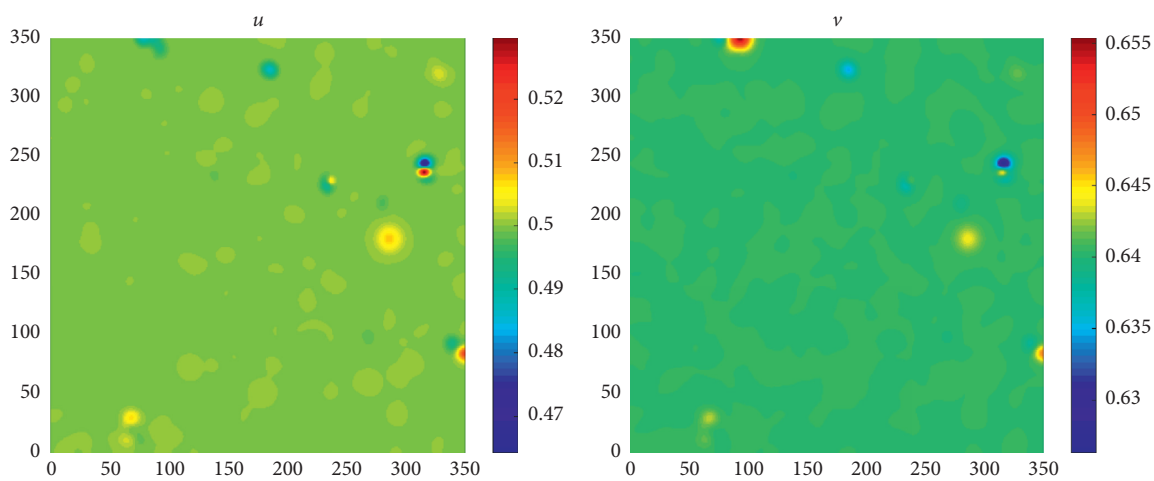


Figure 2: Continued.

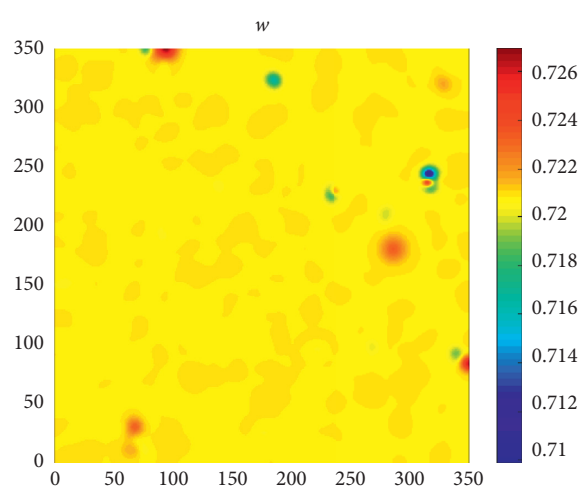


(c)

Figure 2: Spot pattern occurs.



(a) (b)



(c)

Figure 3: Nebulous pattern occurs.

We get $k_c^2 = 0.12$, $c = -0.1022$ when $k_0 = 2$, $k_1 = 1$, $k_2 = 2$, $k_3 = 1$, $k_4 = 1$, $k_5 = 2$, $d_1 = 0.1$, $d_2 = 0.2$, $d_3 = 0.5$, and $\mu = 1.5$ with the perturbation $1/((X+10)(Y+20))$ which means Turing instability, and then the rainbow stripe pattern (Figures 1(b)–1(d)) and dispersion curve occur (Figure 1(a)). In addition, spot pattern occurs (Figure 2) when $k_0 = 3, k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, k_5 = 5, d_1 = 0.1$, $d_2 = 0.2, d_3 = 0.3, \mu = 1$ with the perturbation $\sin(X^2 + Y^2)$, and nebulous pattern occurs (Figure 3) when $k_0 = 2, k_1 = 1$, $k_2 = 2, k_3 = 1, k_4 = 1, k_5 = 2$, $d_1 = 1$, $d_2 = 2$, $d_3 = 1$, $\mu = 1$ with the perturbation $\sin(X^2 + Y^2)$.

5. Conclusion

In this article, we present the theoretical analysis and numerical simulation of the Turing instability with multivariable. It is found that the reaction-diffusion systems with multivariable have rich spatial dynamics by performing a series of numerical simulations. We also give a general method to derive the amplitude equation with multivariable in theory which can be used to solve some problems about pattern formation with multiple variables in the further study. In addition, the mechanism of pattern formation with multiple variables is on the way and can be derived based on the above theory in this paper; however, it is a very complex process, and we will investigate it in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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