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Research Article

The Properties of Generalized Collision Branching Processes

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We consider basic properties regarding uniqueness, extinction, and explosivity for the Generalized Collision Branching Processes (GCBP). Firstly, we investigate some important properties of the generating functions for GCB q-matrix in detail. Then for any given GCB q-matrix, we prove that there always exists exactly one GCBP. Next, we devote to the study of extinction behavior and hitting times. Some elegant and important results regarding extinction probabilities, the mean extinction times, and the conditional mean extinction times are presented. Moreover, the explosivity is also investigated and an explicit expression for mean explosion time is established.

1. Introduction

In this paper, we mainly consider extinction and explosivity for the Generalized Collision Branching Processes (GCBP). The particles in the system that evolves can be described as follows. Collisions between particles occur at random, and whenever m particles collide, they are removed and replaced by j "offsprings" with probability p_j $(j \geq 0)$, independently of other collisions. In any small time interval $(t, t+\Delta t)$, there is a positive probability $\theta\Delta t + o(\Delta t)$ that a collision occurs, and the chance of 2 or more collisions occurring in that time interval is $o(\Delta t)$.

Assume that there are i particles present at time t and all interactions are equally likely. Then, there will be j particles with probability $\binom{i}{m}\theta p_{j-i+m}\Delta t + o(\Delta t)$ after time Δt. In this paper, we take $X(t)$ be the number of particles present at time t and therefore $X(t)$ to be a continuous-time Markov chain with nonzero transition rates $q_{ij} = \binom{i}{m} b_{j-i+m}, j \geq i-m$, $i \geq m$, where $b_m = -\theta(1-p_m)$ and $b_j = \theta p_j$ for $j \neq m$.

This leads us to the following formal definition.

Definition 1. A q-matrix $Q = (q_{ij}; i, j \in \mathbb{Z}_+)$ is called a generalized collision branching q-matrix (henceforth referred to as a GCB q-matrix) if it takes the following form:

$$q_{ij} = \begin{cases} \binom{i}{m} b_{j-i+m}, & \text{if } i \geq m, j \geq i-m, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where

$$\begin{aligned} & b_j \geq 0 \, (j \neq m), \\ & \sum_{j=m+1}^{\infty} b_j > 0, \\ & 0 < -b_m = \sum_{j \neq m} b_j < +\infty, \end{aligned} \tag{2}$$

together with $b_k > 0 (k = 0, 1, \ldots, m-1)$.

The conditions $b_0 > 0$ and $\sum_{j=m+1}^{\infty} b_j > 0$ are essential, while condition $b_k > 0 (k = 1, \ldots, m-1)$ is imposed for convenience; all our conclusions hold true with some minor and obvious adjustments if this latter condition is removed.

Guided by this fact, we formally define this generalized collision branching process as follows.

Definition 2. A generalized collision branching process (henceforth referred to simply as a GCBP) is a continuous-time

Markov chain, taking values in $\mathbb{Z}_+$, whose transition function $P(t) = (p_{ij}(t); i, j \in \mathbb{Z}_+)$ satisfies the forward equation

$$P'(t) = P(t)Q, \quad (3)$$

where Q is a GCB q-matrix as defined in (1) and (2).

In order to avoid discussing some trivial cases, we shall assume that $\mathbb{Z}_+$ is an irreducible class for our q-matrix Q as well as for the corresponding Feller minimal Q-function throughout this paper excepting where we consider the absorbing case.

Good references of Asmussen and Hering [1], Athreya and Jagers [2], Athreya and Ney [3], Chen et al. [4], Ezhov [5], Harris [6], Kalinkin [7], Li [8, 9], Li and Wang [10], Sevast'yanov [11] considered kinds of generalized branching models. Whilst for more other recent excellent developments, we can see Chen et al. [12, 13], Li [14] and Yu et al. [15–17], Ren et al. [18], Xiong and Yang [19], Zhang [20], Zhang [21] and Zhang et al. [22], and so on. In this paper, we consider a more challenging and practical meaning model, which involved $m > 2$ particles collision, and hence, investigating the properties of such model is of great significance. In such case, we assume that m is the smallest positive integer such that all states $\{m, m+1, \ldots,\}$ communicate; in other words, $G = \{m, m+1, \ldots,\}$ is an irreducible class for the GCB q-matrix Q. The more general jump rates will be discussed in subsequence papers.

The structure of this paper is as follows. Some preliminary results are obtained in Section 2. In Section 3, we show that there always exists exactly one GCBP for a given GCB q-matrix Q. And then the extinction behavior and hitting times are considered in the Section 4, where some elegant and important results regarding extinction probabilities and mean extinction times and explosion times are obtained.

2. Preliminaries

In order to investigate properties of GCBPs, we introduce the generating function $B(s)$ of the sequence $\{b_k; k \ge 0\}$ in (1) and (2) as

$$B(s) = \sum_{j=0}^{\infty} b_j s^j, \quad |s| \le 1. \quad (4)$$

The function plays an extremely important role in the following discussion. It is easy to see that $B(0) = b_0 > 0$. Furthermore, $B(s)$ is well defined at least on $[-1, 1]$.

It is clear that $B'(1) > -\infty$. Moreover, the number of solutions to equation $B(s) = 0$ in $s \in [0, 1)$ is determined by the sign of $B'(1)$, and we will give the simple results in the following. However, their proofs are obvious and thus omitted in this paper.

Lemma 1. *The equation $B(s) = 0$ has at least m roots $q_0, q_1, \ldots, q_{m-1}$ in $[-1, 1]$, where $|q_k| \le q_0$ and $0 < q_0 \le 1$ is a positive root. Specially,*

(i) *If $B'(1) \le 0$, then $B(s) = 0$ has exactly m roots in $[-1, 1]$ with $q_0 = 1$ and $B(s) > 0$ for all $s \in [0, 1)$.*

(ii) *If $0 < B'(1) \le +\infty$, then $B(s) = 0$ has exactly $m+1$ roots $1, q_0, q_1, \ldots, q_{m-1}$ in $[-1, 1]$ with $|q_k| \le q_0$ and that $q_0 < 1$; moreover, $B(s) > 0$ for $s \in [0, q_0)$ and $B(s) < 0$ for $s \in (q_0, 1)$.*

(iii) *$B(s)$ can be expressed as*

$$B(s) = (q_0 - s)(s - q_1)\cdots(s - q_{m-1}) \cdot \left(q_0 - \sum_{k=1}^{\infty} q_k s^k\right), \quad (5)$$

where q_0, $q_k \ge 0 (k \ge 1)$. Moreover, $\sum_{k=1}^{\infty} q_k \le q_0$ if $B'(1) < 0$, while $\sum_{k=1}^{\infty} q_k = q_0$ if $B'(1) \ge 0$. Furthermore, if $\{m, m+1, \ldots,\}$ is irreducible for Q, then $q_1 > 0$.

Throughout this paper, we will always denote q_0 be the smallest nonnegative root of $B(s) = 0$ on $(0, 1]$. Moreover, it is easy to see that $q_0 = 1$ iff $B'(1) \le 0$ from Lemma 1.

Lemma 2. *Suppose that Q is a GCB q-matrix as defined in (1) and (2) and let $P(t) = (p_{ij}(t); i, j \ge 0)$ and $\Phi(\lambda) = (\phi_{ij}(\lambda); i, j \ge 0)$ be the Feller minimal Q-function and its Q-resolvent, respectively. Then for any $i \ge 0$, $t \ge 0$, $\lambda > 0$, and $|s| < 1$, we have*

$$\frac{\partial F_i(s,t)}{\partial t} = \frac{1}{m!} B(s) \cdot \frac{\partial^m F_i(s,t)}{\partial s^m}, \quad (6)$$

or equivalently

$$\lambda \Phi_i(s,\lambda) - s^i = \frac{1}{m!} B(s) \cdot \frac{\partial^m \Phi_i(s,\lambda)}{\partial s^m}, \quad (7)$$

where $F_i(s,t) = \sum_{j=0}^{\infty} p_{ij}(t) s^j$ and $\Phi_i(s,\lambda) = \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j$.

Proof. By the Kolmogorov forward equation (3), for any i, $j \ge 0$,

$$p'_{ij}(t) = \sum_{k=m}^{j+m} p_{ik}(t) \binom{k}{m} b_{j-k+m}. \quad (8)$$

Multiplying s^j on both sides of the above equality and summing over $j \in \mathbb{Z}_+$, we immediately obtain (6). Finally, (7) is the Laplace transform of (6).

3. Uniqueness

In this section, we mainly consider the uniqueness of GCBPs.

Lemma 3. *Let $(p_{ij}(t), i, j \in \mathbb{Z}_+)$ and $(\phi_{ij}(\lambda), i, j \in \mathbb{Z}_+)$ be the Feller minimal Q-function and Q-resolvent, respectively, where Q is a GCB q-matrix. Then for any $i \ge m$ and $|s| < 1$, we have*

(i) *$\int_0^{\infty} p_{ij}(t)dt < +\infty (i, j \ge m)$ and thus $\lim_{t \to \infty} p_{ij}(t) = 0 (i, j \ge m)$.*

(ii) *For any $i \ge m$ and $s \in [0, 1)$,*

$$\sum_{j=m}^{\infty} p_{ij}(t)\binom{j}{m}\cdot s^{j-m} < +\infty, \tag{9}$$

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s)\sum_{j=m}^{\infty} p_{ij}(t)\binom{j}{m}\cdot s^{j-m}, \tag{10}$$

and

$$\sum_{j=m}^{\infty} \phi_{ij}(\lambda)\binom{j}{m}\cdot s^{j-m} < +\infty, \tag{11}$$

$$\lambda\sum_{j=0}^{\infty} \phi_{ij}(\lambda)s^j - s^i = B(s)\sum_{j=m}^{\infty} \phi_{ij}(\lambda)\binom{j}{m}\cdot s^{j-m}. \tag{12}$$

Proof. It is easily seen that all states $G = \{m, m+1, \ldots,\}$ are transient, and thus, (i) follows. This simple fact can also be easily obtained analytically. Indeed, by Kolmogorov forward equation, we have

$$p'_{i0}(t) = p_{im}(t)b_0, \quad i \geq m, \tag{13}$$

which implies that $\int_0^{\infty} p_{im}(t)dt < +\infty$ since $b_0 > 0$. Hence, by the irreducibility of all states $\{m, m+1, \ldots,\}$ we know that $\int_0^{\infty} p_{ij}(t)dt < +\infty$ for all $i, j \geq m$.

We now prove (9). Firstly, we know that the Feller minimal Q-resolvent can be obtained by the following (Laplace transform version) forward integral iteration:

$$\begin{cases} \phi_{ij}^{(0)}(\lambda) = \dfrac{\delta_{ij}}{\lambda + q_j}, \\ \\ \phi_{ij}^{(n+1)}(\lambda) = \dfrac{\delta_{ij}}{\lambda + q_j} + \displaystyle\sum_{k\neq j}\phi_{ik}^{(n)}\cdot\dfrac{q_{kj}}{\lambda + q_j}, \quad n \geq 0, \end{cases} \tag{14}$$

and that $\phi_{ij}^{(n)}(\lambda) \uparrow \phi_{ij}(\lambda)$ as $n \longrightarrow \infty$ for all $i, j \in \mathbb{E}$.

Now, we consider our GCB q-matrix Q on $\mathbb{Z}_+$, and we still denote $\{\phi_{ij}^{(n)}(\lambda), i, j \in \mathbb{Z}_+\}$ to be the corresponding Feller minimal resolvent. Firstly, we claim that for any $n \geq 0$, $i \geq 0$ and $0 < s < 1$,

$$\sum_{j=m}^{\infty} \phi_{ij}^{(n)}(\lambda)\binom{j}{m}s^{j-m} < \infty. \tag{15}$$

For $j < m$, (15) is trivially true, so we assume $j \geq m$. We use mathematically induction on n to prove the conclusion. Obviously, it is true for $n = 0$.

Next, by (14), we can easily get that

$$\begin{aligned} &\sum_{j=0}^{\infty} \lambda\phi_{ij}^{(n+1)}(\lambda)s^j - \sum_{j=m}^{\infty}\binom{j}{m}b_m\phi_{ij}^{(n+1)}(\lambda)s^j \\ &\quad = s^i + \sum_{k=m}^{\infty}\phi_{ik}^{(n)}(\lambda)\binom{k}{m}s^{k-m}\cdot\left(\sum_{j\neq m}^{\infty} b_j s^j\right). \end{aligned} \tag{16}$$

Define $A_{ij}^{(n+1)}(\lambda) = \phi_{ij}^{(n+1)}(\lambda) - \phi_{ij}^{(n)}(\lambda)\,(n \geq 0)$, then $A_{ij}^{(n)}(\lambda) \geq 0$ and

$$\lim_{n\longrightarrow\infty} A_{ij}^{(n)}(\lambda) = 0, \quad \text{for all } i, j \in \mathbb{Z}_+. \tag{17}$$

Applying the notation, (16) can be rewritten as

$$\begin{aligned} \lambda\sum_{j=0}^{\infty}\phi_{ij}^{(n+1)}(\lambda)s^j &= s^i + B(s)\sum_{k=m}^{\infty}\phi_{ij}^{(n)}(\lambda)\binom{k}{m}s^{k-m} \\ &\quad + b_m s^m\sum_{j=m}^{\infty} A_{ij}^{(n+1)}(\lambda)\binom{j}{m}s^{j-m}. \end{aligned} \tag{18}$$

By (14), $A_{ij}^{(n+1)}(\lambda) = \sum_{k\neq j} A_{ik}^{(n)}(\lambda)q_{kj}/(\lambda + q_j)$, $n \geq 0$, then

$$\begin{aligned} \sum_{j=0}^{\infty} A_{ij}^{(n+1)}(\lambda)(\lambda + q_j)s^j &= \sum_{k=2}^{\infty} A_{ik}^{(n)}(\lambda)\binom{k}{m}s^{k-2} \\ &\quad \cdot\left(b_0 + b_1 s + \sum_{m=1}^{\infty} b_{m+2}s^{m+2}\right). \end{aligned} \tag{19}$$

It follows from the above two expressions that

$$\sum_{j=m}^{\infty} A_{ij}^{(n+1)}(\lambda)\binom{j}{m}s^{j-m} \leq \frac{B(s) - b_m s^m}{-b_m s^m}\sum_{j=m}^{\infty} A_{ij}^{(n)}(\lambda)\binom{j}{m}s^{k-m}, \tag{20}$$

and so (15) follows from the induction principle.

Also, letting $s \uparrow 1$ in (20) yields that

$$\sum_{j=m}^{\infty} A_{ij}^{(n+1)}(\lambda)\binom{j}{m} \leq \sum_{j=m}^{\infty} A_{ij}^{(n)}(\lambda)\binom{j}{m}, \quad n \geq 1. \tag{21}$$

However, it is easily seen that

$$\sum_{k=m}^{\infty} A_{ik}^{(1)}(\lambda)\binom{k}{m} \leq -\frac{1}{b_m}, \quad n \geq 1, \tag{22}$$

and thus, by (18), we have

$$\sum_{k=m}^{\infty} A_{ik}^{(n)}(\lambda)\binom{k}{m} \leq -\frac{1}{b_m}, \quad n \geq 1. \tag{23}$$

It follows from the Dominated Convergence Theorem and (20) yields that for $0 < s < 1$,

$$\lim_{n\longrightarrow\infty}\sum_{j=m}^{\infty} A_{ij}^{(n+1)}\binom{j}{m}s^{j-m} = 0. \tag{24}$$

Letting $n \longrightarrow \infty$ in (18) and applying the above equality leads to the conclusion that for $0 < s < 1$,

$$\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)s^j = s^i + B(s)\lim_{n\longrightarrow\infty}\sum_{k=m}^{\infty}\phi_{ik}^{(n)}(\lambda)\binom{k}{m}s^{k-m}. \tag{25}$$

However, for all $0 < 1 - \varepsilon \leq s < 1$, we may find an $\varepsilon > 0$ such that $B(s) \neq 0$. Thus,

$$\lim_{n\to\infty}\sum_{k=m}^{\infty}\phi_{ik}^{(n)}(\lambda)\binom{k}{m}s^{k-m}<\infty,\quad 1-\varepsilon\le s<1. \tag{26}$$

Applying the Monotone Convergence Theorem and Dominated Convergence Theorem yields

$$\sum_{k=m}^{\infty}\phi_{ik}(\lambda)\binom{k}{m}s^{k-m}=\lim_{n\to\infty}\sum_{k=m}^{\infty}\phi_{ik}^{(n)}(\lambda)\binom{k}{m}s^{k-m}<\infty,\quad 1-\varepsilon\le s<1. \tag{27}$$

It easy to see that the above equality holds for all $0<s<1$. Thus, (12) yields from (25). Moreover, (10) is the Laplace transform of (12), which implies that (10) holds for almost all $t\ge 0$. Furthermore, note that the left-hand side of (11) is a continuous function of $t>0$; thus, (10) holds for all $t\ge 0$.

Theorem 1. *The GCB q-matrix is regular iff $B'(1)\le 0$.*

Proof. Firstly, we suppose that $B'(1)\le 0$ and let $P(t)=\{p_{ij}(t),\ i,j\ge 0\}$ be the minimal Q-transition function. Substituting (1) into (3) gives

$$p_{ij}'(t)=\sum_{k=m}^{j+m}p_{ik}(t)\binom{k}{m}b_{j-k+m},\quad i,j\ge 0. \tag{28}$$

It easily yields that for $0\le s<1$,

$$\sum_{j=0}^{\infty}p_{ij}'(t)s^{j}=B(s)\sum_{k=m}^{\infty}\binom{k}{m}p_{ik}(t)s^{k-m},\quad i\ge 0, \tag{29}$$

the right-hand side being strictly positive for $s\in(0,1)$ follows from the Lemma 1. Moreover, it is easy to dictate that for all $t\ge 0$,

$$\sum_{j=0}^{\infty}\left|p_{ij}'(t)\right|\le 2q_i, \tag{30}$$

where $q_i:=-q_{ii}=\binom{i}{m}b_m<\infty$. Therefore, the series $\sum_{j=0}^{\infty}p_{ij}'(t)s^j$ converges uniformly on $[0,\infty)$ for every $s\in[0,1)$, and since the derivatives $p_{ij}'(t)$ are all continuous, the derivative of $\sum_{j=0}^{\infty}p_{ij}(t)s^j$ exists and equals $\sum_{j=0}^{\infty}p_{ij}(t)s^j$. Thus, we may integrate (29) to obtain

$$\sum_{j=0}^{\infty}p_{ij}(t)s^{j}-s^{i}\ge 0,\quad i\ge 0,\ 0\le s<1. \tag{31}$$

Letting $s\uparrow 1$ in (31) yields $\sum_{j=0}^{\infty}p_{ij}(t)\ge 1$, which implies that the equality holds for all $i\ge 0$. Therefore, the minimal Q-transition function is honest, and hence, Q is regular.

Conversely, by the Theorem 3.6 of Li and Chen [9], it is easy to obtain the conclusion since $\sum_{k=m}^{\infty}1/\binom{k}{m}<\infty$. The proof is complete.

By Theorem 1, we can see that if $B'(1)\le 0$, then the GCBP is regular. In the sequel, we will prove that for any given GCB q-matrix Q, there always exists exactly one Q-process satisfying the Kolmogorov forward equation (3).

Theorem 2. *There exists exactly one GCBP.*

Proof. It follows from Theorem 1, we only need to consider the case $0<B'(1)\le+\infty$. In order to prove the uniqueness of the GCBP, we will verify Reuter's condition, i.e., we need to prove that the equation

$$\begin{cases} Y(\lambda I-Q)=0, & 0\le Y<+\infty,\\ \sum_{j\in\mathbb{Z}_+}y_j<+\infty, \end{cases} \tag{32}$$

has only the trivial solution, and then cover all $\lambda>0$.

Let $Y=(y_i;\ i\ge 0)$ be a nontrivial solution corresponding to $\lambda=1$, then $y_0>0$ and by (32),

$$\eta_j=\sum_{i=m}^{j+m}\eta_i\binom{i}{m}b_{j-i+m},\quad j\ge 0, \tag{33}$$

with

$$\begin{gathered}\eta_j\ge 0\ (j\ge 0),\\ \sum_{j=0}^{\infty}\eta_j<+\infty.\end{gathered} \tag{34}$$

It is clear that the nontriviality of the solution η implies that

$$\eta_j>0. \tag{35}$$

$\sum_{j=0}^{\infty}\eta_j$ is well defined for all $s\in[0,1]$ since (34) holds, which implies that

$$\eta_j=\sum_{i=m}^{\infty}\eta_j s^j<+\infty,\quad 0\le s<1, \tag{36}$$

because it follows from the root test, these series have the same radius of convergence. Applying Fubini's theorem together with (33) and (36) yields that

$$\sum_{j=0}^{\infty}\eta_j s^j=B(s)\sum_{i=m}^{\infty}\binom{i}{m}\eta_i s^{i-m},\quad 0\le s<1. \tag{37}$$

Therefore, $\sum_{j=0}^{\infty}\eta_j s^j$ and $\sum_{i=m}^{\infty}\binom{i}{m}\eta_i s^{i-m}$ are strictly positive for all $s\in(0,1)$ based on (33)–(36), and thus $B(s)>0$ for all $s\in(0,1)$, since $0<B'(1)\le+\infty$, which contradicts with Lemma 1.

4. Extinction and Explosion

From the previous section, we have obtained that the GCBP is uniquely determined by its q-matrix, so we will examine some of its properties in this section. Let $\{X(t),\ t\ge 0\}$ be the unique GCBP, and denote $P(t)=\{p_{ij}(t),\ i,\ j\ge 0\}$ be its transition function. Define the extinction times τ_k for $k=0,1,\dots,m-1$ as

$$\tau_k=\begin{cases}\inf\{t>0, X(t)=k\}, & \text{if } X(t)=k \text{ for some } t>0,\\ +\infty, & \text{if } X(t)\ne k \text{ for all } t>0,\end{cases} \tag{38}$$

and denote the corresponding extinction probabilities by

$$a_{ik} = P\left(\tau_k < +\infty \mid X(0) = i\right) = \lim_{t\longrightarrow\infty} p_{ik}(t), \tag{39}$$

and the overall extinction probability by $a_k = P(\tau < \infty \mid X(0) = i) = \sum_{k=0}^{m-1} a_{ik}$. Also let $E_i(\cdot)$ denote the expectation conditional on $X(0) = i$.

Theorem 3. *The extinction probabilities $a_{ik}(k = 0, 1, \ldots, m-1)$ satisfy*

$$a_{i0} + q_k a_{i1} + \cdots + q_k^{m-1} a_{im-1} = q_k^i, \quad k = 0, 1, \ldots, m-1. \tag{40}$$

More specifically,

$$a_{i0} + a_{i1} + \cdots + a_{im-1} = 1, \quad \text{if } B'(1) \le 0, \tag{41}$$

$$a_{i0} + q_k a_{i1} + \cdots + q_k^{m-1} a_{im-1} = q_k^i < 1, \quad \text{if } 0 < B'(1) \le +\infty. \tag{42}$$

Proof. Firstly, it is clear that all states $\{m, m+1, \ldots\}$ are transient. For all i, $k \ge m$, we have $\lim_{t\longrightarrow\infty} p_{ik}(t) = 0$ follows from $\int_0^\infty p_{ik}(t)\mathrm{d}t < +\infty$. Thus, letting $t \longrightarrow \infty$ in (30) and using the Dominated Convergence Theorem, we obtain that $a_{i0} + q a_{i1} + \cdots + q^{m-1} a_{im-1} \ge s^i$ for $s \in [0, 1)$. Letting $s \uparrow 1$, we immediately obtain (41) since $a_{i0} + a_{i1} + \cdots + a_{im-1} \le 1$.

We now prove (42). It follows from Lemma 1 that we have $q < 1$ since $0 < B'(1) \le \infty$. Putting $s = q$ in (11) and noting that $B(q) = 0$, we discover that $\sum_{j=0}^\infty p'_{ij}(t) q^j = 0$ for any $t > 0$, implying that $\sum_{j=0}^\infty \int_0^t p'_{ij}(u)\mathrm{d}u \cdot q^j = 0$. Thus, for any $t > 0$,

$$\sum_{j=0}^\infty p_{ij}(t) q^j = q^i, \quad i \ge m. \tag{43}$$

Letting $t \longrightarrow \infty$, we have

$$\begin{aligned}&\lim_{t\longrightarrow\infty} p_{i0}(t) + q \cdot \lim_{t\longrightarrow\infty} p_{i1}(t) + \cdots + q^{m-1} \cdot \lim_{t\longrightarrow\infty} p_{im-1}(t)\\ &\quad + \lim_{t\longrightarrow\infty} \sum_{j=m}^\infty p_{ij}(t) q^j = q^i, \quad i \ge m.\end{aligned} \tag{44}$$

Noting that all of the limits exist, we may apply the Dominated Convergence Theorem in the last term on the left-hand side to obtain (42) since $q < 1$.

By Theorem 3, we know that the process is absorbed with probability less than 1 if $0 < B'(1) \le +\infty$. Our next result establishes that the process must explode if absorption does not occur in such cases.

Theorem 4. *For the Feller minimal GCBP,*

$$E_i(\tau) = m! \cdot \int_0^1 \frac{(1-y)^{m-1}\left(a_{i0} + a_{i1} y + \cdots + a_{im-1} y^{m-1} - y^i\right)}{B(y)} \mathrm{d}y. \tag{45}$$

Therefore, $E_i(\tau)$ is finite for any $i \ge m$ iff

$$\int_0^1 \frac{a_{i0} + a_{i1} y + \cdots + a_{im-1} y^{m-1} - y^i}{B(y)} \mathrm{d}y < \infty. \tag{46}$$

Proof. It follows from (10), for all $s \in [0, 1)$, we have

$$\frac{1}{B(s)} \sum_{j=0}^\infty p'_{ij}(t) s^j = \sum_{j=m}^\infty p_{ij}(t) \binom{j}{m} \cdot s^{j-m}, \tag{47}$$

i.e.,

$$\frac{\partial F_i(t,s)}{\partial t} = \frac{1}{m!} \cdot B(s) \cdot \frac{\partial^m F_i(t,s)}{\partial s^m}, \tag{48}$$

where $F'_i(t,s) = \sum_{j=0}^\infty p'_{ij}(t) s^j$. The apparent singularity at $s = q$ on the left-hand side is removable, because the series on the right-hand side certainly converges for all $s \in [0, 1)$. Moreover, the left-hand side is continuous and strictly positive (indeed increasing) on this interval. Therefore, integrating (48) with respect to s iteration m times and applying Fubini's theorem yields that for any $s \in [0, 1)$,

$$\begin{aligned}F_i(t,s) &= p_{i0}(t) + p_{i1}(t) s + \cdots + p_{im-1}(t) s^{m-1}\\ &\quad + m! \cdot \int_0^s \frac{(s-y)^{m-1}}{B(y)} \cdot F'_i(t,y)\mathrm{d}y.\end{aligned} \tag{49}$$

Letting $s \uparrow 1$ in (49), we can get that the equality (49) also holds for $s = 1$, and

$$\sum_{j=m}^\infty p_{ij}(t) = m! \cdot \int_0^1 \frac{(1-y)^{m-1}}{B(y)} \cdot F'_i(t,y)\mathrm{d}y. \tag{50}$$

Then the proof is complete if (46) holds since

$$\begin{aligned}E_i(\tau) &= \int_0^\infty \left(m! \cdot \int_0^1 \frac{(1-y)^{m-1} F'_i(t,y)}{B(y)} \right)\mathrm{d}t\\ &= m! \cdot \int_0^1 \frac{(1-y)^{m-1}\left(a_{i0} + a_{i1} y + \cdots + a_{im-1} y^{m-1} - y^i\right)}{B(y)} \mathrm{d}y.\end{aligned} \tag{51}$$

Lemma 4. *Let $(p_{ij}(t), i, j \in \mathbb{Z}_+)$ and $(\phi_{ij}(\lambda), i, j \in \mathbb{Z}_+)$ be the Feller minimal Q-function and Q-resolvent where Q is a GCB q-matrix.*

(i) For any i, $k \ge m$,

$$\int_0^\infty p_{ik}(t)\mathrm{d}t = \frac{1}{\binom{k}{m}} \cdot \frac{G_i^{k-m}(0)}{(k-m)!}. \tag{52}$$

(ii) For any $i \ge m$,

$$\int_0^\infty \sum_{k=m}^\infty p_{ik}(t)\mathrm{d}t = \sum_{k=m}^\infty \frac{1}{\binom{k}{m}} \cdot \frac{G_i^{(k-m)}(0)}{(k-m)!} < \infty, \quad (53)$$

and hence, considering the integrand is nonnegative, we obtain that

$$\lim_{t\longrightarrow\infty} \sum_{k=m}^\infty p_{ik}(t) = 0. \quad (54)$$

Proof. By (10), we have

$$\sum_{j=0}^\infty p_{ij}(t)s^j - s^i = B(s) \cdot \sum_{k=m}^\infty \left(\int_0^t p_{ik}(u)\mathrm{d}u\right)\binom{k}{m} s^{k-m}. \quad (55)$$

Letting $t \longrightarrow \infty$ in the equality (55) for $s \in (-1, 1)$, applying the Dominated Convergence Theorem on the left-hand side and the Monotone Convergence Theorem on the right-hand side, we obtain (53) by the uniqueness of the Taylor expansion. Furthermore, (53) implies (54) is trivial, and hence, the proof is complete.

Theorem 5. *For the Feller minimal GCBP, $E_i(\tau)$ is finite for some (and for all) $i \geq m$ iff $B'(1) \leq 0$, and hence*

$$E_i(\tau) = \sum_{k=m}^\infty \frac{G_i^{(k-m)}(0)}{\binom{k}{m} \cdot (k-m)!}. \quad (56)$$

More specifically, if $0 < B'(1) \leq +\infty$, then $E_i(\tau) = +\infty$ for any $i \geq m$.

Proof. It is easily seen from Theroem 3 and Lemma 1 that if $0 < B'(1) \leq \infty$, then $\sum_{k=0}^{m-1} a_{ik} < 1$ which implies $E_i(\tau) = +\infty$, so let us assume that $B'(1) \leq 0$. For these latter cases, it follows from (55) and applying the Monotone Convergence Theorem yields

$$\begin{aligned} E_i[\tau] &= \int_0^\infty (1 - p_{i0}(t) - \cdots - p_{im-1}(t))\mathrm{d}t \\ &= \int_0^\infty \sum_{k=m}^\infty p_{ik}(t)\mathrm{d}t \\ &= \sum_{k=m}^\infty \frac{1}{\binom{k}{m}} \cdot \frac{G_i^{(k-m)}(0)}{(k-m)!}. \end{aligned} \quad (57)$$

Thus, the proof is complete.

It is easily seen that $E_i(\tau_k) = +\infty (i \geq m, k = 0, 1, \ldots, m-1)$ when the extinction is not certain. Under these circumstances, it is natural to consider the conditional expected extinction times, given by $E_i(\tau_k \mid \tau_k < \infty) = \mu_k / a_{ik}$, where $\mu_k = E_i(\tau_k I_{\{\tau_k < \infty\}})$.

Theorem 6. *For the Feller minimal GCBP starting in state $i (i \geq m)$, $E_i(\tau_k \mid \tau_k < \infty) (k = 0, 1, \ldots, m-1)$ are all finite, and moreover,*

$$E_i(\tau_k \mid \tau_k < \infty) = \frac{\mu_{ik}}{a_{ik}}, \quad k \leq m-1, i \geq m, \quad (58)$$

where $\mu_{ik} (k \leq m-1)$ satisfy the linear equations

$$\sum_{k=0}^{m-1} \mu_{ik} q_j^k = \sum_{k=m}^\infty \frac{1}{\binom{k}{m}} \cdot \frac{G_i^{(k-m)}(0)}{(k-m)!} \cdot q_j^k, \quad j = 0, 1, \ldots, m-1. \quad (59)$$

Proof. First we consider the case $0 < B'(1) \leq +\infty$, and thus, $0 < q_0 < 1$, and $|q_i| < 1$ for $j = 1, \ldots, m-1$, applying the Theorem 3 together with $\sum_{k=0} p_{ik}(t) q^k = q^i$ yields the expression

$$\sum_{k=0}^{m-1} (a_{ik} - p_{ik}(t)) q_j^k = \sum_{k=m} p_{ik}(t) q_j^k, \quad j = 0, 1, \ldots, m-1. \quad (60)$$

On integrating (60) and using $a_{ik} - p_{ik}(t) = P(t < \tau_k < \infty \mid X(0) = i) (k = 0, 1, \ldots, m-1)$, we obtain that

$$\begin{aligned} q_j^k \cdot \int_0^\infty & P(s < \tau_k < \infty \mid X(0) = i)\mathrm{d}s \\ &= \sum_{k=m}^\infty \int_0^t p_{ik}(s)\mathrm{d}s \cdot q_j^k, \quad j = 0, 1, \ldots, m-1. \end{aligned} \quad (61)$$

Noting that $|q_j| < 1$ for $j = 1, \ldots, m-1$, letting $t \longrightarrow \infty$ and applying the monotone convergence theorem yields

$$\mu_{i0} + q_k \mu_{i1} + \cdots + q_k^{m-1} \mu_{im-1} = \sum_{k=m}^\infty \frac{G_i^{(k-m)}(0)}{(k-m)!} \cdot q_k^j. \quad (62)$$

On the other hand, by the definition of τ, $E_i(\tau I_{\{\tau < \infty\}}) = \sum_{k=0}^\infty E_i(\tau_k I_{\{\tau_k < \infty\}})$, and then all of the conclusions follow since $|q_i| < 1$ for $j = 1, \ldots, m-1$.

Next we consider the case $B'(1) \leq 0$, then we have $P(\tau < \infty \mid X(0) = i) = 1$. It follows from Theorem 3 that $a_i = a_{i0} + a_{i1} + \cdots + a_{im-1} = 1$, and hence, the ensuing honesty of the transition function allows us to deduce that

$$\sum_{k=0}^\infty (a_{ik} - p_{ik}(t)) = \sum_{k=m}^\infty p_{ik}(t). \quad (63)$$

Noting that $q_0 = 1$ and $|q_i| < 1$ for $j = 1, \ldots, m-1$, and letting $t \longrightarrow \infty$ again and applying the monotone convergence theorem yields

$$\mu_{i0} + q_k \mu_{i1} + \cdots + q_k^{m-1} \mu_{im-1} = \sum_{k=m}^\infty \frac{G_i^{(k-m)}(0)}{(k-m)!} \cdot q_k^j. \quad (64)$$

We know that (59) still holds for $j = 1, \ldots, m-1$ in this case. Hence, we have (40) with $q_0 = 1$. A similar argument yields the required conclusions.

From now on, we will consider the explosion probabilities and expected explosion times. By Theorem 1, we only need to consider the case that $0 < B'(1) \le \infty$. Denote τ_∞ be the explosion time and let $a_{i\infty} = P(\tau_\infty \mid X(0) = i)$ be the probability of explosion starting in state i. Since we are aiming at the minimal process, $p_{i\infty}(t) := 1 - \sum_{j=0}^{\infty} p_{ij}(t) = P(\tau_\infty \le t \mid X(0) = i)$ is the probability of explosion by time t starting in state i, and $p_{i\infty}(t) \longrightarrow a_{i\infty}$ as $t \longrightarrow \infty$.

Theorem 7. *For the minimal process starting in $i (i \ge m)$, we have the following statements.*

(i) *If $B'(1) \le 0$, then $a_{i\infty} = 0$.*

(ii) *If $0 < B'(1) \le +\infty$, then*

$$\sum_{k=0}^{m-1} a_{ik} E_i(\tau_k \mid \tau_k < \infty) + a_{i\infty} E_i(\tau_\infty \mid \tau_\infty < \infty) = \sum_{k=m}^{\infty} \frac{G_i^{(k-m)}(0)}{\binom{k}{m}(k-m)!}. \tag{65}$$

Proof. If $B'(1) \le 0$, then $a_{i\infty} = 0$ since the minimal process is honest. If $0 < B'(1) \le +\infty$, by Theorem 2 we know that the minimal process is dishonest, i.e., $p_{i\infty}(t) = 1 - \sum_{j=0}^{\infty} p_{ij}(t) > 0$. Letting $t \longrightarrow \infty$ and applying (30) together with Theorem 3 yields our expression for $a_{i\infty}$. Next we write $\sum_{k=0}^{m-1} (a_{ik} - p_{ik}(t)) + a_{i\infty} - p_{i\infty}(t) = \sum_{j=m}^{\infty} p_{ij}(t)$; then we obtain (65) by integrating this equality with respect to t, and noting that $P(\tau_\infty \le t \mid \tau_\infty < \infty, X(0) = i) = p_{i\infty}(t)/a_{i\infty}$.

Finally, we consider the time spent in each state over the lifetime of the process. Let T_k be the total time spent in state $k (k \ge m)$ and let $\mu_{ik} = E_i(T_k) (i \ge m)$. Then,

$$\mu_{ik} = E\left(\int_0^\infty I_{\{X(t)=k\}} \mathrm{d}t \mid X(0) = i\right) = \int_0^\infty p_{ik}(t) \mathrm{d}t. \tag{66}$$

This quantity was evaluated in (29). We have therefore the following result.

Theorem 8. *All of $\mu_{ik} (i \ge m, k \ge m)$ are finite and given by*

$$\mu_{ik} = \frac{G_i^{(k-m)}(0)}{\binom{k}{m} \cdot (k-m)!}. \tag{67}$$

Data Availability

Not applicable.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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