Research Article

G-SIRS Model with Logistic Growth and Nonlinear Incidence

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We present the stochastic SIRS model in the G -expectation space as follows: $dX(t) = [rX(1 - (X/K)) - (\beta XY/(1 + \alpha Y)) + \delta Z]d\langle B \rangle(t) + \bar{\sigma}_1 X dB(t)$, $dY(t) = [(\beta XY/(1 + \alpha Y)) - (\rho + \vartheta + c)Y]d\langle B \rangle(t) + \bar{\sigma}_2 Y dB(t)$, $dZ(t) = [cY - (\mu + \delta)Z]d\langle B \rangle(t) + \bar{\sigma}_3 Z dB(t)$, where $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3$ are three intensities of the G -Brownian motion and disturb the three variables, and $B(1)$ follows G -Normal distribution, namely, $B(1) \sim \mathcal{N}(0, [\underline{\pi}^2, \bar{\pi}^2])$. For any initial condition $(X(0), Y(0), Z(0)) \in D^*$, we prove the new model admits a unique solution $(X(t), Y(t), Z(t))$ for $t \geq 0$ and the solution $(X(t), Y(t), Z(t))$ satisfies $\nu(\omega: (X(t), Y(t), Z(t)) \in R_+^3, t \geq 0) = 1$.

1. Introduction

Liu [1] presented the SIRS model without random perturbation as follows:

$$\begin{cases} dX(t) = \left[rX \left(1 - \frac{X}{K} \right) - \frac{\beta XY}{1 + \alpha Y} + \delta Z \right] dt, \\ dY(t) = \left[\frac{\beta XY}{1 + \alpha Y} - (\rho + \vartheta + \gamma)Y \right] dt, \\ dZ(t) = [\gamma Y - (\mu + \delta)Z] dt, \end{cases} \quad (1)$$

where $X(t)$ reflects the susceptible number, $Y(t)$ is the infected number, and $Z(t)$ denotes the recovered number at time t . Model (1) took into account logistic growth and nonlinear incidence. The parameters $(\alpha, \beta, \rho, \vartheta, K, \gamma, \delta, \mu)$ in model (1) have practical significance, please refer to reference [1].

The possible region of (1) is $\{(X, Y, Z) \in R_+^3: X + Y + Z \leq K\} = D^*$. The basic reproduction number for system (1) is $\mathfrak{R}_0 = (\beta K / (\rho + \vartheta + \gamma))$.

In the actual environment, various diseases are disturbed by random factors, and there are many models that

reflect this stochastic phenomenon, for example, [2–6]. Rajasekar and Pitchaimani [7] assumed this random interference is described by three independent Wiener processes. Specifically, they proposed the following SIRS model:

$$\begin{cases} dX(t) = \left[rX \left(1 - \frac{X}{K} \right) - \frac{\beta XY}{1 + \alpha Y} + \delta Z \right] dt + \sigma_1 X dW_1(t), \\ dY(t) = \left[\frac{\beta XY}{1 + \alpha Y} - (\rho + \vartheta + \gamma)Y \right] dt + \sigma_2 Y dW_2(t), \\ dZ(t) = [\gamma Y - (\mu + \delta)Z] dt + \sigma_3 Z dW_3(t). \end{cases} \quad (2)$$

Peng in [8, 9] constructed the interesting G -Brownian motion in nonlinear expectation space, see [10]. Many important properties on G -Brownian motion were investigated, for example, [11]. As far as we know, there is no research on model (1) in the nonlinear expectation space. Some notations and concepts used in this paper are similar to those in references [11, 12].

2. G-SIRS Model

We consider the stochastic SIRS model in the G -expectation space and propose the G -SIRS model (GSIRSM for short) as follows:

$$\begin{cases} dX(t) = \left[rX \left(1 - \frac{X}{K} \right) - \frac{\beta XY}{1 + \alpha Y} + \delta Z \right] d\langle B \rangle(t) + \bar{\sigma}_1 X dB(t), \\ dY(t) = \left[\frac{\beta XY}{1 + \alpha Y} - (\rho + \vartheta + \gamma) Y \right] d\langle B \rangle(t) + \bar{\sigma}_2 Y dB(t), \\ dZ(t) = [\gamma Y - (\mu + \delta) Z] d\langle B \rangle(t) + \bar{\sigma}_3 Z dB(t), \end{cases} \quad (3)$$

where $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3$ are three intensities of the G -Brownian motion, which disturb the three variables, and $B(t)$ satisfies $B(1) \sim \mathcal{N}(0, [\underline{\pi}^2, \bar{\pi}^2])$, $\hat{\mathbb{E}}[B(1)^2] = \bar{\pi}^2$, $\hat{\mathbb{E}}[-B(1)^2] = -\underline{\pi}^2$.

Note that model (3) has nonlinear incidence. We denote $v(c) = \inf_{P \in \mathcal{Q}} E_P[1_c]$ and $V(c) = \sup_{P \in \mathcal{Q}} E_P[1_c]$, where $\hat{\mathbb{E}}[\cdot] = \sup_{P \in \mathcal{Q}} E_P[\cdot]$.

It is very important to prove that the solution (X, Y, Z) of model (3) is of global existence and is nonnegative. We first show system (3) is global and positive. Many asymptotic properties of this system (3) deserve further investigation in the future.

Theorem 1. $\forall (X(0), Y(0), Z(0)) \in D^*$ and $t \geq 0$, $(X(t), Y(t), Z(t))$ in (3) are unique and satisfy

$$v(\omega: (X(t), Y(t), Z(t)) \in R_+^3, t \in [0, +\infty)) = 1. \quad (4)$$

Proof. Since the coefficients of (3) are locally Lipschitz continuous, then $\forall (X(0), Y(0), Z(0)) \in D^*$, there exists a local solution $(X(t), Y(t), Z(t))$ on $t \in [0, \lambda_e)$ quasi surely (q.s.), where λ_e represents the explosion time. To show $\lambda_e = +\infty$ q.s., we prove $(X(t), Y(t), Z(t))$ does not explode to infinity in a finite time. Suppose $k_0 > 1$ is large enough such that (s.t) $(X(0), Y(0), Z(0))$ lies in the interval $[(1/k_0), k_0]^3$. For $k \geq k_0$, define

$$\lambda_k = \inf \left\{ t \in [0, \lambda_e): X(t) \notin \left(\frac{1}{k}, k \right) \text{ or } Y(t) \notin \left(\frac{1}{k}, k \right) \text{ or } Z(t) \notin \left(\frac{1}{k}, k \right) \right\}, \quad (5)$$

where λ_k is increasing as $k \rightarrow \infty$. We have $\lambda_\infty = \lim_{k \rightarrow \infty} \lambda_k$, therefore $\lambda_\infty \leq \lambda_e$ quasi surely. Suppose we guarantee that $\lambda_\infty = \infty$ q.s., then $\lambda_e = \infty$ and $v(\omega: (X(t), Y(t), Z(t)) \in R_+^3) = 1$ q.s. If we assume $0 < V(\lambda_\infty < +\infty)$, then there exists a pair of constants $\chi > 0$ and $\varepsilon \in (0, 1)$ s.t.

$$V(\lambda_\infty \leq \chi) \geq v(\lambda_\infty \leq \chi) \geq \varepsilon. \quad (6)$$

Then, $\exists k_1 \geq k_0$ s.t.

$$V(\Pi_k) := V(\lambda_k \leq \chi) \geq \varepsilon, \quad \text{for all } k \geq k_1. \quad (7)$$

Set a function $U_1: R_+^3 \rightarrow R_+$ by

$$U_1(X, Y, Z) = (X - \ln X) + (Y - \ln Y) + (Z - \ln Z) - 3. \quad (8)$$

We note the function $g(x) = (x - \ln x) - 1 \geq 0$ for any $x > 0$. Using the G -Ito lemma for the function U_1 , we get

$$\begin{aligned} dU_1 &= \frac{\partial U_1}{\partial x} dX + \frac{\partial U_1}{\partial y} dY + \frac{\partial U_1}{\partial z} dZ \\ &\quad + \frac{1}{2} \left[\frac{\partial^2 U_1}{\partial x^2} (dX)^2 + \frac{\partial^2 U_1}{\partial y^2} (dY)^2 + \frac{\partial^2 U_1}{\partial z^2} (dZ)^2 \right] \\ &= \left(1 - \frac{1}{X} \right) dX + \left(1 - \frac{1}{Y} \right) dY + \left(1 - \frac{1}{Z} \right) dZ \\ &\quad + \frac{1}{2} [\bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \bar{\sigma}_3^2] d\langle B \rangle(t) \\ &=: \mathcal{L}U_1 d\langle B \rangle(t) + \Theta(X, Y, Z) dB(t), \end{aligned} \quad (9)$$

where

$$\begin{aligned} \mathcal{L}U_1 &= (X - 1) \left[\left(\frac{rK - rX}{K} \right) - \frac{\beta Y}{1 + \alpha Y} \right] + \frac{\delta(X - 1)Z}{X} \\ &\quad + (Y - 1) \left[\frac{\beta X}{1 + \alpha Y} - (\rho + \vartheta + \gamma) \right] \\ &\quad + \frac{(Z - 1)\gamma Y}{Z} - (Z - 1)(\mu + \delta) + \frac{1}{2} \sum_{i=1}^3 \bar{\sigma}_i^2 \\ &= \left(r + \frac{r}{K} \right) X - \frac{r}{K} X^2 - \frac{\beta X}{1 + \alpha Y} + \frac{\beta Y}{1 + \alpha Y} \\ &\quad - \rho Y - \vartheta Y - \frac{\gamma}{Z} Y - \frac{\delta}{X} Z - \mu Z \\ &\quad - r + \rho + \vartheta + \gamma + \mu + \delta + \frac{1}{2} \sum_{i=1}^3 \bar{\sigma}_i^2, \end{aligned}$$

$$\Theta(X, Y, Z) = \bar{\sigma}_1^2 (X - 1) + \bar{\sigma}_2^2 (Y - 1) + \bar{\sigma}_3^2 (Z - 1). \quad (10)$$

We note that the region $D^* = \{X + Y + Z \leq K\}$ and all the parameters are positive, then we have

$$\begin{aligned} \mathcal{L}U_1 &\leq \left(r + \frac{r}{K} \right) X + \frac{\beta Y}{1 + \alpha Y} + \rho + \vartheta + \gamma + \mu + \delta + \frac{1}{2} \sum_{i=1}^3 \bar{\sigma}_i^2 \\ &\leq \left(r + \frac{r}{K} \right) X + \beta Y + \rho + \vartheta + \gamma + \mu + \delta + \frac{1}{2} \sum_{i=1}^3 \bar{\sigma}_i^2 \\ &\leq \Lambda K + \rho + \vartheta + \gamma + \mu + \delta + \frac{1}{2} \sum_{i=1}^3 \bar{\sigma}_i^2, \end{aligned} \quad (11)$$

where $\Lambda = \max\{(r + (r/K)), \beta\}$. We denote

$$C = \Lambda K + \rho + \vartheta + \gamma + \mu + \delta + \frac{1}{2} \sum_{i=1}^3 \bar{\sigma}_i^2. \quad (12)$$

Therefore,

$$\begin{aligned} dU_1 &= \mathcal{L}U_1 d\langle B \rangle(t) + \Theta(X, Y, Z)dB(t) \\ &\leq C d\langle B \rangle(t) + \Theta(X, Y, Z)dB(t). \end{aligned} \quad (13)$$

Integrate (13) from 0 to $\lambda_k \wedge \chi$,

$$\begin{aligned} U_1(X(\lambda_k \wedge \chi), Y(\lambda_k \wedge \chi), Z(\lambda_k \wedge \chi)) \\ \leq U_1(X(0), Y(0), Z(0)) + C \cdot \langle B \rangle(\lambda_k \wedge \chi) \\ + \int_0^{\lambda_k \wedge \chi} \Theta(X(t), Y(t), Z(t))dB(t), \end{aligned} \quad (14)$$

and take the G -expectation,

$$\begin{aligned} \widehat{\mathbb{E}}[U_1(X(\lambda_k \wedge \chi), Y(\lambda_k \wedge \chi), Z(\lambda_k \wedge \chi))] \\ \leq U_1(X(0), Y(0), Z(0)) + C \cdot \widehat{\mathbb{E}}[\langle B \rangle(\lambda_k \wedge \chi)] \\ = U_1(X(0), Y(0), Z(0)) + C \cdot \bar{\pi}^2 \cdot (\lambda_k \wedge \chi). \end{aligned} \quad (15)$$

Note the set $\Pi_k(\omega) := \{\omega: \lambda_k(\omega) \leq \chi\}$ and (7), then $V(\Pi_k(\omega)) \geq \varepsilon$ for all $k \geq k_1$. We see that the definition of λ_k , then for every $\omega \in \Pi_k(\omega)$, there exist at least $X(\lambda_k(\omega))$ or $Y(\lambda_k(\omega))$ or $Z(\lambda_k(\omega))$ equals to k or $(1/k)$. For example, if $X(\lambda_k(\omega)) = k$ or $X(\lambda_k) = (1/k)$, then $U_1(X(\lambda_k), Y(\lambda_k), Z(\lambda_k)) = k - 1 - \ln k + Y(\lambda_k) - 1 - \ln Y(\lambda_k) + Z(\lambda_k) - 1 + \ln Z(\lambda_k) \geq k - 1 - \ln k$, or $U_1(X(\lambda_k), Y(\lambda_k), Z(\lambda_k)) = (1/k) - 1 + \ln k + Y(\lambda_k) - 1 - \ln Y(\lambda_k) + Z(\lambda_k) - 1 + \ln Z(\lambda_k) \geq (1/k) - 1 + \ln k$. Thus,

$$U_1(X(\lambda_k), Y(\lambda_k), Z(\lambda_k)) \geq \min\left\{\frac{1}{k} - 1 + \ln k, k - 1 - \ln k\right\}. \quad (16)$$

From (7) and (14)–(16), we have

$$\begin{aligned} \widehat{\mathbb{E}}[I_{\Pi_k(\omega)} U_1(X(\lambda_k), Y(\lambda_k), Z(\lambda_k))] \\ = \widehat{\mathbb{E}}[I_{\Pi_k(\omega)} U_1(X(\lambda_k \wedge \chi), Y(\lambda_k \wedge \chi), Z(\lambda_k \wedge \chi))] \\ \leq U_1(X(0), Y(0), Z(0)) + C \cdot \widehat{\mathbb{E}}[I_{\Pi_k(\omega)} \langle B \rangle(\lambda_k \wedge \chi)] \\ \leq U_1(X(0), Y(0), Z(0)) + C \cdot \bar{\pi}^2 \cdot \chi < +\infty, \end{aligned} \quad (17)$$

$$\begin{aligned} \widehat{\mathbb{E}}[I_{\Pi_k(\omega)} U_1(X(\lambda_k), Y(\lambda_k), Z(\lambda_k))] \\ \geq \widehat{\mathbb{E}}\left[\left(\frac{1}{k} + \ln k - 1\right) \wedge (k - 1 - \ln k) \cdot I_{\Pi_k(\omega)}\right] \\ = \left[\left(\frac{1}{k} + \ln k - 1\right) \wedge (k - 1 - \ln k)\right] \cdot \widehat{\mathbb{E}}[I_{\Pi_k(\omega)}] \\ = \left[\left(\frac{1}{k} + \ln k - 1\right) \wedge (k - 1 - \ln k)\right] \cdot V(\Pi_k(\omega)) \\ \geq \left[\left(\frac{1}{k} + \ln k - 1\right) \wedge (k - 1 - \ln k)\right] \cdot \varepsilon. \end{aligned} \quad (18)$$

Therefore, from inequalities (17) and (18), we have

$$\begin{aligned} \left[\left(\frac{1}{k} + \ln k - 1\right) \wedge (k - 1 - \ln k)\right] \cdot \varepsilon \\ \leq \widehat{\mathbb{E}}[I_{\Pi_k(\omega)} U_1(X(\lambda_k), Y(\lambda_k), Z(\lambda_k))] < +\infty. \end{aligned} \quad (19)$$

Letting $k \rightarrow \infty$, we find out inequality (19) is a contradiction. Thus, $V(\lambda_\infty < +\infty) = 0$, namely, $v(\lambda_\infty = +\infty) = 1$ and $v(\omega: (X(t), Y(t), Z(t)) \in R_+^3, t \geq 0) = 1$. $\square$

3. Discussion

Although the endemic equilibrium for (1) exists, the endemic equilibrium of the stochastic versions (2) and (3) do not exist. From stochastic stability of Has'minskii [13], Rajasekar and Pitchaimani [7] exemplified that system (2) admits an ergodic stationary distribution. However, in the G -expectation space, we first need to obtain the new ergodic stationary distribution theorem similar to the theory of Has'minskii and use it to show the ergodic property for G -system (3). We also hope to discuss the disease is extinct for a long time in model (3). We need to find sufficient conditions for extinction of the disease for (3). However, because of the lack of a theorem which is similar to Theorem 1.16 in [14], we cannot get the corresponding results immediately for G -system (3). We will investigate the existence of ergodic stationary distribution and the sufficient conditions of extinction for G -stochastic system (3) in the future research. By the way, some more realistic and impulsive perturbations models, as well as a nonautonomous case for system (3) are also worth continuing to probe. In addition, numerical simulations for the system will be further investigated.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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