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Research Article

Egoroff's Theorem and Lusin's Theorem for Capacities in the Framework of g-Expectation

Zhaojun Zong[1] **Feng Hu**[1,2] **and Xiaoxin Tian**[1]

[1]*School of Statistics, Qufu Normal University, Qufu 273165, China*
[2]*Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong, China*

Correspondence should be addressed to Feng Hu; hufengqf@163.com

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In the classical real analysis theory, Egoroff's theorem and Lusin's theorem are two of the most important theorems. The σ-additivity of measures plays a crucial role in the proofs of these theorems. Later, many researchers have carried out lots of studies on Egoroff's theorem and Lusin's theorem when the measure is monotone and nonadditive (see, e.g., Li and Yasuda (2004) and Li and Mesiar (2011)). In this paper, we study Egoroff's theorem and Lusin's theorem for capacities in the framework of g-expectation. We give some different assumptions that provide Egoroff's theorem and Lusin's theorem in the framework of g-expectation.

1. Introduction

In the classical real analysis theory, Egoroff's theorem and Lusin's theorem are two of the most important theorems. The σ-additivity of measures plays a crucial role in the proofs of these theorems. But in fact, the σ-additivity of measures has been abandoned in some areas because many uncertain phenomena cannot be well modelled by using additive measures.

The research studies on Egoroff's theorem in nonadditive measure theory were carried out by Wang and Klir [1]; Li [2]; Li and Yasuda [3]; and Murofushi et al. [4]. These results faithfully contribute to nonadditive measure theory. Li [2] introduced the concept of *condition (E)* of set function and proved an essential result: a necessary and sufficient condition that Egoroff's theorem remains valid for monotone set function is that the monotone set function fulfils *condition (E)*. Murofushi et al. [4] defined the concept of *Egoroff condition* and proved that it is a necessary and sufficient condition for Egoroff's theorem with respect to nonadditive measures. Li and Yasuda [3] studied Egoroff's theorem on finite monotone nonadditive measure space by using *condition (E)*.

In nonadditive measure theory, Lusin's theorem was generalized by Wu and Ha [5] under the conditions of continuity and autocontinuity. Further research on this matter was performed by Jiang and Suzuki [6]. Kawabe [7] investigated regularity and Lusin's theorem for Riesz space-valued fuzzy measures. Li and Mesiar [8] proved Lusin's theorem on monotone measure spaces, assuming that the monotone measure fulfils *condition (E)* and has $(p.g.p.)$ that was introduced by Dobrakov and Farkova [9].

The original motivation for studying nonlinear expectation and g-expectation comes from expected utility theory, which is the foundation of modern mathematical economics. Chen and Epstein [10] gave an application of dynamically consistent nonlinear expectation to recursive utility. Peng [11, 12] and Rosazza Gianin [13] investigated some applications of dynamically consistent nonlinear expectations and g-expectations to static and dynamic pricing mechanisms and risk measures. Hu et al. [14] studied Fubini's theorem for nonadditive measures in the framework of g-expectation.

In this paper, we study Egoroff's theorem and Lusin's theorem for capacities induced by g-expectation. We give the sufficient conditions that provide Egoroff's theorem and Lusin's theorem in the framework of g-expectation. The remainder of this paper is organized as follows: In Section 2, we introduce some notations, assumptions, notions, lemmas, and propositions that are used in this paper. In Section 3, we give Egoroff's theorem, Lusin's theorem, and continuous function approximation theorem in the framework of g-expectation including the proofs.

2. Preliminaries

In this section, we shall present some notations, assumptions, notions, lemmas, and propositions that are used in this paper.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion with respect to filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the Brownian motion and all P-null subsets, i.e.,

$$\mathcal{F}_t = \sigma\{W_s; s \leq t\} \vee \mathcal{N}, \quad (1)$$

where $\mathcal{N}$ is the set of all P-null subsets. Fix a real number $T > 0$.

Let us introduce the following spaces:

$L^2(\Omega, \mathcal{F}_T, P) = \{\xi\colon \xi \text{ is } F_T - \text{measurable random variable such that } E[|\xi|^2] < \infty\}$

$L^2(0, T; P; \mathbb{R}^d) = \{V\colon V_t \text{ is } \mathbb{R}^d\text{-valued and } \mathcal{F}_t\text{-adapted process such that } E[\int_0^T |V_t|^2 dt] < \infty\}$

$S^2(0, T; P; \mathbb{R}) = \{V\colon V_t \text{ is continuous process in } L^2(0, T; P; \mathbb{R}) \text{ such that } E[\sup_{0\leq t\leq T} |V_t|^2] < \infty\}$

Now, we consider the following 1-dimensional backward stochastic differential equation (BSDE):

$$y_t = \xi + \int_t^T g(t, y_s, z_s) ds - \int_t^T z_s dW_s, \quad t \in [0, T]. \quad (2)$$

Let

$$g\colon \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \longmapsto \mathbb{R}, \quad (3)$$

such that for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $g(\cdot, y, z)$ is $\mathcal{F}_t$-progressively measurable. We make the following assumptions:

(H1) $E[\int_0^T |g(t, 0, 0)|^2 dt] < \infty$.

(H2) There exists a constant $\mu > 0$ such that for any $\omega \in \Omega$, $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq \mu(|y_1 - y_2| + |z_1 - z_2|). \quad (4)$$

(H3) For any $\omega \in \Omega$, $t \in [0, T]$ and $y \in \mathbb{R}$, $g(t, y, 0) = 0$.

(H4) g is subadditive with respect to y and z, i.e., for any $\omega \in \Omega$, $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$,

$$g(t, y_1 + y_2, z_1 + z_2) \leq g(t, y_1, z_1) + g(t, y_2, z_2). \quad (5)$$

Lemma 1 (see Pardoux and Peng [15]). *Suppose that g satisfies (H1) and (H2). Then, for any $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, BSDE (2) has a unique pair of adapted processes $(y_t, z_t) \in S^2(0, T; P; \mathbb{R}) \times L^2(0, T; P; \mathbb{R}^d)$.*

Definition 1 (g-expectation, see Peng [16]). Suppose that g satisfies (H2) and (H3). For any $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, let (y_t, z_t) be the solution of BSDE (2) with terminal value ξ. Consider the mapping $\varepsilon_g[\cdot] : L^2(\Omega, \mathcal{F}_T, P) \longmapsto \mathbb{R}$, denoted by $\varepsilon_g[\xi] = y_0$. We call $\varepsilon_g[\xi]$ the g-expectation of ξ.

From Peng [16], we know that that g-expectation keeps many properties of mathematical expectation:

(i) $\varepsilon_g[c] = c$, if c is a constant

(ii) $\varepsilon_g[\xi_1] \geq \varepsilon_g[\xi_2]$, if $\xi_1 \geq \xi_2$

For more details of the properties of g-expectation, we can see Briand et al. [17]; Chen et al. [18, 19]; Jiang [20]; He et al. [21]; Hu [22]; Zong and Hu [23, 24]; and Zong et al. [25].

Proposition 1 (see Briand et al. *[17]). Suppose that g satisfies (H2) and (H3). For any $\xi, \eta \in L^2(\Omega, \mathcal{F}_T, P)$, there exists a positive constant C such that*

$$\left|\varepsilon_g[\xi] - \varepsilon_g[\eta]\right|^2 \leq CE\left[|\xi - \eta|^2\right]. \quad (6)$$

Definition 2 (see Choquet [26]). A capacity is a real-valued set function $V\colon \mathcal{F} \longmapsto [0, 1]$ satisfying

(1) $V(\varnothing) = 0$, $V(\Omega) = 1$

(2) $V(A) \leq V(B)$, whenever and $A, B \in \mathcal{F}$

Define the conjugate $\overline{V}$ of V by $\overline{V}(A) := 1 - V(\Omega \backslash A)$, $\forall A \in \mathcal{F}$. Obviously, $\overline{V}$ is also a capacity and $\overline{\overline{V}} = V$.

Definition 3. Suppose that V is a capacity. Then,

(i) Countably subadditive:

$$V\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} V(A_n), \quad \forall A_n \in \mathcal{F}. \quad (7)$$

(ii) Continuity from above: for any $A_n, A \in \mathcal{F}$ $(n = 1, 2, \ldots)$, $\lim_{n\longrightarrow\infty} V(A_n) = V(A)$, whenever $A_n \searrow A$.

(iii) Continuity from below: for any $A_n, A \in \mathcal{F}$ $(n = 1, 2, \ldots)$, $\lim_{n\longrightarrow\infty} V(A_n) = V(A)$, whenever $A_n \nearrow A$.

(iv) Continuity: V is continuous from below and above.

Definition 4 (see Wang and Klir [1]). Let F be the class of all finite real-valued measurable functions on $(\Omega, \mathcal{F}, V)$, and let $f, f_n \in F$ $(n = 1, 2, \ldots)$:

(i) $\{f_n\}$ converges almost everywhere to f on Ω ($f_n \xrightarrow{\text{a.e.}} f$): there is a set $E \in \mathcal{F}$ such that $V(E) = 0$ and $f_n \longrightarrow f$ on $\Omega \backslash E$

(ii) $\{f_n\}$ converges pseudo almost everywhere to f on Ω ($f_n \xrightarrow{\text{p.a.e.}} f$): there is a set $Q \in \mathcal{F}$, such that $V(\Omega \backslash Q) = 1$ and $f_n \longrightarrow f$ on $\Omega \backslash Q$

(iii) $\{f_n\}$ converges almost uniformly to f on Ω ($f_n \xrightarrow{\text{a.u.}} f$): for any $\delta > 0$, there is a set $E_\delta \in \mathcal{F}$, such that $V(\Omega \backslash E_\delta) < \delta$ and f_n converges to f uniformly on E_δ

(iv) $\{f_n\}$ converges to f pseudo almost uniformly on Ω ($f_n \xrightarrow{\text{p.a.u.}} f$): there exists $\{Q_k\} \subset \mathcal{F}$ with $\lim_{k\longrightarrow\infty} V(\Omega\backslash Q_k) = 1$ such that f_n converges to f on $\Omega\backslash Q_k$ uniformly for any fixed $k = 1, 2, \ldots$

Remark 1. It is easy to prove that

(1) $f_n \xrightarrow{\text{a.e.}} f$ with respect to V if and only if $f_n \xrightarrow{\text{p.a.e.}} f$ with respect to $\overline{V}$

(2) $f_n \xrightarrow{\text{a.u.}} f$ with respect to V if and only if $f_n \xrightarrow{\text{p.a.u.}} f$ with respect to $\overline{V}$

Define

$$V_g(A) := \varepsilon_g[I_A], \quad \forall A \in \mathcal{F}_T. \tag{8}$$

It is easy to check that V_g is a capacity.

Remark 2. By Proposition 1, we can obtain that suppose g satisfies (H2) and (H3), $A_n, A \in \mathcal{F}_T$ $(n = 1, 2, \ldots)$; then

(1) $V_g(A_n) \searrow V_g(A)$, whenever $A_n \searrow A$

(2) $V_g(A_n) \nearrow V_g(A)$, whenever $A_n \nearrow A$

Thus, V_g is a continuous capacity. Similarly, $\overline{V_g}$ is a continuous capacity.

The following proposition is a special case of Corollary 3.5 by Peng [12].

Proposition 2. *Suppose that g satisfies (H2)–(H4). Then,* $V_g(A_1 \cup A_2) \le V_g(A_1) + V_g(A_2)$ $\forall A_1, A_2 \in \mathcal{F}_T$.

Remark 3. Suppose that g satisfies (H2)–(H4). By Remark 2 and Proposition 2, we have

$$\begin{aligned} V_g\left(\bigcup_{k=1}^{\infty} A_k\right) &= \lim_{n\longrightarrow\infty} V_g\left(\bigcup_{k=1}^{n} A_k\right) \le \lim_{n\longrightarrow\infty} \sum_{k=1}^{n} V_g(A_k) \\ &= \sum_{k=1}^{\infty} V_g(A_k). \end{aligned} \tag{9}$$

Thus, V_g is countably subadditive.

3. Main Results

In this section, we study Egoroff's theorem, Lusin's theorem, and continuous function approximation theorem in the framework of g-expectation.

Theorem 1 (Egoroff's Theorem). *Suppose that g satisfies (H2)–(H4),* f_n *and f are* $\mathcal{F}_T$*-measurable random variables. Then,*

(1) If $f_n \xrightarrow{a.e.} f$ *with respect to* V_g*, then* $f_n \xrightarrow{a.u.} f$ *with respect to* V_g

(2) If $f_n \xrightarrow{p.a.e.} f$ *with respect to* $\overline{V}_g$*, then* $f_n \xrightarrow{p.a.u.} f$ *with respect to* $\overline{V}_g$

Proof. Firstly, we prove Theorem 1 (1). Let D be the set of these points ω at which $\{f_n\}$ does not converge to f. Then,

$$D = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\omega : \left|f_n(\omega) - f(\omega)\right| \ge \frac{1}{k}\right\}. \tag{10}$$

Since $f_n \xrightarrow{\text{a.e.}} f$ with respect to V_g, we have $V_g(D) = 0$. Thus, for any fixed positive integer k,

$$V_g\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\omega : \left|f_n(\omega) - f(\omega)\right| \ge \frac{1}{k}\right\}\right) = 0. \tag{11}$$

Noting the fact that

$$\bigcup_{n=N}^{\infty} \left\{\omega : \left|f_n(\omega) - f(\omega)\right| \ge \frac{1}{k}\right\} \searrow \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\omega : \left|f_n(\omega) - f(\omega)\right| \ge \frac{1}{k}\right\}, \tag{12}$$

and by Remark 2, we have

$$\begin{aligned} &\lim_{N\longrightarrow\infty} V_g\left(\bigcup_{n=N}^{\infty} \left\{\omega : \left|f_n(\omega) - f(\omega)\right| \ge \frac{1}{k}\right\}\right) \\ &= V_g\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\omega : \left|f_n(\omega) - f(\omega)\right| \ge \frac{1}{k}\right\}\right) \\ &= 0. \end{aligned} \tag{13}$$

Therefore for any $\delta > 0$ and any positive integer k, there exists a positive integer N_k, such that

$$V_g\left(\bigcup_{n=N_k}^{\infty} \left\{\omega : \left|f_n(\omega) - f(\omega)\right| \ge \frac{1}{k}\right\}\right) < \frac{\delta}{2^k}. \tag{14}$$

Let

$$E_\delta := \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} \left\{\omega : \left|f_n(\omega) - f(\omega)\right| < \frac{1}{k}\right\}. \tag{15}$$

By Remark 3, we have

$$\begin{aligned} V_g(\Omega\backslash E_\delta) &= V_g\left(\bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} \left\{\omega : \left|f_n(\omega) - f(\omega)\right| \ge \frac{1}{k}\right\}\right) \\ &\le \sum_{k=1}^{\infty} V_g\left(\bigcup_{n=N_k}^{\infty} \left\{\omega : \left|f_n(\omega) - f(\omega)\right| \ge \frac{1}{k}\right\}\right) \\ &< \sum_{k=1}^{\infty} \frac{\delta}{2^k} \\ &= \delta. \end{aligned} \tag{16}$$

Thus, f_n converges to f uniformly on E_δ. The proof of Theorem 1 (1) is complete.

From Theorem 1 (1) and by Remark 1, we can easily obtain Theorem 1 (2).

From now on, for studying Lusin's theorem, we consider the following path spaces: $\Omega = C_0^d(\mathbb{R}^+)$ is the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t\ge0}$ with $\omega_0 = 0$, equipped with the distance

$$\rho\left(\omega^1, \omega^2\right) := \sum_{n=1}^{\infty} 2^{-n} \left[\left(\max_{t\in[0,n]} \left|\omega_t^1 - \omega_t^2\right|\right) \wedge 1\right]. \tag{17}$$

We set $\Omega_T := \{\omega_{\cdot\wedge T} : \omega \in \Omega\}$. It is clear that (Ω, ρ) and (Ω_T, ρ) are both complete separable metric spaces. Let $\mathcal{O}$ and $\mathcal{C}$ be the classes of open sets and closed sets in (Ω, ρ), respectively. Similarly, $\mathcal{O}_T$ and $\mathcal{C}_T$ are the classes of open sets and closed sets in (Ω_T, ρ), respectively.

We consider the canonical process: $\omega_t = W_t(\omega)$, $t \in [0,\infty)$, for $\omega \in \Omega$. Let $\mathscr{F}$ be the smallest σ-algebra containing $\mathscr{O}$, and let $\mathscr{F}_T$ be the smallest σ-algebra containing $\mathscr{O}_T$. We can choose a probability measure $\widetilde{P}$ such that $(W_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion under $(C_0^d(\mathbb{R}^+), \mathscr{F}, \widetilde{P})$. □

Definition 5 (see Wu and Ha [5]). A capacity V is called regular, if for every $A \in \mathscr{F}$ and $\delta > 0$, there exists a closed set F_δ and an open set G_δ of Ω, such that

$$\begin{gathered} F_\delta \subset A \subset G_\delta, \\ V(G_\delta \backslash F_\delta) < \delta. \end{gathered} \tag{18}$$

Lemma 2. *Suppose that g satisfies (H2)–(H4), then V_g is regular on $\mathscr{F}_T$.*

Proof. Let $\mathscr{A}$ be the class of all sets $A \in \mathscr{F}_T$ such that for any $\delta > 0$, there exists a closed set F_δ and an open set G_δ of Ω_T satisfying

$$\begin{gathered} F_\delta \subset A \subset G_\delta, \\ V_g(G_\delta \backslash F_\delta) < \delta. \end{gathered} \tag{19}$$

To prove this lemma, it is sufficient to show that $\mathscr{F}_T \subset \mathscr{A}$.

Firstly, we verify that $\mathscr{A}$ is an algebra. It is easy to know that $\Omega_T \in \mathscr{A}$. Suppose $A, B \in \mathscr{A}$, then for any $\delta > 0$, there exist closed sets $F_{1,\delta}, F_{2,\delta} \in \Omega_T$ and open sets $G_{1,\delta}, G_{2,\delta} \in \Omega_T$ such that

$$\begin{gathered} F_{1,\delta} \subset A \subset G_{1,\delta}, \\ V_g\left(G_{1,\delta} \backslash F_{1,\delta}\right) < \delta; \\ F_{2,\delta} \subset B \subset G_{2,\delta}, \\ V_g\left(G_{2,\delta} \backslash F_{2,\delta}\right) < \delta. \end{gathered} \tag{20}$$

So we have

$$F_{1,\delta} G_{2,\delta}^c \subset A \backslash B \subset G_{1,\delta} F_{2,\delta}^c. \tag{21}$$

$F_{1,\delta} G_{2,\delta}^c$ is a closed set of Ω_T, $G_{1,\delta} F_{2,\delta}^c$ is an open set of Ω_T, and

$$\begin{aligned} V_g\left(G_{1,\delta} F_{2,\delta}^c \backslash F_{1,\delta} G_{2,\delta}^c\right) &= V_g\left(G_{1,\delta} F_{2,\delta}^c F_{1,\delta}^c G_{2,\delta}\right) \\ &= V_g\left(\left(G_{1,\delta} \backslash F_{1,\delta}\right) \bigcap \left(G_{2,\delta} \backslash F_{2,\delta}\right)\right) \\ &\leq \min\left\{V_g\left(G_{1,\delta} \backslash F_{1,\delta}\right), V_g\left(G_{2,\delta} \backslash F_{2,\delta}\right)\right\} \\ &< \delta. \end{aligned} \tag{22}$$

That is, $A \backslash B \subset \mathscr{A}$. So $\mathscr{A}$ is an algebra of Ω_T.

Next, we prove that $\mathscr{A}$ is closed under the formation of pairwise disjoint countable unions. Let $\{A_n\}_{n=1}^\infty \subset \mathscr{A}$ be the sequence of pairwise disjoint set and $\delta > 0$ be given. From the definition of $\mathscr{A}$ and $A_n \in \mathscr{A}$, we know that for each given n, there exist an open set G_n and a closed set F_n of Ω_T such that

$$\begin{gathered} F_n \subset A_n \subset G_n, \\ V_g(G_n \backslash F_n) < \frac{\delta}{2^{n+1}}. \end{gathered} \tag{23}$$

Noting the fact that

$$\bigcup_{n=1}^\infty F_n \backslash \bigcup_{n=1}^k F_n \searrow \varnothing, \tag{24}$$

and by Remark 2, we have

$$\lim_{k \longrightarrow \infty} V_g\left(\bigcup_{n=1}^\infty F_n \backslash \bigcup_{n=1}^k F_n\right) = 0. \tag{25}$$

Thus, there exists a positive integer k_0 such that

$$V_g\left(\bigcup_{n=1}^\infty F_n \backslash \bigcup_{n=1}^k F_n\right) < \frac{\delta}{2}. \tag{26}$$

Denote $G_\delta := \bigcup_{n=1}^\infty G_n$ and $F_\delta := \bigcup_{n=1}^{k_0} F_n$; then, G_δ is an open set of Ω_T, F_δ is a closed set of Ω_T, and

$$F_\delta \subset \bigcup_{n=1}^\infty A_n \subset G_\delta. \tag{27}$$

By Remark 3, we have

$$\begin{aligned} V_g(G_\delta \backslash F_\delta) &= V_g\left(\bigcup_{n=1}^\infty F_n \backslash \bigcup_{n=1}^{k_0} F_n\right) \\ &\leq V_g\left(\left(\bigcup_{n=1}^\infty (G_n \backslash F_n)\right) \bigcup \left(\bigcup_{n=1}^\infty F_n \backslash \bigcup_{n=1}^{k_0} F_n\right)\right) \\ &\leq V_g\left(\bigcup_{n=1}^\infty (G_n \backslash F_n)\right) + V_g\left(\left(\bigcup_{n=1}^\infty F_n \backslash \bigcup_{n=1}^{k_0} F_n\right)\right) \\ &\leq \sum_{n=1}^\infty V_g(G_n \backslash F_n) + V_g\left(\left(\bigcup_{n=1}^\infty F_n \backslash \bigcup_{n=1}^{k_0} F_n\right)\right) \\ &< \delta. \end{aligned} \tag{28}$$

That is,

$$\bigcup_{n=1}^\infty A_n \in \mathscr{A}. \tag{29}$$

So $\mathscr{A}$ is a σ-algebra of Ω_T.

In real analysis theory, we know that for any closed set $F \in \mathscr{C}_T$, there exists a sequence of open sets $\{E_n\}_{n=1}^\infty$ such that

$$E_n \backslash F \searrow \varnothing, \quad \text{as } n \longrightarrow \infty. \tag{30}$$

Therefore, by Remark 2, we have $\lim_{n\longrightarrow\infty} V_g(E_n \backslash F) = 0$. Thus, $\mathscr{C}_T \subset \mathscr{A}$. Since $\mathscr{A}$ is closed under the formation of complements, we have $\mathscr{O}_T \subset \mathscr{A}$. This shows that $\mathscr{A}$ is a σ-algebra containing $\mathscr{O}_T$. So $\mathscr{F}_T \subset \mathscr{A}$. □

Remark 4. Suppose that g satisfies (H2)–(H4).

(1) By Lemma 2, we know that for any $A \in \mathscr{F}_T$, there exist an increasing sequence $\{F_n\}_{n=1}^\infty$ of closed sets

and a decreasing sequence $\{G_n\}_{n=1}^{\infty}$ of open sets such that for every $n = 1, 2, \ldots$, $F_n \subset A \subset G_m$,

$$V_g(G_n \backslash A) < \frac{1}{n}, \quad V_g(A \backslash F_n) < \frac{1}{n}. \tag{31}$$

(2) By Theorem 1 (1) and Lemma 2, we know that if $f_n \xrightarrow{\text{a.e.}} f$ with respect to V_g, then for any $\delta > 0$, there exists a closed set $F_\delta \in \mathscr{C}_T$ such that $V_g(\Omega_T \backslash F_\delta) < \delta$ and f_n converges to f uniformly on F_δ.

(3) By Theorem 1 (1) and Lemma 2, we know that if $f_n \xrightarrow{\text{a.e.}} f$ with respect to V_g, then there exists an increasing sequence of closed sets $\{H_k\}_{k=1}^{\infty} \subset \mathscr{F}_T$ such that

$$V_g\left(\Omega_T \backslash \bigcup_{k=1}^{\infty} H_k\right) = 0, \tag{32}$$

and f_n converges to f on H_k uniformly for any fixed $k = 1, 2, \ldots$.

In the following, we present Lusin's theorem in the framework of g-expectation.

Theorem 2 (Lusin's Theorem). *Suppose that g satisfies (H2)–(H4) and f is an $\mathscr{F}_T$-measurable random variable. Then, for each $\delta > 0$, there exists a closed set $F_\delta \in \mathscr{C}_T$ such that $V_g(\Omega_T \backslash F_\delta) < \delta$ and f is continuous on F_δ.*

Proof. We prove this theorem stepwise in the following two situations.

(a) Suppose that f is a simple function, i.e., $f = \sum_{k=1}^{n} c_k \chi_{E_k}$, where χ_{E_k} is the characteristic function of E_k and $\Omega_T = \bigcup_{k=1}^{n} E_k$ (a disjoint finite union). For any $\delta > 0$, by Lemma 2, we know that for each k, there exists a closed set F_k of Ω_T such that $F_k \subset E_k$ and

$$V_g(E_k \backslash F_k) < \frac{\delta}{n}. \tag{33}$$

Let

$$F_\delta := \bigcup_{k=1}^{n} F_k. \tag{34}$$

Then, F_δ is a closed set. By Remark 3, we have

$$\begin{aligned} V_g(\Omega_T \backslash F_\delta) &= V_g\left(\bigcup_{k=1}^{n} E_k \backslash \bigcup_{k=1}^{n} F_k\right) \\ &\leq V_g\left(\bigcup_{k=1}^{n} (E_k \backslash F_k)\right) \\ &\leq \sum_{k=1}^{n} V_g(E_k \backslash F_k) \\ &< \delta. \end{aligned} \tag{35}$$

Obviously, f is continuous on F_δ.

(b) Let f be an $\mathscr{F}_T$-measurable random variable. Then, there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of simple functions such that $\varphi_n \longrightarrow f$ on Ω, as $n \longrightarrow \infty$. With the help of Remark 4 (3), we know that there exists an increasing sequence of closed sets $\{H_k\}_{k=1}^{\infty} \subset \mathscr{F}_T$ such that

$$V_g\left(\Omega_T \backslash \bigcup_{k=1}^{\infty} H_k\right) = 0, \tag{36}$$

and φ_n converges to f on H_k uniformly for any fixed $k = 1, 2, \ldots$. Applying (a), we can prove that for any fixed n, there exists a closed set $F_k^{(n)}$ of Ω_T satisfying that $F_k^{(n)} \subset H_k$ such that

$$V_g\left(H_k \backslash F_k^{(n)}\right) < \frac{\delta}{2^{n+k}}, \quad k = 1, 2, \ldots, \tag{37}$$

and φ_n is continuous on $F_k^{(n)}$. Let

$$F_\delta := \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_k^{(n)}. \tag{38}$$

Then, F_δ is a closed set. By Remark 3, we have

$$\begin{aligned} V_g(\Omega_T \backslash F_\delta) &= V_g\left(\left(\Omega_T \backslash \bigcup_{k=1}^{\infty} H_k\right) \bigcup \left(\bigcup_{k=1}^{\infty} H_k \backslash \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_k^{(n)}\right)\right) \\ &\leq V_g\left(\Omega_T \backslash \bigcup_{k=1}^{\infty} H_k\right) + V_g\left(\bigcup_{k=1}^{\infty} H_k \backslash \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} F_k^{(n)}\right) \\ &\leq V_g\left(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \left(H_k \backslash F_k^{(n)}\right)\right) \\ &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} V_g\left(H_k \backslash F_k^{(n)}\right) \\ &< \delta. \end{aligned} \tag{39}$$

At last, we show that f is continuous on F_δ. In fact, φ_n is continuous and converges to f uniformly on F_δ. So for any $\varepsilon > 0$ and any $\omega, \omega_0 \in F_\delta$, there exist a positive integer n_o and a positive constant ς such that

$$\left|\varphi_{n_0}(\omega) - f(\omega)\right| < \frac{\varepsilon}{3}, \quad \left|\varphi_{n_0}(\omega) - \varphi_{n_0}(\omega_0)\right| < \frac{\varepsilon}{3}, \tag{40}$$

when $|\omega - \omega_0| < \varsigma$. Thus, we have

$$\begin{aligned} \left|f(\omega) - f(\omega_0)\right| &= \left|f(\omega) - \varphi_{n_0}(\omega) + \varphi_{n_0}(\omega) - \varphi_{n_0}(\omega_0) + \varphi_{n_0}(\omega_0) - f(\omega_0)\right| \\ &\leq \left|f(\omega) - \varphi_{n_0}(\omega)\right| + \left|\varphi_{n_0}(\omega) - \varphi_{n_0}(\omega_0)\right| + \left|\varphi_{n_0}(\omega_0) - f(\omega_0)\right| \\ &< \varepsilon. \end{aligned} \tag{41}$$

So f is continuous on F_δ. □

Remark 5. Suppose that g satisfies (H2)–(H4). By Theorem 2 and Lemma 2, we know that for any fixed $n = 1, 2, \ldots$, there exists a closed sequence $\{F_n\}_{n=1}^{\infty} \subset \mathscr{F}_T$ such that f is continuous on F_n and

$$V_g(\Omega_T \backslash F_n) < \frac{1}{n}. \quad (42)$$

At last, we show continuous function approximation theorem in the framework of g-expectation.

Theorem 3 (Continuous Function Approximation Theorem). *Suppose that g satisfies (H2)–(H4) and f is an $\mathscr{F}_T$-measurable random variable. Then, there exists a continuous function sequence $\{\phi_n\}_{n=1}^{\infty}$ on Ω such that $\phi_n \xrightarrow{a.e.} f$ with respect to V_g. Furthermore, if $|f| \le M$, then $|\phi_n| \le M$ $(n = 1, 2, \ldots)$, where M is a positive constant.*

Proof. By Remark 5, we know that for every $k = 1, 2, \ldots$, there exists a closed set F_k of Ω_T such that f is continuous on F_k and $V_g(\Omega_T \backslash F_k) < (1/k)$ By Tietze's extension theorem in Royden [27], for every $k = 1, 2, \ldots$, there exists a continuous function ψ_k on Ω such that $\psi_k(\omega) = f(\omega)$, for $\omega \in F_k$. And if $|f| \le M$, then $|\psi_k| \le M$. Therefore, for any $\varepsilon > 0$, we have

$$\{\omega: |\psi_k(\omega) - f(\omega)| \ge \varepsilon\} \subset \Omega_T \backslash F_k, \quad (43)$$

And, hence, for any $k = 1, 2, \ldots$,

$$V_g(\{\omega: |\psi_k(\omega) - f(\omega)| \ge \varepsilon\}) \le V_g(\Omega_T \backslash F_k) < \frac{1}{k}. \quad (44)$$

Thus, we have

$$\lim_{k \longrightarrow \infty} V_g(\{\omega: |\psi_k(\omega) - f(\omega)| \ge \varepsilon\}) = 0. \quad (45)$$

From the above fact, we can choose a subsequence $\{\psi_{k_n}\}_{n=1}^{\infty}$ of $\{\psi_k\}_{k=1}^{\infty}$ such that

$$V_g\left(\left\{\omega: |\psi_{k_n}(\omega) - f(\omega)| \ge \frac{1}{2^n}\right\}\right) < \frac{1}{2^n}. \quad (46)$$

Let

$$E_n := \left\{\omega: |\psi_{k_n}(\omega) - f(\omega)| \ge \frac{1}{2^n}\right\}. \quad (47)$$

Then,

$$\sum_{n=1}^{\infty} V_g(E_n) < \infty. \quad (48)$$

Next, we prove

$$V_g\left(\bigcap_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} \{\omega: |\psi_{k_{n+\nu}}(\omega) - f(\omega)| \ge \varepsilon\}\right) = 0. \quad (49)$$

Indeed, for any $\varepsilon > 0$, there exists a positive integer n_0 such that for any $n \ge n_0$, $(1 \backslash 2^n) < \varepsilon$ and

$$\begin{aligned} &V_g\left(\bigcap_{n=1}^{\infty} \bigcup_{\nu=1}^{\infty} \{\omega: |\psi_{k_{n+\nu}}(\omega) - f(\omega)| \ge \varepsilon\}\right) \\ &\le \sum_{m=n}^{\infty} V_g\left(\left\{\omega: |\psi_{k_m}(\omega) - f(\omega)| \ge \varepsilon\right\}\right) \\ &\le \sum_{m=n}^{\infty} V_g(E_m) \\ &\longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (50)$$

That is, $\psi_{kn} \xrightarrow{\text{a.e.}} f$ with respect to V_g, we take $\phi_n = \psi_{k_n}$, $n = 1, 2, \ldots$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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