Research Article

A Study on a New Class of Backward Stochastic Differential Equation

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The existence and uniqueness for a new type of backward stochastic differential equation when the generator includes the values of solutions of the past, the present, and the future are obtained in this paper. An important comparison theorem for this sort of BSDEs is also proved.

1. Introduction

Pardoux and Peng in [1] first provided the famous backward stochastic differential equations (BSDEs). The existence and uniqueness for the BSDEs are proved by them. Since then, BSDEs have been discussed and applied to many fields, e.g., Chen and Epstein [2] and Karoui et al. [3–7]. A lot of research has focused on the assumptions on the generator, such as [8–12]. Recently, Delong and Imkeller in [13, 14] obtained many interesting results about the time-delayed equation in which the generator at time t only depends on the past solution. Peng and Yang in [15] discussed anticipated BSDEs, in which the generator includes present and future solutions.

Therefore, the natural questions are as follows: can we discuss the backward stochastic differential equations when the generator includes not only the past and the present but also the future solutions? The comparison theorem for it is still true? Indeed, these questions are answered in the affirmative in this paper. The equation is called delay and anticipated BSDEs (DABSDEs for short). We obtain existence and uniqueness for delay and anticipated BSDEs,

which can be seen as a general version of Delong and Imkeller in [13] or Peng and Yang in [15].

2. Main Notations

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a natural filtration $(\mathcal{F}_v)_{v \geq 0}$ and $(B_v)_{v \geq 0}$ be a standard Brownian motion. We denote the norm in R^m by $|\cdot|$. Given $T > 0$, denote the following:

- (i) $L^2(\mathcal{F}_T; R^n) = \{\xi: \mathcal{F}_T\text{-measurable and } E[|\xi|^2] < \infty\}$.
- (ii) $L^2_{\mathcal{F}}(0, T; R^n) = \{\varphi_t: \mathcal{F}_t\text{-adapted and } E[\int_0^T |\varphi_t|^2 dt] < \infty\}$.
- (iii) $S^2_{\mathcal{F}}(0, T; R^n) = \{\chi_s: \chi_s \in L^2_{\mathcal{F}}(0, T; R^n) \text{ and continuous, as well as } E[\sup_{0 \leq t \leq T} |\chi_t|^2] < \infty\}$.

In the case $n = 1$, they are abbreviated to $L^2(\mathcal{F}_T)$, $L^2_{\mathcal{F}}(0, T)$, and $S^2_{\mathcal{F}}(0, T)$, respectively.

3. Delay and Anticipated Backward Stochastic Differential Equations

We propose a new type of BSDEs as follows:

$$\begin{cases} -dY_t = g(t, Y_{t-d_1(t)}, Z_{t-d_2(t)}, Y_t, Z_t, Y_{t+d_3(t)}, Z_{t+d_4(t)})dt - Z_t dB_t, & 0 \leq t \leq T, \\ Y_t = \xi_t, & T \leq t \leq T+D, \\ Z_t = \eta_t, & T \leq t \leq T+D, \end{cases} \quad (1)$$

where $d_i(\cdot), i = 1, \dots, 4$, are four continuous functions s.t.

(D1) $\exists D \geq 0$ s.t.

$$0 \leq t - d_1(t) \leq t; 0 \leq t - d_2(t) \leq t; t + d_3(t) \leq T + D; t + d_4(t) \leq T + D, 0 \leq t \leq T.$$

(2)

(D2) $\exists L \geq 0$ s.t., for all nonnegative and integrable $f(\cdot)$,

$$\begin{aligned} \int_t^T f(v - d_1(v))dv &\leq L \int_t^{T+D} f(v)dv; \int_t^T f(v - d_2(v))dv \\ &\leq L \int_t^{T+D} f(v)dv, \\ \int_t^T f(v + d_3(v))dv &\leq L \int_t^{T+D} f(v)dv; \int_t^T f(v + d_4(v))dv \\ &\leq L \int_t^{T+D} f(v)dv. \end{aligned} \quad (3)$$

Note that the simple examples for $d_i(s)$ satisfying (D1) and (D2) are constant delay and $d_i(s) = s$.

We call system (1) delay and anticipated backward stochastic differential equations (delay and anticipated BSDEs).

Our aim is to search out a pair of processes $(Y, Z) \in S_{\mathcal{F}}^2(0, D+T; R^m) \times L_{\mathcal{F}}^2(0, D+T; R^{m \times d})$ which satisfies the delay and anticipated BSDEs (1).

For all $s \in [0, T]$, we suppose $g(\omega, s, \phi, \mu, \gamma, z, \psi, \nu): \Omega \times [0, T] \times L_{\mathcal{F}}^2(0, s; R^m) \times L_{\mathcal{F}}^2(0, s; R^{m \times d}) \times R^m \times R^{m \times d} \times L^2$

$(\mathcal{F}_r; R^m) \times L^2(\mathcal{F}_{r'}; R^{m \times d}) \longrightarrow L^2(\mathcal{F}_s; R^m)$, $s \leq r, r' \leq T+D$, and satisfies

(A1) $\exists C > 0$, $\phi, \phi' \in L_{\mathcal{F}}^2(0, s; R^m); \mu, \mu' \in L_{\mathcal{F}}^2(0, s; R^{m \times d}); \gamma, \gamma' \in R^m; z, z' \in R^{m \times d}; \psi, \psi' \in L_{\mathcal{F}}^2(s, T+D; R^m); \nu, \nu' \in L_{\mathcal{F}}^2(s, T+D; R^{m \times d}); r, t \in [0, s]; \bar{r}, \bar{t} \in [s, T+D]$, such that

$$\begin{aligned} &|g(s, \phi_r, \mu_t, \gamma, z, \psi_{\bar{r}}, \nu_{\bar{t}}) - g(s, \phi'_r, \mu'_t, \gamma', z', \psi'_{\bar{r}}, \nu'_{\bar{t}})| \\ &\leq C(|\phi_r - \phi'_r| + |\mu_t - \mu'_t| + |\gamma - \gamma'| + |z - z'| \\ &\quad + E[|\psi_{\bar{r}} - \psi'_{\bar{r}}| + |\nu_{\bar{t}} - \nu'_{\bar{t}}| | \mathcal{F}_s]). \end{aligned} \quad (4)$$

$$(A2) E[\int_0^T |g(s, 0, 0, 0, 0, 0, 0)|^2] < \infty.$$

Theorem 1. Suppose g satisfies (A1) and (A2), as well as $d_i(t), i = 1, \dots, 4$, satisfy (D1) and (D2). Then, the DABSDE (1) has a unique solution for $\xi \in S_{\mathcal{F}}^2(T, D+T; R^m)$ and $\eta \in L_{\mathcal{F}}^2(T, D+T; R^{m \times d})$, namely, there exists unique $(\bar{Y}, \bar{Z}) \in S_{\mathcal{F}}^2(0, D+T; R^m) \times L_{\mathcal{F}}^2(0, D+T; R^{m \times d})$ satisfying the DABSDEs.

Proof. We choose suitable β which satisfies $0 < (e^{\beta T}/\beta) \leq 1/(32C^2(2L+1))$ and define a norm in $L_{\mathcal{F}}^2(0, D+T; R^m)$:

$$\|\mu(\cdot)\|_{\beta} = \sqrt{E\left[\int_0^{D+T} |\mu_t|^2 e^{\beta t} dt\right]}. \quad (5)$$

We set

$$\begin{cases} Y_u = \xi_T + \int_u^T g(v, \gamma_{v-d_1(v)}, z_{v-d_2(v)}, \gamma_v, z_v, \gamma_{v+d_3(v)}, z_{v+d_4(v)})dv - \int_u^T Z_v dB_v, & 0 \leq u \leq T; \\ Y_u = \xi_u, & T \leq u \leq T+D; \\ Z_u = \eta_u, & T \leq u \leq T+D. \end{cases} \quad (6)$$

We set a mapping $M[(y, z)]: = (Y, Z): L_{\mathcal{F}}^2(0, D+T; R^m \times R^{m \times d}) \longrightarrow L_{\mathcal{F}}^2(0, D+T; R^m \times R^{m \times d})$. Let us show M is a contraction mapping under $\|\cdot\|_{\beta}$. $\forall (y, z), (y'', z'') \in L_{\mathcal{F}}^2(0, T+D; R^m \times R^{m \times d})$; define $(Y, Z) = M(y, z)$, $(Y', Z') = M(y', z')$, and

$$\begin{aligned} (\hat{Y}, \hat{Z}) &= (y - y', z - z'), \\ (\hat{Y}', \hat{Z}') &= (Y - Y', Z - Z'). \end{aligned} \quad (7)$$

Using Itô's lemma for $|\hat{Y}_u|^2 e^{\beta u}$, $0 \leq u \leq T$, and taking expectation,

$$\begin{aligned}
& |\widehat{Y}_0|^2 + \beta E \left[\int_0^T e^{\beta u} |\widehat{Y}_u|^2 ds \right] + E \left[\int_0^T e^{\beta u} |\widehat{Z}_u|^2 du \right] \\
&= E \left[2 \int_0^T e^{\beta u} \left(g(u, y_{u-d_1}(u), z_{u-d_2}(u), y_u, z_u, y_{u+d_3}(u), z_{u+d_4}(u)) \right. \right. \\
&\quad \left. \left. - g(u, y_{u-d_1}(u'), z_{u-d_2}(u'), y'_u, z'_u, y_{u+d_3}(u'), z_{u+d_4}(u')) \right) \widehat{Y}_u du \right] \\
&\leq \frac{\beta}{2} E \left[\int_0^T e^{\beta u} |\widehat{Y}_u|^2 du \right] + \frac{2}{\beta} E \left[\int_0^T e^{\beta u} \left| g(u, y_{u-d_1}(u), z_{u-d_2}(u), y_u, z_u, y_{u+d_3}(u), z_{u+d_4}(u)) \right. \right. \\
&\quad \left. \left. - g(u, y_{u-d_1}(u'), z_{u-d_2}(u'), y'_u, z'_u, y_{u+d_3}(u'), z_{u+d_4}(u')) \right|^2 du \right].
\end{aligned} \tag{8}$$

Thus,

$$\begin{aligned}
& E \left[\int_0^T e^{\beta u} \left(\frac{\beta}{2} |\widehat{Y}_u|^2 + |\widehat{Z}_u|^2 \right) du \right] \\
&\leq \frac{2}{\beta} E \left[\int_0^T e^{\beta u} \left| g(u, y_{u-d_1}(u), z_{u-d_2}(u), y_u, z_u, y_{u+d_3}(u), z_{u+d_4}(u)) \right. \right. \\
&\quad \left. \left. - g(u, y_{u-d_1}(u'), z_{u-d_2}(u'), y'_u, z'_u, y_{u+d_3}(u'), z_{u+d_4}(u')) \right|^2 du \right].
\end{aligned} \tag{9}$$

First, we note that

$$\begin{aligned}
\int_0^T e^{\beta s} |\widehat{y}_{s-d_i}(s)|^2 ds &= \int_0^T e^{\beta(s-d_i(s))} |\widehat{y}_{s-d_i}(s)|^2 e^{\beta d_i(s)} ds \\
&\leq e^{\beta T} \int_0^T e^{\beta s} |\widehat{y}_s|^2 ds.
\end{aligned} \tag{10}$$

Because $d_i(s), i = 1, \dots, 4$, satisfy (D2) and g satisfies (A1), we have

$$\begin{aligned}
& E \left[\int_0^T e^{\beta s} \left(\frac{\beta}{2} |\widehat{Y}_s|^2 + |\widehat{Z}_s|^2 \right) ds \right] \\
&\leq \frac{2C^2}{\beta} E \left[\int_0^T e^{\beta s} \left(|\widehat{y}_{s-d_1}(s)| + |\widehat{z}_{s-d_2}(s)| + |\widehat{y}_s| + |\widehat{z}_s| + E \left[|\widehat{y}_{s+d_3}(s)| + |\widehat{z}_{s+d_4}(s)| \mid \mathcal{F}_s \right] \right)^2 ds \right] \\
&\leq \frac{8C^2}{\beta} E \left[\int_0^T e^{\beta u} \left(2|\widehat{y}_{u-d_1}(u)|^2 + 2|\widehat{z}_{u-d_2}(u)|^2 + |\widehat{z}_u|^2 + |\widehat{y}_u|^2 + 2|\widehat{y}_{u+d_3}(u)|^2 + 2|\widehat{z}_{u+d_4}(u)|^2 \right) du \right] \\
&\leq \frac{8C^2(2L+1)e^{\beta T}}{\beta} E \left[\int_0^{T+D} e^{\beta u} (|\widehat{y}_u|^2 + |\widehat{z}_u|^2) du \right].
\end{aligned} \tag{11}$$

Since β satisfies $(e^{\beta T}/\beta) \leq 1/(32C^2(2L+1))$, then

$$\begin{aligned}
& E \left[\int_0^{T+D} e^{\beta s} (|\widehat{Y}_s|^2 + |\widehat{Z}_s|^2) ds \right] \leq E \left[\int_0^{T+D} e^{\beta s} \left(\frac{\beta}{2} |\widehat{Y}_s|^2 + |\widehat{Z}_s|^2 \right) ds \right] \\
&\leq \frac{1}{4} E \left[\int_0^{T+D} e^{\beta s} (|\widehat{y}_s|^2 + |\widehat{z}_s|^2) ds \right].
\end{aligned} \tag{12}$$

Therefore,

$$\|(\widehat{Y}, \widehat{Z})\|_{\beta} \leq \frac{1}{2} \|(\widehat{y}, \widehat{z})\|_{\beta}. \tag{13}$$

Consequently, M is a strict contraction mapping. From the fixed point theorem, the DABSDE (1) has a unique solution.

We use Itô's lemma for $|Y_v|^2 e^{\alpha v}, t \leq v \leq T$, where α is an arbitrary constant, and take conditional expectation, similar to Proposition 4.4 in [15]; then,

$$\begin{aligned}
& E \left[\sup_{0 \leq t \leq T} |Y_t|^2 + \int_t^T |Z_s|^2 ds \mid \mathcal{F}_t \right] \leq C' E \left[|\xi_T|^2 \right. \\
&\quad \left. + \int_0^T |g(s, 0, 0, 0, 0, 0, 0)|^2 ds + \int_T^{T+D} (|\xi_v|^2 + |\eta_v|^2) dv \right],
\end{aligned} \tag{14}$$

where C' only depends on C in (A1), L in (D2), and T . Therefore, $Y \in S_{\mathcal{F}}^2(0, D+T; R^m)$. $\square$

Example 1. Consider a typical delay and anticipated backward stochastic differential equation

$$\begin{cases} -dY_u = (Z_{u-(\pi/2)} + Z_{u+\pi})du - Z_u dB_u, & 0 \leq u \leq T, \\ Y_u = \int_0^T \sin t dB_t, & T \leq u \leq T+D, \\ Z_u = \sin T, & T \leq u \leq T+D, \end{cases} \quad (15)$$

with $Y_T = \int_0^T \sin t dB_t$. We can get the unique solution of DABSDE (15) which is

$$\begin{cases} Y_u = \int_0^u \sin s dB_s - (\cos u + \sin u) + \cos T + \sin T, & 0 \leq u \leq T, \\ Z_u = \sin u, & 0 \leq u \leq T. \end{cases} \quad (16)$$

4. Comparison Theorem for Delay and Anticipated BSDEs

Next, we deduce the comparison theorem for one-dimensional DABSDEs. Denote $(i)_t^{(i)} := Y_t^{(i)}, i = 1, 2, 3$. Let $(Y^{(j)}, Z^{(j)})(j = 1, 2)$ be the solution of the following one-dimensional delay and anticipated BSDEs:

$$\begin{cases} -dY_s^{(j)} = g_j(s, Y_{s-d_1}^{(j)}, Y_s^{(j)}, Z_s^{(j)}, Y_{s+d_3}^{(j)})ds - Z_s^{(j)}dB_s, & s \in [0, T], \\ Y_s^{(j)} = \xi_s^{(j)}, & s \in [T, D+T]. \end{cases} \quad (17)$$

$$Y_t^{(2)} \leq Y_t^{(1)}, \quad \text{a.e., a.s.} \quad (18)$$

In particular, we also have the strict comparison theorem:

Theorem 2. Assume that g_1, g_2 satisfy (A1) and (A2), $\xi^{(1)}, \xi^{(2)} \in S_{\mathcal{F}}^2(T, D+T), d_k(t), k = 1, 3$, satisfy (D1), (D2), and $\forall y \in R, z \in R^d, g_2(t, \cdot, y, z, \cdot)$ is strictly increasing. If $\xi^{(2)} \leq \xi^{(1)}, s \in [T, D+T]$ and $g_1(\cdot, \phi_r, y, z, \phi_{\bar{r}}) \geq g_2(\cdot, \phi_r, y, z, \phi_{\bar{r}}), \phi_r \in L_{\mathcal{F}}^2(0, t), \phi_{\bar{r}} \in L_{\mathcal{F}}^2(t, T+D), r \in [0, t], \bar{r} \in [t, D+T]$, then

$$Y_0^{(1)} = Y_0^{(2)} \iff \begin{cases} g_1(t, Y_{t-d_1}^{(1)}, Y_t^{(1)}, Z_t^{(1)}, Y_{t+d_3}^{(1)}) = g_2(t, Y_{t-d_1}^{(1)}, Y_t^{(1)}, Z_t^{(1)}, Y_{t+d_3}^{(1)}), & t \in [0, T], \\ \xi_t^{(1)} = \xi_t^{(2)}, & t \in [T, D+T]. \end{cases} \quad (19)$$

Proof. Set

$$\begin{cases} Y_u^{(3)} = \xi_T^{(2)} + \int_u^T g_2(s, Y_{s-d_1}^{(1)}, Y_s^{(3)}, Z_s^{(3)}, Y_{s+d_3}^{(1)})ds - \int_u^T Z_s^{(3)}dB_s, & u \in [0, T], \\ Y_u^{(3)} = \xi_u^{(2)}, & u \in [T, D+T]. \end{cases} \quad (20)$$

Denote $\hat{g}_u := g_1(u, Y_{u-d_1}^{(1)}, Y_u^{(1)}, Z_u^{(1)}, Y_{u+d_3}^{(1)}) - g_2(u, Y_{u-d_1}^{(1)}, Y_u^{(1)}, Z_u^{(1)}, Y_{u+d_3}^{(1)})$ and $\hat{\xi} := \xi^{(1)} - \xi^{(2)}, \hat{y} := Y^{(1)} - Y^{(3)}, \hat{z} := Z^{(1)} - Z^{(3)}$. Then, $(\hat{y}, \hat{z})$ is the solution of the following DABSDEs:

$$\begin{cases} \hat{y}_t = \hat{\xi}_T + \int_t^T (\hat{g}_v + a_v \hat{y}_v + b_v \hat{z}_v)dv - \int_t^T \hat{z}_v dB_v, & 0 \leq t \leq T, \\ \hat{y}_t = \hat{\xi}_t, & T \leq t \leq D+T, \end{cases} \quad (21)$$

where

$$a_v = \begin{cases} \frac{g_1(v, Y_{v-d_1(v)}^{(1)}, Y_v^{(1)}, Z_v^{(1)}, Y_{v+d_3(v)}^{(1)}) - g_2(v, Y_{v-d_1(v)}^{(1)}, Y_v^{(3)}, Z_v^{(1)}, Y_{v+d_3(v)}^{(1)})}{Y_v^{(1)} - Y_v^{(3)}}, & Y_v^{(1)} \neq Y_v^{(3)}, \\ 0, & Y_v^{(1)} = Y_v^{(3)}, \end{cases} \quad (22)$$

$$b_v = \begin{cases} \frac{g_1(v, Y_{v-d_1(v)}^{(1)}, Y_v^{(3)}, Z_v^{(1)}, Y_{v+d_3(v)}^{(1)}) - g_2(v, Y_{v-d_1(v)}^{(1)}, Y_v^{(3)}, Z_v^{(3)}, Y_{v+d_3(v)}^{(1)})}{Z_v^{(1)} - Z_v^{(3)}}, & Z_v^{(1)} \neq Z_v^{(3)}, \\ 0, & Z_v^{(1)} = Z_v^{(3)}, \end{cases}$$

Since g_2 follows assumption (A1), then $|a_v| \leq C$ and $|b_v| \leq C$. Denote

$$q_s := \exp \left[-\frac{1}{2} \int_0^s |b_v|^2 dv + \int_0^s a_v dv + \int_0^s b_v dB_v \right]. \quad (23)$$

Using Itô's lemma for $q_s \hat{y}_s$ and taking expectation, then

$$\hat{y}_t = E \left[\hat{\xi}_T q_T + \int_t^T \hat{g}_s q_s ds \mid \mathcal{F}_t \right]. \quad (24)$$

From this, we have $Y^{(3)} \leq Y^{(1)}$ a.e., a.s. Similarly, we set

$$\begin{cases} Y_u^{(4)} = \int_u^T g_2(s, Y_{s-d_1(s)}^{(3)}, Y_s^{(4)}, Z_s^{(4)}, Y_{s+d_3(s)}^{(3)}) ds - \int_u^T Z_s^{(4)} dB_s + \xi_u^{(2)}, & 0 \leq u \leq T, \\ Y_u^{(4)} = \xi_u^{(2)}, & T \leq u \leq D+T. \end{cases} \quad (25)$$

Because $Y_t^{(1)} \geq Y_t^{(3)}$ a.e., a.s., and $g_2(\cdot)$ is strictly increasing, from the classical comparison theorem, then $Y^{(4)} \leq Y^{(3)}$. When $n = 5, 6, \dots$, we investigate the DABSDs:

$$\begin{cases} Y_u^{(n)} = \xi_u^{(2)} + \int_u^T g_2(v, Y_{v-d_1(v)}^{(n-1)}, Y_v^{(n)}, Z_v^{(n)}, Y_{v+d_3(v)}^{(n-1)}) dv - \int_u^T Z_v^{(n)} dB_v, & 0 \leq u \leq T, \\ Y_u^{(n)} = \xi_u^{(2)}, & T \leq u \leq T+D. \end{cases} \quad (26)$$

Since $g_2(t, \cdot, y, z, \cdot)$ is strictly increasing, we have $Y_t^{(4)} \geq Y_t^{(5)} \geq \dots \geq Y_t^{(n)} \geq \dots$. From the proving method of Theorem 1, $(Y^{(n)})$ and $(Z^{(n)})_{n \geq 4}$ are Cauchy sequences in

$L^2_{\mathcal{F}}(0, D+T)$ and in $L^2_{\mathcal{F}}(0, T)$, $n \geq 4$. Write their limits as Y and Z ; then, $(Y, Z) \in L^2_{\mathcal{F}}(0, D+T) \times L^2_{\mathcal{F}}(0, T)$, and when $n \rightarrow \infty$,

$$\begin{aligned} & E \left[\int_t^T \left| g_2(s, Y_{s-d_1(s)}^{(n-1)}, Y_s^{(n)}, Z_s^{(n)}, Y_{s+d_3(s)}^{(n-1)}) - g_2(s, Y_{s-d_1(s)}, Y_s, Z_s, Y_{s+d_3(s)}) \right|^2 e^{\beta s} ds \right] \\ & \leq 4C^2 E \left[\int_t^T \left(|Y_v^{(n)} - Y_v|^2 + 2L |Y_v^{(n-1)} - Y_v|^2 + |Z_v^{(n)} - Z_v|^2 \right) e^{\beta v} dv \right] \rightarrow 0. \end{aligned} \quad (27)$$

Therefore, (Y, Z) satisfies the following delay and anticipated BSDEs:

$$\begin{cases} Y_t = \xi_T^{(2)} + \int_t^T g_2(s, Y_{s-d_1(s)}, Y_s, Z_s, Y_{s+d_3(s)}) ds - \int_t^T Z_s dB_s, & 0 \leq t \leq T, \\ Y_t^{(n)} = \xi_t^{(2)}, & T \leq t \leq T + D. \end{cases} \quad (28)$$

By Theorem 1, $Y = Y^{(2)}$.

Since $Y_t^{(1)} \geq Y_t^{(3)} \geq \dots \geq Y_t^{(n)} \geq \dots$, it holds immediately $Y_t^{(1)} \geq Y_t^{(2)}$.

Similar to the deducing technique of Peng and Yang [15], next we prove the strict comparison theorem.

If $Y_0^{(1)} = Y_0^{(2)}$, then we have

$$\begin{aligned} g_1(u, Y_{u-d_1(u)}^{(1)}, Y_u^{(1)}, Z_u^{(1)}, Y_{u+d_3(u)}^{(1)}) \\ = g_2(u, Y_{u-d_1(u)}^{(2)}, Y_u^{(1)}, Z_u^{(1)}, Y_{u+d_3(u)}^{(2)}), \end{aligned} \quad (29)$$

Since $Y_0^{(1)} \geq Y_0^{(3)} \geq Y_0^{(2)}$, then $Y_0^{(1)} = Y_0^{(3)}$ and

$$\begin{aligned} g_1(t, Y_{t-d_1(t)}^{(1)}, Y_t^{(1)}, Z_t^{(1)}, Y_{t+d_3(t)}^{(1)}) \\ = g_2(t, Y_{t-d_1(t)}^{(1)}, Y_t^{(1)}, Z_t^{(1)}, Y_{t+d_3(t)}^{(1)}), \end{aligned} \quad (30)$$

as well as $g_2(\cdot)$ is strictly increasing, and we can easily get satisfying results.

If $g_1(u, Y_{u-d_1(u)}^{(1)}, Y_u^{(1)}, Z_u^{(1)}, Y_{u+d_3(u)}^{(1)}) = g_2(u, Y_{u-d_1(u)}^{(1)}, Y_u^{(1)}, Z_u^{(1)}, Y_{u+d_3(u)}^{(1)})$, $0 \leq u \leq T$, and $\xi_u^{(1)} = \xi_u^{(2)}$, $T \leq u \leq T + D$, then

$$\hat{y}_t = Y_t^{(1)} - Y_t^{(3)} = E \left[\hat{\xi}_T q_T + \int_t^T \hat{g}_s q_s ds \mid \mathcal{F}_t \right] \equiv 0. \quad (31)$$

Therefore, from Theorem 1, $Y^{(1)} = Y^{(2)}$; in particular, $Y_0^{(1)} = Y_0^{(2)}$, a.e., a.s. $\square$

The following result is to show the important relation between the delayed time and the solution of delay and anticipated BSDEs.

Theorem 3. Suppose $(Y^{(l)}, Z^{(l)})$, $l = 1, 2$, are solutions of two delay and anticipated BSDEs:

$$\begin{cases} -dY_s^{(l)} = g(s, Y_{s-d_1^{(l)}(s)}^{(l)}, Y_s^{(l)}, Z_s^{(l)}, Y_{s+d_3^{(l)}(s)}^{(l)}) ds - Z_s^{(l)} dB_s, & s \in [0, T], \\ Y_s^{(l)} = \xi_s^{(l)}, & s \in [T, T + D]. \end{cases} \quad (32)$$

Let $\xi \in S^2(T, D + T)$, $d_1^{(l)}$ and $d_3^{(l)}$ satisfy (D1) and (D2), and the function $g(\cdot)$ satisfies (A1) and (A2). If $d_1^{(1)}(v) \geq d_1^{(2)}(v)$, $d_3^{(2)}(v) \geq d_3^{(1)}(v)$, $0 \leq v \leq T$, then $\exists \lambda: = \lambda(C, L, T) > 0$ s.t.

$$\begin{aligned} |Y_t^{(1)} - Y_t^{(2)}|^2 \leq \lambda \int_t^T (d_1^{(1)}(v) - d_1^{(2)}(v) + d_3^{(2)}(v) - d_3^{(1)}(v)) dv \\ \times E \left[\int_T^{T+D} |\xi_r|^2 dr + |\xi_T|^2 + \int_t^T |g(r, 0, 0, 0, 0)|^2 dr \mid \mathcal{F}_t \right]. \end{aligned} \quad (33)$$

Proof. Denote $y = Y^{(1)} - Y^{(2)}$, $z = Z^{(1)} - Z^{(2)}$, and then apply Itô's lemma for $|y_s|^2 e^{\beta s}$:

$$\begin{aligned} |y_u|^2 + E \left[\int_u^T (\beta |y_v|^2 + |z_v|^2) e^{\beta(v-u)} ds \mid \mathcal{F}_u \right] \\ = E \left[\int_u^T 2y_s \left| g(v, Y_{v-d_1^{(1)}(v)}^{(1)}, Y_v^{(1)}, Z_v^{(1)}, Y_{v+d_3^{(1)}(v)}^{(1)}) \right. \right. \\ \left. \left. - g(v, Y_{v-d_1^{(2)}(v)}^{(2)}, Y_v^{(2)}, Z_v^{(2)}, Y_{v+d_3^{(2)}(v)}^{(2)}) \right| e^{\beta(v-u)} dv \mid \mathcal{F}_u \right], \end{aligned} \quad (34)$$

namely,

$$\begin{aligned}
& |y_t|^2 + E \left[\int_t^T \exp(\beta(s-t)) (|z_s|^2 + 0.5\beta|y_s|^2) ds \mid \mathcal{F}_t \right] \\
& \leq 2\beta^{-1} E \left[\int_t^T \left| g\left(s, Y_{s-d_1^{(1)}(s)}^{(1)}, Y_s^{(1)}, Z_s^{(1)}, Y_{s+d_3^{(1)}(s)}^{(1)}\right) \right. \right. \\
& \quad \left. \left. - g\left(s, Y_{s-d_1^{(2)}(s)}^{(2)}, Y_s^{(2)}, Z_s^{(2)}, Y_{s+d_3^{(2)}(s)}^{(2)}\right) \right|^2 e^{\beta(s-t)} ds \mid \mathcal{F}_t \right] \\
& \leq \frac{8C^2}{\beta} E \left[\int_t^T \left(\left| Y_{s-d_1^{(1)}(s)}^{(1)} - Y_{s-d_1^{(2)}(s)}^{(2)} \right|^2 + |y_s|^2 + |z_s|^2 \right. \right. \\
& \quad \left. \left. + E \left[\left| Y_{s+d_3^{(1)}(s)}^{(1)} - Y_{s+d_3^{(2)}(s)}^{(2)} \right|^2 \mid \mathcal{F}_s \right] \right) e^{\beta(s-t)} ds \mid \mathcal{F}_t \right] \\
& \leq \frac{8C^2}{\beta} E \left[\int_t^T \left(2 \left| y_{s-d_1^{(1)}(s)} \right|^2 + 2 \left| Y_{s-d_1^{(1)}(s)}^{(2)} - Y_{s-d_1^{(2)}(s)}^{(2)} \right|^2 + |y_s|^2 + |z_s|^2 \right. \right. \\
& \quad \left. \left. + 2E \left[\left| y_{s+d_3^{(1)}(s)} \right|^2 \mid \mathcal{F}_s \right] + 2\hat{E} \left[\left| Y_{s+d_3^{(1)}(s)}^{(2)} - Y_{s+d_3^{(2)}(s)}^{(2)} \right|^2 \mid \mathcal{F}_s \right] \right) e^{\beta(s-t)} ds \mid \mathcal{F}_t \right] \\
& \leq \frac{8C^2 + 24C^2L}{\beta} E \left[\int_t^T \exp(\beta(s-t)) |y_s|^2 ds \mid \mathcal{F}_t \right] + \frac{8C^2}{\beta} E \left[\int_t^T \exp(\beta(s-t)) |z_s|^2 ds \mid \mathcal{F}_t \right] \\
& \quad + \frac{16C^2}{\beta} E \left[\int_t^T E \left[\left| \int_{s+d_3^{(1)}(s)}^{s+d_3^{(2)}(s)} g(r, Y_{r-d_1(r)}^{(2)}, Y_r^{(2)}, Z_r^{(2)}, Y_{r+d_3(r)}^{(2)}) dr \right|^2 \mid \mathcal{F}_s \right] e^{\beta(s-t)} ds \mid \mathcal{F}_t \right] \\
& \quad + \frac{16C^2}{\beta} E \left[\int_t^T \left| \int_{s-d_1^{(1)}(s)}^{s-d_1^{(2)}(s)} g(r, Y_{r-d_1(r)}^{(2)}, Y_r^{(2)}, Z_r^{(2)}, Y_{r+d_3(r)}^{(2)}) dr \right|^2 e^{\beta(s-t)} ds \mid \mathcal{F}_t \right].
\end{aligned} \tag{35}$$

We set $\beta = 8C^2$; then,

$$\begin{aligned}
& |y_t|^2 \leq (1 + 3L)e^{\beta(T-t)} E \left[\int_t^T |y_s|^2 ds \mid \mathcal{F}_t \right] \\
& \quad + 2E \left[\int_t^T E \left[\left| \int_{s+d_3^{(1)}(s)}^{s+d_3^{(2)}(s)} g(r, Y_{r-d_1(r)}^{(2)}, Y_r^{(2)}, Z_r^{(2)}, Y_{r+d_3(r)}^{(2)}) dr \right|^2 \mid \mathcal{F}_s \right] e^{\beta(s-t)} ds \mid \mathcal{F}_t \right] \\
& \quad + 2E \left[\int_t^T \left| \int_{s-d_1^{(1)}(s)}^{s-d_1^{(2)}(s)} g(r, Y_{r-d_1(r)}^{(2)}, Y_r^{(2)}, Z_r^{(2)}, Y_{r+d_3(r)}^{(2)}) dr \right|^2 e^{\beta(s-t)} ds \mid \mathcal{F}_t \right] \\
& \leq (1 + 3L)e^{\beta(T-t)} E \left[\int_t^T |y_s|^2 ds \mid \mathcal{F}_t \right] \\
& \quad + 2E \left[\int_t^T (d_3^{(2)}(s) - d_3^{(1)}(s)) E \left[\left| \int_{s+d_3^{(1)}(s)}^{s+d_3^{(2)}(s)} g(r, Y_{r-d_1(r)}^{(2)}, Y_r^{(2)}, Z_r^{(2)}, Y_{r+d_3(r)}^{(2)}) dr \right|^2 \mid \mathcal{F}_s \right] e^{\beta(s-t)} ds \mid \mathcal{F}_t \right] \\
& \quad + 2E \left[\int_t^T (d_1^{(1)}(s) - d_1^{(2)}(s)) \int_{s-d_1^{(1)}(s)}^{s-d_1^{(2)}(s)} \left| g(r, Y_{r-d_1(r)}^{(2)}, Y_r^{(2)}, Z_r^{(2)}, Y_{r+d_3(r)}^{(2)}) \right|^2 dr e^{\beta(s-t)} ds \mid \mathcal{F}_t \right] \\
& \leq e^{\beta(T-t)} (1 + 3L) E \left[\int_t^T |y_s|^2 ds \mid \mathcal{F}_t \right] + 8 \int_t^T ((d_1^{(1)}(s) - d_1^{(2)}(s)) + (d_3^{(2)}(s) - d_3^{(1)}(s))) e^{\beta(s-t)} ds \\
& \quad \cdot E \left[\int_t^T (C^2 |Y_{r-d_1(r)}^{(2)}|^2 + C^2 |Y_r^{(2)}|^2 + C^2 |Z_r^{(2)}|^2 + C^2 |Y_{r+d_3(r)}^{(2)}|^2 + |g(r, 0, 0, 0, 0)|^2) dr \mid \mathcal{F}_t \right] \\
& \leq (1 + 3L)e^{\beta(T-t)} E \left[\int_t^T |y_s|^2 ds \mid \mathcal{F}_t \right] + 8 \int_t^T e^{\beta(s-t)} ((d_1^{(1)}(s) - d_1^{(2)}(s)) + (d_3^{(2)}(s) - d_3^{(1)}(s))) ds \\
& \quad \cdot E \left[\int_t^T (C^2 (2L + 1) |Y_r^{(2)}|^2 + C^2 |Z_r^{(2)}|^2 + |g(r, 0, 0, 0, 0)|^2) dr + \int_T^{T+D} |\xi_r|^2 dr \mid \mathcal{F}_t \right].
\end{aligned} \tag{36}$$

Therefore, there exists $\lambda': = \lambda'(C, L, K) > 0$ s.t.

$$\begin{aligned} |y_t|^2 \leq & \lambda' E \left[\int_t^T |y_s|^2 ds \mid \mathcal{F}_t \right] + \lambda' \int_t^T ((d_1^{(1)}(s) - d_1^{(2)}(s)) \\ & + (d_3^{(2)}(s) - d_3^{(1)}(s))) ds \\ & \times E \left[|\xi_T|^2 + \int_T^{D+T} |\xi_r|^2 dr + \int_t^T |g(r, \mathbf{0})|^2 dr \mid \mathcal{F}_t \right]. \end{aligned} \quad (37)$$

Thus, from Gronwall inequality,

$$\begin{aligned} |y_t|^2 \leq & \lambda \int_t^T (d_1^{(1)}(s) - d_1^{(2)}(s) + d_3^{(2)}(s) - d_3^{(1)}(s)) ds \\ & \times E \left[|\xi_T|^2 + \int_t^T |g(r, 0, 0, 0, 0)|^2 dr + \int_T^{T+D} |\xi_r|^2 dr \mid \mathcal{F}_t \right]. \end{aligned} \quad (38)$$

□

$$\begin{cases} Y_v = \theta'_v, & v \in [-\tau, 0], \\ Z_v = \chi'_v, & v \in [-\tau, 0], \\ -dY_v = g(v, Y_{v-d_1(v)}, Z_{v-d_2(v)}, Y_v, Z_v, Y_{v+d_3(v)}, Z_{v+d_4(v)}) dv - Z_v dB_v, & v \in [0, T], \\ Y_v = \theta_v, & v \in [T, D+T], \\ Z_v = \chi_v, & v \in [T, D+T], \end{cases} \quad (39)$$

where $\tau > 0, D > 0$, and $d_i(\cdot), i = 1, \dots, 4$, are four positive continuous functions s.t.

$$(D3) \exists D \geq 0 \text{ s.t.}$$

$$\begin{aligned} -\tau \leq v - d_1(v) \leq v; \quad -\tau \leq v - d_2(v) \leq v; \quad v + d_3(v) \\ \leq T + D; \quad v + d_4(v) \leq T + D, \quad v \in [0, T]. \end{aligned} \quad (40)$$

$$(D4) \exists K \geq 0 \text{ s.t., for all nonnegative and integrable functions, } f(\cdot) \text{ satisfy}$$

$$\begin{aligned} \int_t^T f(u - d_1(u)) du & \leq K \int_{-\tau}^{D+T} f(u) du; \quad \int_t^T f(u - d_2(u)) du \\ & \leq K \int_{-\tau}^{D+T} f(u) du; \\ \int_t^T f(u + d_3(u)) du & \leq K \int_t^{D+T} f(u) du; \\ \int_t^T f(u + d_4(u)) du & \leq K \int_t^{D+T} f(u) du, \end{aligned} \quad (41)$$

The existence and uniqueness of equation (39) can be obtained by the similar approach of proving Theorem 1.

Data Availability

No data were used to support this study.

Remark 1. If $g(v, Y_{v-d_1(v)}, Z_{v-d_2(v)}, Y_v, Z_v, Y_{v+d_3(v)}, Z_{v+d_4(v)}) = f(v, Y_v, Z_v, Y_{v+d_3(v)}, Z_{v+d_4(v)})$, then the delay and anticipated BSDE (1) is anticipated BSDEs in [15], and it is the result of Peng and Yang in [15].

Remark 2. In fact, the delay in (1) can go below $t = 0$, namely, the DABSDE could have the following form:

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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