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Research Article

On Zero Left Prime Factorizations for Matrices over Unique Factorization Domains

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In this paper, zero prime factorizations for matrices over a unique factorization domain are studied. We prove that zero prime factorizations for a class of matrices exist. Also, we give an algorithm to directly compute zero left prime factorizations for this class of matrices.

1. Introduction

Multidimensional linear systems theory has a wide range of applications in circuits, systems, control of networked systems, signal processing, and other areas (see, e.g., [1, 2]). Multivariate polynomial matrix theory is a well-established tool for these systems, since many problems in the analysis and synthesis of control systems can be well solved using multivariate polynomial matrix techniques [1–3].

In recent years, n-D polynomial matrix factorizations have been widely studied [4–10]. In [11, 12], the zero left prime factorization problem was raised. This problem has been solved in [4–6]. The minor left prime factorization problem has been solved in [7, 10]. In the algorithms given in [7, 10], a fitting ideal of some module over the multivariate (n-D) polynomial ring needs to be computed. It is a little complicated.

It is well known that a multivariate polynomial ring over a field is a unique factorization domain. Then, the following problem is interesting.

Problem 1. How to decide if a matrix with full row rank over a unique factorization domain has a zero left prime factorization?

In this paper, we will give a partial solution to this problem.

2. Preliminaries

Let R be a unique factorization domain. The set of all $l \times m$ matrices with entries from R is denoted by $R^{l\times m}$. Let $F \in R^{l\times m}\,(l < m)$. We denote the greatest common divisor of all $l \times l$ minors of F by $d(F)$. Let $C \in R^{l\times l}$ be a submatrix of F. By deleting C from F, we get a submatrix of F. This submatrix is denoted by $F\backslash C$.

Let $C \in R^{m\times m}$. $\mathrm{adj}(C)$ denotes the adjoint matrix of C. $\mathrm{acof}_{ij}(C)$ denotes the i, jth algebraic cofactor of C.

Definition 1. Let $F \in R^{l\times m}\,(l < m)$, and let $C \in R^{l\times l}$ be a submatrix of F. A minor of F consisting of $l-1$ columns from C and one column from $F\backslash C$ is said to be a related minor of C.

The following definition is from the multidimensional systems theory [13].

Definition 2. Let $F \in R^{l\times m}$ be of full row rank. Then, F is said to be zero left prime (ZLP) if the $l \times l$ minors of F generate the unit ideal R. Suppose F has a factorization $F = CF_1$, where $C \in R^{l\times l}$ and $F_1 \in R^{l\times m}$. If F_1 is ZLP, then this factorization is said to be a zero left prime factorization.

3. Main Results

First, we need a lemma.

Lemma 1. *Let $F = (C, \overline{C}) \in R^{l\times m}\,(l < m)$, where $C \in R^{l\times l}$ and $\overline{C} \in R^{l\times (m-l)}$. Then, the elements of $\mathrm{adj}\,C \cdot \overline{C}$ are just all related minors of C (up to a sign).*

Proof. Let $C = (c_{ij})_{l\times l}$ and $\overline{C} = (\overline{c}_{ij})_{l\times(m-l)}$. Let $\operatorname{adj} C \cdot \overline{C} = (b_{ij})_{l\times(m-l)}$. Then,

$$b_{ij} = \operatorname{acof}_{1i}(C)\overline{c}_{1j} + \cdots + \operatorname{acof}_{li}(C)\overline{c}_{lj} = \det\begin{pmatrix} c_{11} & \cdots & c_{1i-1} & \overline{c}_{1j} & c_{1i+1} & \cdots & c_{1l} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{l1} & \cdots & c_{li-1} & \overline{c}_{lj} & c_{li+1} & \cdots & c_{ll} \end{pmatrix}, \quad (1)$$

by Laplace Theorem. Thus, b_{ij} is a related minor of C (up to a sign). It is clear that they are just all related minors of C (up to a sign).

Now, we prove the main theorem of this paper. □

Theorem 1. *Let $F \in R^{l\times m}\,(l<m)$. If there exists an $l\times l$ submatrix C of F such that* $\det C$ *is a common factor of all related minors of C, then there exists $F_1 \in R^{l\times m}$ such that $F = CF_1$ and F_1 is ZLP; i.e., F has a ZLP factorization.*

Proof. We can change the order of the columns of F such that the submatrix C consists of the left l columns of F. Thus, there exists an invertible matrix $Q \in R^{m\times m}$ such that $FQ = (C, \overline{C})$, where $C \in R^{l\times l}$ and $\overline{C} \in R^{l\times(m-l)}$. Since $\det C$ is a common factor of all related minors of C, by Lemma 1, we have $C^{-1}\overline{C} = \operatorname{adj} C \cdot \overline{C}/\det C \in R^{l\times(m-l)}$. Let

$$Q_1 = \begin{pmatrix} I_l & -C^{-1}\overline{C} \\ 0 & I_{m-l} \end{pmatrix}. \quad (2)$$

Then, $Q_1 \in R^{m\times m}$. We have

$$FQQ_1 = (C, \overline{C})Q_1 = (C, \overline{C})\begin{pmatrix} I_l & -C^{-1}\overline{C} \\ 0 & I_{m-l} \end{pmatrix} = (C, O). \quad (3)$$

Then,

$$F = (C, O)Q_1^{-1}Q^{-1} \ (\text{by } (3)) = C(I_l, O)Q_1^{-1}Q^{-1}. \quad (4)$$

Let $F_1 = (I_l, O)Q_1^{-1}Q^{-1} \in R^{l\times m}$. Then, $F = CF_1$. Since F_1 consists of the upper l rows of invertible matrix $Q_1^{-1}Q^{-1}$, we have F_1 is ZLP. □

Corollary 1. *Let $F \in R^{l\times m}\,(l<m)$. If there exists an $l\times l$ submatrix C of F such that* $\det C$ *is a common factor of all related minors of C, then* $\det C = d(F)$.

Proof. Clearly, $d(F) \mid \det C$. By Theorem 1, there exists $F_1 \in R^{l\times m}$ such that $F = CF_1$. By Cauchy–Binet formula, we have $\det C \mid d(F)$. Therefore, $\det C = d(F)$. □

Corollary 2. *Let $F \in R^{l\times m}\,(l<m)$. If there exists an $l\times l$ submatrix C of F such that* $\det C$ *is a common factor of all related minors of C, then F is equivalent to (C, O).*

Proof. By Theorem 1, there exists $F_1 \in R^{l\times m}$ such that $F = CF_1$ and F_1 is ZLP. By Quillen–Suslin theorem, there exists $F_2 \in R^{(m-l)\times m}$ such that $(F_1^T, F_2^T)^T$ is an invertible matrix. Since $F = CF_1 = (C, O)(F_1^T, F_2^T)^T$, we have F being equivalent to (C, O).

Now, let $F \in R^{l\times m}\,(l<m)$. Suppose there exists an $l\times l$ submatrix C of F such that $\det C = d(F)$. We can give an algorithm to directly compute the ZLP factorization of F. □

Algorithm 1

(i) Compute all $l\times l$ minors of F and $d(F)$.

(ii) Find an $l\times l$ submatrix C of F such that $\det C = d(F)$.

(iii) Compute invertible matrix Q such that $FQ = (C, \overline{C})$.

(iv) Let $Q_1 = \begin{pmatrix} I_l & -C^{-1}\overline{C} \\ 0 & I_{m-l} \end{pmatrix}$ and $F_1 = (I_l, O)Q_1^{-1}Q^{-1}$. Then, $F = CF_1$.

Now, we give an example to illustrate this algorithm.

Example 1. Let $R = \mathbb{Z}[x, y]$, and let

$$F = \begin{pmatrix} 6x^2y + 2xy & 2x & 2xy \\ 6x^2y^2 + 6x^2y + 2xy^2 + 5xy & 2xy + 2x & 2xy^2 + 2xy + y \end{pmatrix}. \quad (5)$$

Then, $d(F) = 2xy$. Let

$$C = \begin{pmatrix} 2x & 2xy \\ 2xy + 2x & 2xy^2 + 2xy + y \end{pmatrix}. \quad (6)$$

Then, C is a 2×2 submatrix of F and $\det C = d(F)$. Let

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (7)$$

Then, $FQ = (C, \overline{C})$, where

$$\overline{C} = \begin{pmatrix} 6x^2y + 2xy \\ 6x^2y^2 + 6x^2y + 2xy^2 + 5xy \end{pmatrix}. \quad (8)$$

Thus, $-C^{-1}\overline{C} = \begin{pmatrix} -y \\ -3x \end{pmatrix}$. Let

$$Q_1 = \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & -3x \\ 0 & 0 & 1 \end{pmatrix}. \quad (9)$$

Then,

$$Q_1^{-1}Q^{-1} = \begin{pmatrix} y & 1 & 0 \\ 3x & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (10)$$

Let

$$F_1 = \begin{pmatrix} y & 1 & 0 \\ 3x & 0 & 1 \end{pmatrix}. \quad (11)$$

Then, $F = CF_1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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