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Research Article

Characterization of Nonstationary Phase Noise Using the Wigner–Ville Distribution

Chagai Levy, Monika Pinchas, and Yosef Pinhasi

Department of Electrical and Electronic Engineering, Ariel University, Ariel 40700, Israel

Correspondence should be addressed to Chagai Levy; chagailevy@gmail.com

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Oscillators and atomic clocks, as well as lasers and masers, are affected by physical mechanisms causing amplitude fluctuations, phase noise, and frequency instabilities. The physical properties of the elements composing the oscillator as well as external environmental conditions play a role in the coherence of the oscillatory signal produced by the device. Such instabilities demonstrate frequency drifts, modulation, and spectrum broadening and are observed to be nonstationary processes in nature. Most of the tools which are being used to measure and characterize oscillator stability are based on signal processing techniques, assuming time invariance within a temporal window, during which the signal is assumed to be stationary. This letter proposes a new time-frequency approach for the characterization of frequency sources. Our technique is based on the Wigner–Ville time-frequency distribution, which extends the spectral measures to include the temporal nonstationary behavior of the processes affecting the coherence of the oscillator and the accuracy of the clock. We demonstrate the use of the technique in the characterization of nonstationary phase noise in oscillators.

1. Introduction

In recent years, due to the high data rates required to be transferred in communication networks, the demand for high-accuracy satellite navigation systems and the development of high-resolution radars, oscillators, and accurate frequency clocks with very high precision are required. Stable oscillators, including atomic ones such as rubidium and even cesium are often used in communication and navigation facilities [1, 2]. Apart from the above applications, many other electronic systems require frequency generators. In such systems, time synchronization and an accurate frequency are required [3, 4].

Oscillators and clocks, as well as lasers and masers, are subjected to environmental conditions, internal malfunctions, and inherent physical phenomena causing instabilities in the oscillatory signal being produced [5]. These instabilities are expressed by amplitude and phase fluctuations causing frequency deviations, drift, and spectrum broadening. Instead of being stationary coherent, the generated signal is observed to be a stochastic process characterized by statistical and spectral features, which are nonstationary time-varying [6, 7]. Therefore, their coherent features in the time and frequency domain should be investigated.

Phase noise is the frequency domain representation of rapid, short-term, and random fluctuations in the phase of a waveform. Historically, there have been two conflicting yet widely used definitions for phase noise. Some authors define phase noise to be the spectral density of a signal's phase only [8], while the other definition refers to the phase spectrum (which pairs up with the amplitude spectrum) resulting from the spectral estimation of the signal itself. The IEEE defines phase noise as $L(f) = S_{\varphi}(f)/2$, where the "phase instability" is given in terms of the power spectral density $S_{\varphi}(f)$ of a signal's phase deviation $\varphi(t)$ defined for $f > 0$ [9]. However, the standard phase noise [9] does not give the possibility of identifying and interpreting nonstationary effects.

In lasers and masers, the radiation line-width limit is calculated via the Schawlow–Townes expression [10–12], which assumes a stationary process [13–17] although it is not stationary in general.

In this letter, we propose a time-frequency metric for the characterization of oscillators and frequency generators, also

considering nonstationary variations. It is based on the Wigner–Ville distribution [18, 19], proposed in 1932 for the characterization of quantum fluctuations. This time-frequency distribution is applicable for the analysis of nonstationary signals [20–23], radar signals [24, 25], biomedical signals [26–28], analysis of time-varying filters [29, 30], and image processing [31–33]. Up to now, the Wigner–Ville distribution was not yet used in the characterization of nonstationary oscillators such as evaluation of the phase noise. In this letter, we introduce an analytic expression of the Wigner–Ville time-frequency distribution for the characterization of the temporal coherency of nonstationary signals produced by classical and quantum oscillators. Our proposed expression can reveal the time-varying frequencies generally seen in the clock error noise under nonstationary conditions and in other platforms such as oscillator under vibrations or shocks.

The analysis is in general considering noise generated by different mechanisms. It is valid also for multinoise sources applied simultaneously to multinodes of the oscillator system. The multinoise sources are combination of two basic classes of noise, additive and parametric, in which their behavior and the underlying physical mechanisms are strikingly different. Each noise caused by an internal behavior of an oscillator or by a near-dc process modulates the carrier in phase.

2. The Oscillator Model

The model for the oscillator includes variation in the phase and given by [5, 34]

$$r(t) \simeq A\cos\left[2\pi\nu_0 t + \varphi(t)\right], \tag{1}$$

where A is the amplitude, ν_0 is the central frequency of the oscillation, and $\varphi(t)$ represents the random fluctuation in phase. The phase variations $\varphi(t)$ is given as a summation of seven terms [35, 36]:

$$\begin{aligned}\varphi(t) = 2\pi\nu_0 \Bigg[& x_0 + y_0 t + a\frac{t^2}{2} + \mu_3\frac{t^3}{6} + \sigma_1 B_{H_1}(t) \\ & + \sigma_2 \int_0^t B_{H_2}(s)ds + \sigma_3 \int_0^t \frac{(t-s)^2}{2} dB_{H_3}(s)\Bigg],\end{aligned} \tag{2}$$

where

(1) x_0 and y_0 are the initial phase and frequency offsets, respectively.

(2) a is defined as the frequency drift coefficient.

(3) μ_3 is a linear variation coefficient added to the frequency drift.

(4) $\left\{B_{H_i}(t), t \geq 0\right\}$, $i = 1, 2, 3$ are three independent, one-dimensional standard Wiener processes (Brownian motion) [35]. Following [37], the flicker noise is usually omitted from the mathematical model since it is not a rational process. The Wiener process is often referred to as an integral of white noise and is a nonstationary process.

(5) The constants σ_1, σ_2, and σ_3 represent the magnitude of the white frequency noise, the random walk frequency noise, and the frequency drift, respectively, which are three contributors to the noise in a typical oscillator. The constants σ_1, σ_2, and σ_3 are determined by the intensity of each of these respective noise components, resulting from internal electronic noise mechanisms in an oscillator, such as thermal (Johnson) noise, shot noise, and parametric noise, in which a near-dc process modulates the phase of the carrier [38].

(6) The frequency drift is a constant $\sigma_3 = 0$ [35].

The stochastic term of the phase variations $\varphi(t)$ is given by the term

$$\Delta\varphi(t) \approx 2\pi\nu_0\left[\sigma_1 B_{H_1}(t) + \sigma_2\int_0^t B_{H_2}(s)ds\right], \tag{3}$$

while the deterministic term of the phase is expressed as shift and drift in the oscillator frequency. Hereafter, we concentrate in the stochastic term (3) only and analyse its time-frequency characteristics using the Wigner–Ville distribution.

The stochastic term (3) is the phase noise causing broadening in the line width.

3. The Wigner–Ville Distribution

In this section, we will present briefly the Wigner–Ville distribution and its spectrum.

One of the most representative joint representations of time-frequency analysis is the Wigner–Ville distribution (WD), which is a quadratic signal representation introduced by Wigner in 1932 [19] and later applied by Ville in 1948 [39] for signal time-frequency analysis.

In the following, we will denote the expectation value of x as $E\{x\}$.

The statistical autocorrelation of $\Delta\varphi(t)$ is denoted by $\overline{R}_{\Delta\varphi}(t,\tau)$ and defined as follows:

$$\overline{R}_{\Delta\varphi}(t,\tau) = E\{\Delta\varphi(t+\tau/2)\Delta\varphi^*(t-\tau/2)\}, \tag{4}$$

where the shifting parameter is denoted by τ.

The Wigner spectrum is defined as the Fourier transform of the statistical autocorrelation $\overline{R}_{\Delta\varphi}(t,\tau)$ [40]:

$$\overline{W}_{\Delta\varphi}(t,f) = \int_{-\infty}^{\infty} \overline{R}_{\Delta\varphi}(t,\tau)e^{-i2\pi f\tau}d\tau, \tag{5}$$

where f denotes the frequency variable.

4. Time-Frequency Analysis of Phase Noise Using Wigner Spectrum

This section proposes a new time-frequency closed form expression for the characterization of nonstationary phase noise in oscillators. Our technique is based on the Wigner spectrum, which extends the spectral measures to consist of the temporal nonstationary behavior of the phase process affecting the accuracy of the clock.

Theorem 1. *We derive the Wigner spectrum (5) of the phase variation given in (3) and find that the closed form expression for a nonstationary phase noise is*

$$\overline{W}_{\Delta\varphi}(t,f) = (2\pi\nu_0)^2 \left\{ \frac{\sigma_1^2}{4(\pi f)^2}[1-\cos(4\pi t f)] + \frac{\sigma_2^2}{(2\pi f)^4} - \frac{\sigma_2^2}{(2\pi)^4 f^3}\left[\frac{3}{f}\cos(4\pi t f) + 8\pi t \sin(4\pi t f)\right] - \frac{\sigma_2^2}{2^4 3 f^2}\left[\cos(4\pi t f)\left(\frac{3}{4f^2\pi^4} - \frac{8t^2}{\pi^2}\right) + \sin(4\pi t f)\frac{4t}{f\pi^3}\right]\right\}. \quad (6)$$

Proof 1. By substituting $\Delta\varphi(t)$ (3) into $\overline{R}_{\Delta\varphi}(t,\tau)$ (4), we obtain

$$\overline{R}_{\Delta\varphi}(t,\tau) = (2\pi\nu_0)^2 E\left\{\sigma_1 B_{H_1}(t+\tau/2) + \sigma_2\int_0^{t+\tau/2} B_{H_2}(s)ds\right] \cdot \left[\sigma_1 B^*_{H_1}(t-\tau/2) + \sigma_2\int_0^{t-\tau/2} B^*_{H_2}(s)ds\right\}. \quad (7)$$

Because $\left\{B_{H_i}(t), t\geq 0\right\}$, $i=1,2$ are two independent, one-dimensional standard Wiener processes, we can write (7) as follows:

$$\overline{R}_{\Delta\varphi}(t,\tau) = (2\pi\nu_0)^2 E\Big\{\sigma_1 B_{H_1}(t+\tau/2)\sigma_1 B^*_{H_1}(t-\tau/2) + \sigma_2\int_0^{t+\tau/2} B_{H_2}(s)ds\sigma_2\int_0^{t-\tau/2} B^*_{H_2}(s)ds\Big\}. \quad (8)$$

$\varphi(t)$ is a real signal; therefore, $B^*_{H_1}(t) = B_{H_1}(t)$and $B^*_{H_2}(t) = B_{H_2}(t)$, and thus we can write (8) as follows:

$$\overline{R}_{\Delta\varphi}(t,\tau) = (2\pi\nu_0)^2 E\Big\{\sigma_1 B_{H_1}(t+\tau/2)\sigma_1 B_{H_1}(t-\tau/2) + \sigma_2\int_0^{t+\tau/2} B_{H_2}(s)ds\sigma_2\int_0^{t-\tau/2} B_{H_2}(s)ds\Big\}. \quad (9)$$

The standard Wiener process $\{B_H(t), t\geq 0\}$ in its general form is a continuous-time Gaussian process on $[0,T]$, which starts at zero, has expectation zero for $t\in[0,T]$, and has the following covariance function [41, 42]:

$$E\{B_H(t)B_H(s)\} = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \quad (10)$$

In our case, the process $\{B_H(t), t\geq 0\}$ is the standard Brownian motion, i.e., $H=0.5$. By substituting (10) into (9), we obtain

$$\overline{R}_{\Delta\varphi}(t,\tau) = (2\pi\nu_0)^2 \left\{\frac{\sigma_1^2}{2}(|t+\tau/2| + |t-\tau/2| - |\tau|) + E\left\{\sigma_2\int_0^{t+\tau/2} B_{H_2}(s)ds\sigma_2\int_0^{t-\tau/2} B_{H_2}(s)ds\right\}\right\}. \quad (11)$$

Now, we will insert the expectation value into the joint integral argument on the second term of (11):

$$\overline{R}_{\Delta\varphi}(t,\tau) = (2\pi\nu_0)^2 \left\{\frac{\sigma_1^2}{2}(|t+\tau/2| + |t-\tau/2| - |\tau|) + \sigma_2^2\int_0^{t+\tau/2}\int_0^{t-\tau/2} E\left\{B_{H_2}(s)B_{H_2}(s')\right)\right\}dsds'\right\}. \quad (12)$$

Next, we will preform the expectation value of the second term of (12) by using (10):

$$\overline{R}_{\Delta\varphi}(t,\tau) = (2\pi\nu_0)^2 \left\{\frac{\sigma_1^2}{2}(|t+\tau/2| + |t-\tau/2| - |\tau|) + \sigma_2^2\int_0^{t+\tau/2}\int_0^{t-\tau/2}\frac{1}{2}\left(|s| + |s'| - |s-s'|\right)dsds'\right\}. \quad (13)$$

Finally, we may write (13) in its final form:

$$\overline{R}_{\Delta\varphi}(t,\tau) = (2\pi\nu_0)^2 \left\{\frac{\sigma_1^2}{2}(|t+\tau/2| + |t-\tau/2| - |\tau|) + \frac{\sigma_2^2}{4}|t+\tau/2|\left(t^2-\tau^2/4\right) + \frac{\sigma_2^2}{4}|t-\tau/2|\left(t^2-\tau^2/4\right) + \frac{\sigma_2^2}{12}\left[|\tau|\tau^2 - |t+\tau/2|(t+\tau/2)^2 - |t-\tau/2|(t-\tau/2)^2\right]\right\}. \quad (14)$$

Then, we will turn to preform the closed form expression of $\overline{W}_{\Delta\varphi}(t,f)$ by substituting (14) into (5):

$$\overline{R}_{\Delta\varphi}(t,\tau) = (2\pi\nu_0)^2[a(t,\tau) + b(t,\tau) + c(t,\tau) + d(t,\tau) + e(t,\tau) + f(t,\tau)], \quad (15)$$

where

$$\begin{aligned} a(t,\tau) &= \frac{\sigma_1^2}{2}(|t+\tau/2| + |t-\tau/2| - |\tau|),\\ b(t,\tau) &= \frac{\sigma_2^2}{4}|t+\tau/2|\left(t^2-\tau^2/4\right),\\ c(t,\tau) &= \frac{\sigma_2^2}{4}|t-\tau/2|\left(t^2-\tau^2/4\right),\\ d(t,\tau) &= \frac{\sigma_2^2}{12}|\tau|\tau^2,\\ e(t,\tau) &= -\frac{\sigma_2^2}{12}|t+\tau/2|(t+\tau/2)^2,\\ f(t,\tau) &= -\frac{\sigma_2^2}{12}|t-\tau/2|(t-\tau/2)^2. \end{aligned} \quad (16)$$

Next, we will present the closed form expression for $\overline{W}_{\Delta\varphi}(t,f)$ by solving the Fourier transform of (16):

$$\begin{aligned}\overline{W}_{\Delta\varphi}(t,f) &= (2\pi\nu_0)^2\int_{-\infty}^{\infty}[a(t,\tau)+b(t,\tau)+c(t,\tau)+d(t,\tau)+e(t,\tau)+f(t,\tau)]e^{-i2\pi f\tau}d\tau \\ &= (2\pi\nu_0)^2[A(t,f)+B(t,f)+C(t,f)+D(t,f)+E(t,f)+F(t,f)],\end{aligned} \tag{17}$$

where $[A,B,C,D,E,F]$ denotes the Fourier transform of $[a,b,c,d,e,f]$ given in (16), respectively.

Finally, by solving the Fourier transform of (16) and substituting its closed form expressions into (17), we get (6).

This completes our Proof 1.

It is important to note that expression (6) for the Wigner time-frequency spectrum describes the temporal evolution of the phase-noise spectrum instantaneously. The power spectrum $S_{\Delta\varphi}(f)$ of the phase noise is obtained by integration of the resulted Wigner spectrum (6) within a temporal time "window" T, an operation which is "smoothing" short-term nonstationary instabilities in the phase variations:

$$\begin{aligned}S_{\Delta\varphi}(f) &= \lim_{T\longrightarrow\infty}\frac{1}{T}\int_{-T/2}^{T/2}\overline{W}_{\Delta\varphi}(t,f)dt \\ &= (2\pi\nu_0)^2\left[\frac{\sigma_1^2}{4(\pi f)^2}+\frac{\sigma_2^2}{(2\pi f)^4}\right].\end{aligned} \tag{18}$$

A random process is said to be stationary (in a strict-sense) if all its statistical properties are invariant under an arbitrary time shift, that is, the autocorrelation function depends only on the time difference. Therefore, by using the Wiener–Khintchine theorem, the resulted power spectrum $S(f)$ of a stationary process is dependent only on the frequency f (without any temporal variations).

Please note that the integration of (6) in a finite observation time T produced a frequency dependent power spectrum as expected in stochastic processes, which are wide-sense stationary.

Inspection of equation (18) reveals two different stationary noise processes;

The first term describes the white frequency noise which declines by f^{-2} power law while the second term describes the random walk frequency noise which declines by f^{-4} power law [35, 38].

An estimation of the nonstationary characteristics of (6) can be made by fitting the two noise sources expressed in (18) with the power law function $S_{\varphi}(f)=\sum_i b_i f^i$ for the phase noise spectra [38]. Please note that preforming the estimation gives $b_{-2}=2\nu_0\sigma_1^2$ and $b_{-4}=(\nu_0^2\sigma_2^2/2\pi^2)$, which is exactly the results for b_{-2} and b_{-4} coefficients derived in [35].

5. Numerical Simulations

The Wigner–Ville distribution can be used to identify short-term, nonstationary instabilities in the oscillator phase and follow its spectral evolution in time. Differing from the power spectral density, which is a result of a time assimilation, assuming a stationary behavior within the temporal integration window, the Wigner–Ville distribution is a fundamental time-dependent measure, which can trend phase variations in ultrahigh resolution, depending on the sampling rate only. We employ the Wigner–Ville spectrum to analyse the nonstationary phase noise of a high accuracy Rubidium atomic clocks. Two clock models are used in the experiments: AR13300 [43] and NAC1 [44]. These atomic clocks are designed to generate a harmonic signal at frequency of $\nu_0 = 10$ MHz.

While the NAC1 Rubidium resonator is locked on local Temperature-Compensated Crystal Oscillator (TCXO) and the AR13300 Rubidium resonator is locked on a local Oven-Controlled Crystal Oscillator (OCXO), and thus expected to be more stable (characterized with a lower diffusion parameter σ_1). Table 1 summarizes the atomic clocks characteristics of both models.

The two noise types, white frequency noise and the random walk frequency noise, which are characterized by σ_1 and σ_2, respectively, are those commonly encountered in the measurement of oscillator phase noise. The manufacturer predefined the total power spectrum $S_{\varphi}(f)$ and then σ_1 and σ_2 magnitudes extracted [38].

In Figures 1 and 2, we present a three-dimensional plot of the calculated phase-noise Wigner spectrum of the Rubidium-based TCXO (model NAC1) and the Rubidium-based OCXO (model AR13300) atomic oscillators, respectively. The temporal spectral variations of the OCXO based are revealed to be smaller than that of the NAC1 model. Integration of the Wigner spectrum in time, results in the power spectral density $S_{\Delta\varphi}(f)$, as shown in Figure 3. A comparison is made between the NAC1 and the AR13300. The last is shown to be more stable as expected.

6. Summary and Conclusion

In this letter, we presented an effective technique for phase noise characterization of oscillators in time-frequency domain. While the common measure of phase noise power spectrum fails to follow nonstationary processes, the Wigner–Ville distribution is shown to demonstrate high-accuracy analysis of short time phase instabilities and a comprehensive description of their evolution in high levels of temporal resolution. It is shown that the Wigner–Ville distribution can be utilized for a nonstationary phase noise characterization that can reveal time-varying spectrum, generally seen in the oscillators and clocks, during fast variating conditions, including vibrations, shocks, and

Table 1: Summary table of the diffusion parameters vs. the oscillator type [43, 44] and its floor phase noise.

Oscillator	Model	σ_1	σ_2	Approximate phase noise in 10% frequency shift
Rubidium-based OCXO	AR13300	10^{-4}	10^{-6}	−160 dBc/Hz
Rubidium-based TCXO	NAC1	$2 \cdot 10^{-4}$	10^{-6}	−150 dBc/Hz

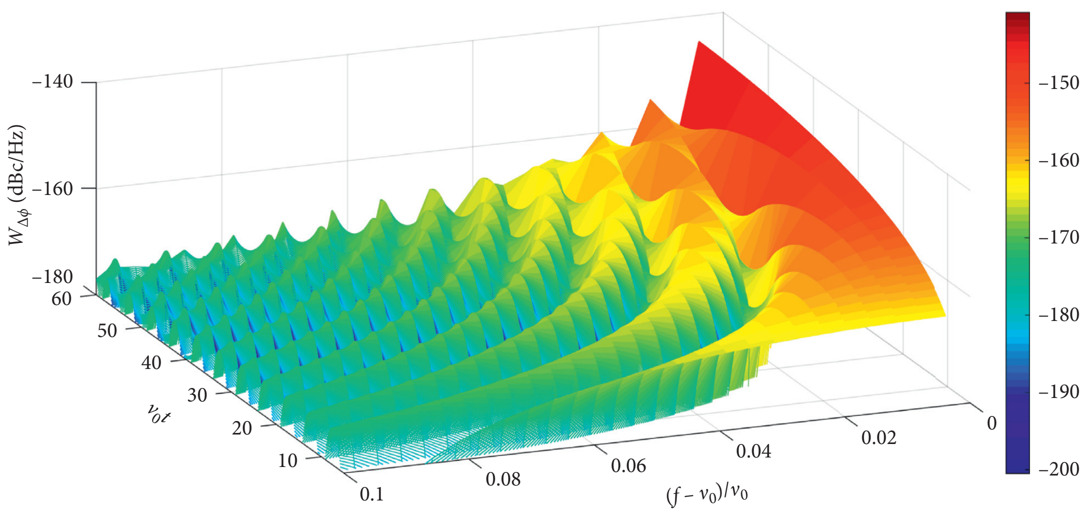

Figure 1: The analytic Wigner–Ville time-dependent spectrum $\overline{W}_{\Delta\varphi}(t, f)$ of NAC1 atomic oscillator.

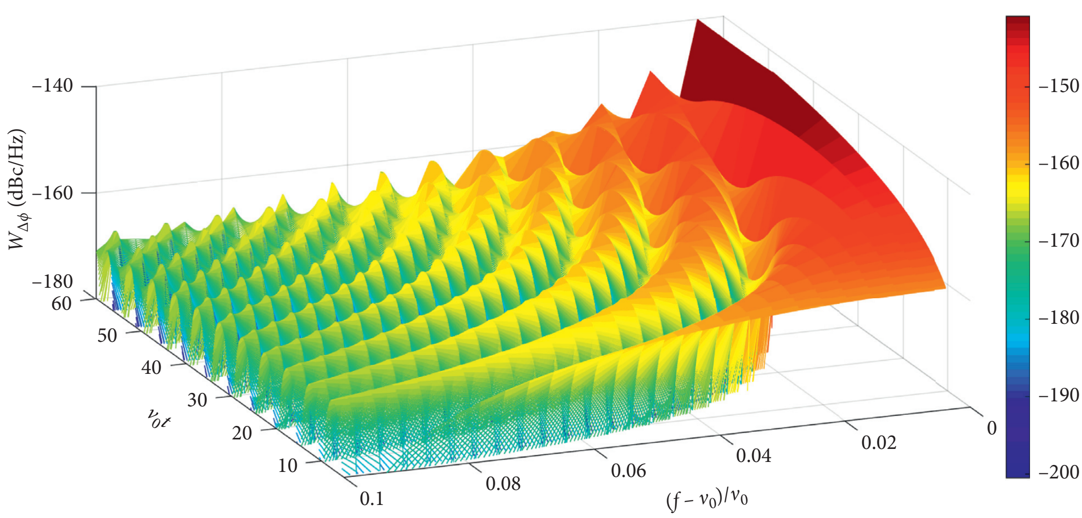

Figure 2: The analytic Wigner–Ville time-dependent spectrum $\overline{W}_{\Delta\varphi}(t, f)$ of AR13300 atomic oscillator.

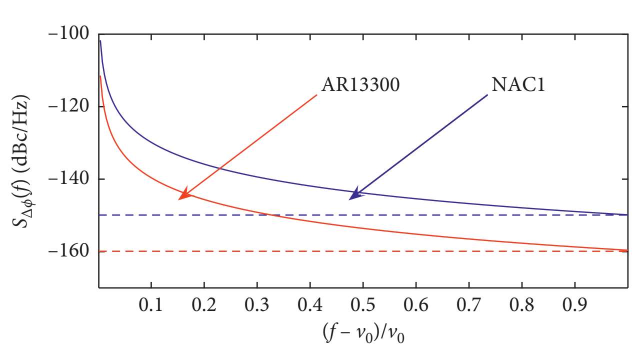

Figure 3: The power spectrum phase-noise $S_{\Delta\varphi}(f)$ of NAC1 and AR13300 obtained by integration of the resulted Wigner spectrum $\overline{W}_{\Delta\varphi}(t, f)$.

temperature variations. We derived an analytic expression for the Wigner–Ville spectrum for nonstationary phase noise and demonstrate its application on atomic clocks. The technique is applicable for the characterization of different oscillators, frequency generators, and radiation sources including lasers and masers.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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