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Research Article

A Necessary Condition for Optimal Control of Forward-Backward Stochastic Control System with Lévy Process in Nonconvex Control Domain Case

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This paper analyzes one kind of optimal control problem which is described by forward-backward stochastic differential equations with Lévy process (FBSDEL). We derive a necessary condition for the existence of the optimal control by means of spike variational technique, while the control domain is not necessarily convex. Simultaneously, we also get the maximum principle for this control system when there are some initial and terminal state constraints. Finally, a financial example is discussed to illustrate the application of our result.

1. Introduction

Stochastic optimal control is an important matter that cannot be neglected in modern control theory in long days. As is known to all, Pontryagin's [1] maximum principle is one of the main ways to settle the stochastic optimal control problem. By introducing the Hamiltonian function, a necessary condition for the optimal control of stochastic control systems was given by him, which was called the maximum condition. From that time, plenty of works on this issue have been done. Peng [2] was the first one to prove the general maximum principle of the forward-backward stochastic control system with diffusion coefficient containing the control variable by the technique of the second-order Taylor expansion and the second-order duality. He [3] was also the first one to demonstrate the maximum principle of forward-backward stochastic control systems from the view of backward stochastic differential equations (BSDE). In Peng's paper [3], the control domain was convex (in local form); Xu [4] extended this conclusion to the case of the nonconvex control domain (in global form), but the control variables were not included in the diffusion coefficient. And these results were extended to the fully coupled case in the form of local and global by Shi and Wu [5, 6] in 1998 and in 2006, respectively. On the basis of these works, Situ [7] was the first to obtain the maximum principle of the forward stochastic control system with global form of random jumps in 1991. Shi and Wu [8] and Shi [9] acquired the maximum principle for a kind of forward-backward stochastic control system with Poisson jumps in the form of local and global, respectively. The fully coupled forward-backward stochastic control system was extended by Liu et al. [10] at the base of Shi and Wu [8], and in the meanwhile, they also obtained the maximum principle with the control system be constrained about initial-terminal state constraints. Considering that in real life, the decision makers could only get partial information but not complete information in most cases; many scholars have paid attention to the partial observable stochastic optimal control problem and have achieved many results (see, for example, [11–13]). Traditionally, when using a stochastic partial differential equation called the Zakai equation to transform a full-information optimal control problem to the partially observable case, scholars will encounter a difficult problem: an infinite-dimensional optimal control problem. Wang and Wu [14] proposed a backward separation approach and replaced the original state and observation equation with the Zakai equation, and lots of complicated stochastic calculi in infinite-dimensional spaces were avoided in this way. Based on this approach, Xiao [15] studied a partially observed optimal control of forward-backward stochastic systems with random jumps and obtained the maximum principle and sufficient conditions of an optimal control under some certain convexity assumptions. Wang et al. [16] proved the maximum principles for forward-backward stochastic control systems with correlated state and observation noises. More recent conclusions of the partially observed stochastic control problem can be seen from the studies conducted by Wang et al. [17], Zhang et al. [18], and Xiong et al. [19].

In these years, through the study of mathematical economics and mathematical finance, many scholars turn their attention to the stochastic control system driven by Lévy process. In 2000, Nualart and Schoutens [20] built a pair of pairwise strongly orthonormal martingales which was called the Teugels martingale. Meanwhile, under some exponential moment conditions, they also obtained a martingale representation in that paper. Under these two important conclusions, for BSDE driven by the Teugels martingale, they [21] proved the existence and uniqueness theorem of its solution in the next year. From then on, a number of important results were proved: Meng and Tang [22] obtained the maximum principle of the forward stochastic control system driven by Lévy process. A necessary and sufficient condition for the existence of the optimal control of backward stochastic control systems driven by Lévy process was deduced by Tang and Zhang [23] through convex variation methods and duality techniques. For the forward-backward stochastic control system driven by Lévy process, there are also a lot of achievements: based on the existence and uniqueness theorem of FBSDEL [24], Zhang et al. [25] obtained a necessary condition of the optimal control and verification theorem, but in their control system, the backward state variables y_t and z_t did not enter the forward part. Wang and Huang [26] extended this result to the fully coupled control system and obtained the continuity result depending on parameters about FBSDEL and the local form maximum principle. Subsequently, Huang et al. [27] studied this control system with terminal state constraints and obtained the corresponding necessary maximum principle using Ekeland's variational. For more recent conclusions about the stochastic control problem driven by Lévy process, please refer to [28–30].

In this paper, we will study the optimal control problem for forward-backward stochastic control systems driven by Lévy process, which could be considered as a nonconvex control domain case that is extended from the result of [25]. With the technique of spike variation and Ekeland's variational principle, the maximum principle of this type of control system and the control system with initial and final state constraints are obtained.

The structure of this paper is as follows. Section 2 describes some of the preparations used in this paper. The maximum principle and the one with initial and terminal state constraints as the major results of this paper will be shown in Sections 3 and 4. As an application of the maximum principle, Section 5 gives an optimal consumption problem in the financial market. Section 6 is the summary of this article.

2. Preliminary Statement

Let $(\Omega, \mathcal{F}_t, P)$ be a complete probability space which satisfied the usual conditions, and the information structure is given by $\mathcal{F}_t$ which is generated by two processes: a standard Brownian motion $\{B_t\}_{0\le t\le T}$ valued in R^d and an independent 1-dimensional Lévy process $\{L_t\}_{0\le t\le T}$ of the form $L_t = b_t + l_t$; here, l_t is a pure jump process. And assume that Lévy measure ν satisfies the following two conditions; thereby, Lévy process $\{L_t\}_{0\le t\le T}$ has moments in all orders.

(i) $\int_R (1\wedge x^2)\nu(\mathrm{d}x) < \infty$.

(ii) $\int_{(-\varepsilon,\varepsilon)^c} e^{\lambda|x|}\nu(\mathrm{d}x) < \infty$, $\forall \varepsilon > 0$ and some $\lambda > 0$.

Denote $L_t^1 = L_t$, $\triangle L_t = L_t - L_{t-}$, and $L_t^i = \sum_{0<s\le t}(\triangle L_s)^i$ for $i\ge 2$. And let $Y_t^i = L_t^i - E[L_t^i]$ $(i\ge 1)$ be the compensated power jump process of order i; then, Teugels martingale is defined by $H_t^i = \sum_{j=1}^i c_{ij}Y_t^j$; the coefficients c_{ij} correspond to orthonormalization of the polynomials $1, x, x^2, \cdots$ with respect to the measure $\mu(\mathrm{d}x) = v(\mathrm{d}x) + \sigma^2\delta_0(\mathrm{d}x)$.

Then, $\{H_t^i\}_{i=1}^\infty$ are pairwise strongly orthogonal martingales, and their predictable quadratic variation processes are $\langle H_t^i, H_t^j\rangle = \delta_{ij}t$, δ_{ij} is an indicator function here. And $[H^i, H^j]_t - \langle H_t^i, H_t^j\rangle$ is an $\mathcal{F}_t$–martingale; for more details of the Teugels martingale, see Nualart and Schoutens [20].

In the following of this section, we shall assume some notations: for a Hilbert space $\mathcal{H}$,

$l^2(\mathcal{H}) := \left\{\phi \mid \mathcal{H} - \text{valued}, \sum_{i=1}^\infty \|\phi^i\|^2 < \infty\right\}$.

$L^2(\Omega, \mathcal{H}) := \{\xi \mid$
$\mathcal{H}$ valued, $\mathcal{F}_T-$ measurable, $E|\xi|^2 < \infty\}$.

$l^2(0,T;\mathcal{H}) := \{\phi_t^i \mid l^2(\mathcal{H})-$
valued, $\mathcal{F}_t -$ measurable, $\sum_{i=1}^\infty E\int_0^T \|\phi_t^i\|^2\mathrm{d}t < \infty\}$.

$M^2(0,T;\mathcal{H}) := \{\phi(\cdot) \mid$
$\mathcal{H} -$ valued, $\mathcal{F}_t -$ measurable, $E\int_0^T |\phi_t|^2\mathrm{d}t < \infty\}$.

For the following FBSDEL,

$$\begin{cases} \mathrm{d}x_t = b(t,x_t,y_t,z_t,r_t)\mathrm{d}t + \sigma(t,x_t,y_t,z_t,r_t)\mathrm{d}B_t + \sum_{i=1}^\infty g^i(t,x_{t-},y_{t-},z_t,r_t)\mathrm{d}H_t^i, \\ -\mathrm{d}y_t = f(t,x_t,y_t,z_t,r_t)\mathrm{d}t - z_t\mathrm{d}B_t - \sum_{i=1}^\infty r_t^i\mathrm{d}H_t^i, \\ x_0 = a,\ y_T = \Phi(x_T), \end{cases} \tag{1}$$

where (x_t, y_t, z_t, r_t) take the value in $\Omega \times [0,T] \times R^n \times R^m \times R^{m\times d} \times l^2(R^m)$ and mappings b, σ, g, and f take the value in R^n, $R^{n\times d}$, $l^2(R^n)$, and R^m, respectively. Convenient for writing, set column vector $\alpha = (x, y, z)^T$ and $A(t,\alpha,r) = (-M^\tau f(t,\alpha,r), Mb(t,\alpha,r), M\sigma(t,\alpha,r))^T$, where M is a $m\times n$ full rank matrix.

Assumption 1

(i) All mappings in equation (1) are uniformly Lipschitz continuous in their own arguments, respectively.

(ii) For all $(\omega,t) \in \Omega\times[0,T]$, $l(\omega,t,0,0,0,0) \in M^2(0,T;R^{n+m+m\times d}) \times l^2(0,T;R^m)$ for $l = b, f, \sigma$, respectively, and $g(\omega,t,0,0,0,0) \in l^2(R^n)$.

(iii) $\langle \Phi(\overline{x}) - \Phi(x), M(\overline{x}-x)\rangle \geq \beta |M\widehat{x}|^2$.

(iv) $\langle A(t,\overline{\alpha},\overline{r}) - A(t,\alpha,r), \overline{\alpha}-\alpha\rangle + \sum_{i=1}^{\infty}\langle M\widehat{g^i}, \widehat{r^i}\rangle \leq -\mu_1|M\widehat{x}|^2 - \mu_2(|M^\tau\widehat{y}|^2 + |M^\tau\widehat{z}|^2 + \sum_{i=1}^{\infty}\|M^\tau\widehat{r^i}\|^2)$, where $\overline{\alpha} = (\overline{x},\overline{y},\overline{z}), \widehat{x} = \overline{x}-x, \widehat{y} = \overline{y}-y, \widehat{z} = \overline{z}-z, \widehat{r^i} = \overline{r}^i - r^i, \widehat{g^i} = g^i(t,\overline{\alpha},\overline{r}) - g^i(t,\alpha,r)$. μ_1, μ_2, and β are nonnegative constants which satisfied $\mu_1+\mu_2>0$, $\mu_2+\beta>0$, and $\mu_1>0, \beta>0$(resp. $\mu_2>0$) when $m>n$ (resp. $n>m$).

Then, the following existence and uniqueness of the solution conclusion holds.

Lemma 1. *There exists a unique solution in $M^2(0,T;\mathscr{H})$ satisfying FBSDEL (1) under Assumption 1.*

The detailed certification process of this conclusion can be seen in [24].

3. Stochastic Maximum Principle

In this section, for any given admissible control $u(\cdot)$, we consider the following stochastic control system:

$$\begin{cases} \mathrm{d}x_t = b(t,x_t,u_t)\mathrm{d}t + \sigma(t,x_t)\mathrm{d}B_t + \sum_{i=1}^{\infty} g^i(t,x_{t-})\mathrm{d}H_t^i, \\ -\mathrm{d}y_t = f(t,x_t,y_t,z_t,r_{t,},u_t)\mathrm{d}t - z_t\mathrm{d}B_t - \sum_{i=1}^{\infty} r_t^i\mathrm{d}H_t^i, \\ x_0 = a, y_T = \Phi(x_T), t\in[0,T], \end{cases} \tag{2}$$

where $a \in R^n$ is given. An admissible control $u(\cdot) \in M^2(0,T;R^p)$ is an $\mathscr{F}_t$-predictable process which takes values in a nonempty subset U of R^p; U_{ad} is the set of all admissible controls.

And the performance criterion is

$$J(u) = E\gamma(y_0), \tag{3}$$

where $\gamma: R^m \longrightarrow R$ is a given Frechet differential function.

Our optimal control problem amounts to determining an admissible control $u^* \in U_{\text{ad}}$ such that

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in U_{\text{ad}}} J(u(\cdot)). \tag{4}$$

In order to get the necessary conditions for the optimal control, we assume u_t^* is the optimal control, and the corresponding solution of (2) is recorded as $(x_t^*, y_t^*, z_t^*, r_t^*)$ and introduce the "spike variational control" as follows:

$$u_t^\varepsilon = \begin{cases} v_t, & \tau\leq t\leq \tau+\varepsilon, \\ u_t^*, & \text{otherwise}, \end{cases} \tag{5}$$

and $(x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon)$ are the state trajectories of u_t^ε; here, v_t is an arbitrary admissible control and ε is a sufficiently small constant.

We also need the following assumption and variational equation (6).

Assumption 2

(i) b, f, g, σ, Φ, and γ are continuously differentiable with respect to (x, y, z, r, u), and the derivatives are all bounded.

(ii) There exists a constant $C>0$, and it holds that $|\gamma_y| \leq C(1+|y|)$.

$$\begin{cases} \mathrm{d}X_t = [b_x(t)X_t + b(t,u_t^\varepsilon) - b(t,u_t^*)]\mathrm{d}t + \sigma_x(t)X_t\mathrm{d}B_t + \sum_{i=1}^{\infty} g_x^i(t)X_t\mathrm{d}H_t^i, \\ -\mathrm{d}Y_t = \left[f_x(t)X_t + f_y(t)Y_t + f_z(t)Z_t + f_r(t)R_t + f(t,u_t^\varepsilon) - f(t,u_t^*)\right]\mathrm{d}t, \\ -Z_t\mathrm{d}B_t - \sum_{i=1}^{\infty} R_t^i\mathrm{d}H_t^i, \\ X_0 = 0, \\ Y_T = \Phi_x(t)X_T. \end{cases} \tag{6}$$

Here, $b_x(t)=b_x(t,x_t^*,u_t^*)$, $\sigma_x(t)=\sigma_x(t,x_t^*)$, $g_x^i(t)=g_x^i(t,x_t^*)$, $b(t,u_t^\varepsilon)=b(t,x_t^\varepsilon,u_t^\varepsilon)$, $b(t,u_t^*)=b\quad(t,x_t^*,u_t^*)$, $f_w(t)=f_w(t,x_t^*,y_t^*,z_t^*,r_t^*,u_t^*)$, $(w=x,y,z\cdot r)$, $f(t,u_t^\varepsilon)=f(x_t^\varepsilon,y_t^\varepsilon,z_t^\varepsilon,r_t^\varepsilon,u_t^\varepsilon)$, and $f(t,u_t^*)=f(x_t^*,\ y_t^*,z_t^*,r_t^*,u_t^*)$.

Lemma 2. *Suppose Assumptions 1 and 2 hold; for the first-order variation X,Y,Z,R, we have the following estimations:*

$$\sup_{0\le t\le T} E|X_t|^2\le C\varepsilon^2, \tag{7}$$

$$\sup_{0\le t\le T} E|X_t|^4\le C\varepsilon^4, \tag{8}$$

$$\sup_{0\le t\le T} E|Y_t|^2\le C\varepsilon^2, \tag{9}$$

$$\sup_{0\le t\le T} E|Y_t|^4\le C\varepsilon^4, \tag{10}$$

$$\sup_{0\le t\le T} E\int_0^T |Z_t|^2 ds\le C\varepsilon^2, \tag{11}$$

$$\sup_{0\le t\le T} E\left(\int_0^T |Z_t|^2 ds\right)^2\le C\varepsilon^4, \tag{12}$$

$$\sup_{0\le t\le T} E\int_0^T \|R_t\|^2 ds\le C\varepsilon^2, \tag{13}$$

$$\sup_{0\le t\le T} E\left(\int_0^T \|R_t\|^2 ds\right)^2\le C\varepsilon^4. \tag{14}$$

Proof. We first prove inequations (7) and (8). For the forward part of the first-order variation equation, we have

$$\begin{aligned}
E|X_t|^2 &= E\left\{\int_0^t [b_x(s)X_s+b(s,u_s^\varepsilon)-b(s,u_s^*)]ds+\sigma_x(s)X_s dB_s+\sum_{i=1}^{\infty} g_x^i(s)X_s dH_s^i\right\}^2\\
&\le 4\left\{E\left(\int_0^t b_x(s)X_s ds\right)^2+E\left(\int_0^t [b(s,u_s^\varepsilon)-b(s,u_s^*)]ds\right)^2+E\int_0^t [\sigma_x(s)X_s]^2 ds+E\int_0^t\left[\sum_{i=1}^{\infty} g_x^i(s)X_s\right]^2 ds\right\}\\
&\le 12C^2TE\int_0^t X_s^2 ds+4E\left(\int_0^t [b(s,u_s^\varepsilon)-b(s,u_s^*)]ds\right)^2.
\end{aligned} \tag{15}$$

Applying Gronwall's inequation, we have

$$E|X_t|^2\le C\varepsilon^2,\quad \text{for all } t\in[0,T]. \tag{16}$$

Similarly, (8) holds.

We next estimate Y_t, Z_t, and R_t; the backward part of the first-order variation equation can be rewritten as

$$\begin{aligned}
Y_t+\int_t^T Z_s dB_s+\int_t^T\sum_{i=1}^{\infty} R_s^i dH_s^i = \Phi_x(t)X_T+\int_t^T [f_x(t)X_t\\
+f_y(t)Y_t+f_z(t)Z_t\\
+f_r(t)R_t\\
+f(t,u_t^\varepsilon)-f(t,u_t^*)]dt.
\end{aligned} \tag{17}$$

Squaring both sides of (17) and using the fact of

$$\begin{aligned}
EY_t\int_t^T Z_s dB_s &= 0,\\
EY_t\int_t^T\sum_{i=1}^{\infty} R_s^i dH_s^i &= 0,\\
E\int_t^T Z_s dB_s\int_t^T\sum_{i=1}^{\infty} R_s^i dH_s^i &= 0,
\end{aligned} \tag{18}$$

we get

$$\begin{aligned}
&E|Y_t|^2+E\int_t^T Z_s^2 ds+E\int_t^T\left(\sum_{i=1}^{\infty} R_s^i\right)^2 dH_s^i\\
&= E\left\{\Phi_x(t)X_T+\int_t^T\left[f_x(t)X_t+f_y(t)Y_t+f_z(t)Z_t+f_r(t)R_t+f(t,u_t^\varepsilon)-f(t,u_t^*)\right]dt\right\}^2\\
&\le 6C^2EX_T^2+6C^2TE\int_t^T X_s^2 ds+6C^2TE\int_t^T Y_s^2 ds+6C^2(T-t)E\int_t^T Z_s^2 ds\\
&\quad+6C^2(T-t)E\int_t^T\left(\sum_{i=1}^{\infty} R_s^i\right)^2 ds+6E\left(\int_t^T (f(s,u_s^\varepsilon)-f(s,u_s^*))ds\right)^2.
\end{aligned} \tag{19}$$

When $t \in [T-\delta, T]$ with $\delta = 1/12C^2$, we have

$$E|Y_t|^2 + \frac{1}{2}E\int_t^T Z_s^2 ds + \frac{1}{2}E\int_t^T \left(\sum_{i=1}^{\infty} R_s^i\right)^2 dH_s^i$$
$$\le 6C^2 EX_T^2 + 6C^2 TE\int_t^T X_s^2 ds + 6C^2 TE \tag{20}$$
$$\cdot \int_t^T Y_s^2 ds + 6E\left(\int_t^T (f(s,u_s^\varepsilon) - f(s,u_s^*))ds\right)^2.$$

Applying Gronwall's inequation, we have

$$\sup_{0\le t\le T} E|Y_t|^2 \le C\varepsilon^2, \quad t \in [T-\delta, T];$$
$$\sup_{0\le t\le T} E\int_0^T |Z_t|^2 ds \le C\varepsilon^2, \quad t \in [T-\delta, T]; \tag{21}$$
$$\sup_{0\le t\le T} E\int_0^T \|R_t\|^2 ds \le C\varepsilon^2, \quad t \in [T-\delta, T].$$

Consider the BSDE of the first-order variation equation in the interval $[t, T-\delta]$:

$$Y_t + \int_t^{T-\delta} Z_s dB_s + \int_t^{T-\delta}\sum_{i=1}^{\infty} R_s^i dH_s^i = Y_{T-\delta}$$
$$+ \int_t^{T-\delta}\left[f_x(t)X_t + f_y(t)Y_t + f_z(t)Z_t\right. \tag{22}$$
$$\left. + f_r(t)R_t + f(t,u_t^\varepsilon) - f(t,u_t^*)\right]dt.$$

Thus,

$$E|Y_t|^2 + E\int_t^{T-\delta} Z_s^2 ds + E\int_t^{T-\delta}\left(\sum_{i=1}^{\infty} R_s^i\right)^2 dH_s^i,$$
$$= E\left\{Y_{T-\delta} + \int_t^{T-\delta}\left[f_x(t)X_t + f_y(t)Y_t + f_z(t)Z_t + f_r(t)R_t\right.\right.$$
$$\left.\left. + f(t,u_t^\varepsilon) - f(t,u_t^*)\right]dt\right\}^2$$
$$\le 6C^2 EX_T^2 + 6C^2 TE\int_t^{T-\delta} X_s^2 ds + 6C^2 TE\int_t^{T-\delta} Y_s^2 ds$$
$$+ 6C^2(T-t)E\int_t^{T-\delta} Z_s^2 ds + 6C^2(T-t)E$$
$$\cdot \int_t^{T-\delta}\left(\sum_{i=1}^{\infty} R_s^i\right)^2 ds + 6E\left(\int_t^{T-\delta}(f(s,u_s^\varepsilon) - f(s,u_s^*))ds\right)^2. \tag{23}$$

So, when $t \in [T-2\delta, T]$ with $\delta = 1/12C^2$, we have

$$\sup_{0\le t\le T} E|Y_t|^2 \le C\varepsilon^2, \quad t \in [T-2\delta, T],$$
$$\sup_{0\le t\le T} E\int_0^T |Z_t|^2 ds \le C\varepsilon^2, \quad t \in [T-2\delta, T], \tag{24}$$
$$\sup_{0\le t\le T} E\int_0^T \|R_t\|^2 ds \le C\varepsilon^2, \quad t \in [T-2\delta, T].$$

After a finite number of iterations, (9), (11), and (13) are obtained. And (10), (12), and (14) can be proved by using a similar method and the following inequalities: $E(\int_t^T Z_s dB_s)^4 \ge \beta_1 E(\int_t^T Z_s^2 ds)^2, \beta_1 > 0$; $E(\int_t^T \sum_{i=1}^{\infty} R_s^i dH_s^i)^4 \ge \beta_2 E(\int_t^T (\sum_{i=1}^{\infty} R_s^i)^2 ds)^2, \beta_2 > 0$. □

Lemma 3. *Under hypothesis, Assumptions 1 and 2, it holds the following four estimations:*

$$\sup_{0\le t\le T} E|x_t^\varepsilon - x_t^* - X_t|^2 \le C_\varepsilon \varepsilon^2, \quad C_\varepsilon \longrightarrow 0, \text{when } \varepsilon \longrightarrow 0, \tag{25}$$

$$\sup_{0\le t\le T} E|y_t^\varepsilon - y_t^* - Y_t|^2 \le C_\varepsilon \varepsilon^2, \quad C_\varepsilon \longrightarrow 0, \text{when } \varepsilon \longrightarrow 0, \tag{26}$$

$$E\int_0^T |z_t^\varepsilon - z_t^* - Z_t|^2 ds \le C_\varepsilon \varepsilon^2, \quad C_\varepsilon \longrightarrow 0, \text{when } \varepsilon \longrightarrow 0, \tag{27}$$

$$E\int_t^T \|r_t^\varepsilon - r_t^* - R_t\|^2 ds \le C_\varepsilon \varepsilon^2, \quad C_\varepsilon \longrightarrow 0, \text{when } \varepsilon \longrightarrow 0. \tag{28}$$

Proof. To prove (25), we observe that

$$\int_0^t b(s, x_s^* + X_s, u_s^\varepsilon)ds + \int_0^t \sigma(s, x_s^* + X_s)dB_s$$
$$+ \int_0^t \sum_{i=1}^{\infty} g^i(s, x_{s-}^* + X_{s-})dH_s^i$$
$$= \int_0^t \left[b(s, x_s^*, u_s^\varepsilon) + \int_0^1 b_x(s, x_s^* + \lambda X_s, u_s^\varepsilon)d\lambda X_s\right]ds$$
$$+ \int_0^t \left[\sigma(s, x_s^*) + \int_0^1 \sigma_x(s, x_s^* + \lambda X_s)d\lambda X_s\right]dB_s$$
$$+ \int_0^t \left[\sum_{i=1}^{\infty} g^i(s, x_s^*) + \int_0^1 \sum_{i=1}^{\infty} g_x^i(s, x_s^* + \lambda X_s)d\lambda X_s\right]dH_s^i$$
$$= \int_0^t b(s, x_s^*, u_s^*)ds + \int_0^t \sigma(s, x_s^*)dB_s + \int_0^t \sum_{i=1}^{\infty} g^i(s, x_s^*)dH_s^i$$
$$+ \int_0^t b_x(s, x_s^*, u_s^*)X_s ds + \int_0^t \sigma_x(s, x_s^*)X_s dB_s$$
$$+ \int_0^t \sum_{i=1}^{\infty} g_x^i(s, x_{s-}^*)dH_s^i$$
$$+ \int_0^t [b(s, x_s^*, u_s^\varepsilon) - b(s, x_s^*, u_s^*)]ds + \int_0^t A^\varepsilon ds$$
$$+ \int_0^t B^\varepsilon dB_s + \int_0^t C^\varepsilon dH_s^i$$
$$= x_t^* - x_0 + X_t + \int_0^t A^\varepsilon ds + \int_0^t B^\varepsilon dB_s + \int_0^t C^\varepsilon dH_s^i, \tag{29}$$

where

$$A^\varepsilon = \int_0^1 [b_x(s, x_s^* + \lambda X_s, u_s^\varepsilon) - b_x(s, x_s^*, u_s^*)] \mathrm{d}\lambda X_s,$$
$$B^\varepsilon = \int_0^1 [\sigma_x(s, x_s^* + \lambda X_s) - \sigma_x(s, x_s^*)] \mathrm{d}\lambda X_s, \quad (30)$$
$$C^\varepsilon = \int_0^1 \left[\sum_{i=1}^{\infty} \left(g_x^i(s, x_s^* + \lambda X_s) - g_x^i(s, x_s^*)\right)\right] \mathrm{d}\lambda X_s.$$

It follows easily from Lemma 2 that

$$\sup_{0\le t\le T} E\left[\left(\int_0^t A^\varepsilon \mathrm{d}s\right)^2 + \left(\int_0^t B^\varepsilon \mathrm{d}B_s\right)^2 + \left(\int_0^t C^\varepsilon \mathrm{d}H_s^i\right)^2\right] = o(\varepsilon^2). \quad (31)$$

Since

$$x_t^\varepsilon - x_0 = \int_0^t b(s, x_s^\varepsilon, u_s^\varepsilon) \mathrm{d}s + \int_0^t \sigma(s, x_s^\varepsilon) \mathrm{d}B_s + \int_0^t \sum_{i=1}^{\infty} g^i(s, x_{s-}^\varepsilon) \mathrm{d}H_s^i, \quad (32)$$

then

$$\begin{aligned} x_t^\varepsilon - x_t^* - X_t &= \int_0^t [b(s, x_s^\varepsilon, u_s^\varepsilon) - b(s, x_s^* + X_s, u_s^\varepsilon)] \mathrm{d}s \\ &\quad + \int_0^t [\sigma(s, x_s^\varepsilon) - \sigma(s, x_s^* + X_s)] \mathrm{d}B_s \\ &\quad + \int_0^t \sum_{i=1}^{\infty} \left(g^i(s, x_{s-}^\varepsilon) - g^i(s, x_{s-}^* + X_{s-})\right) \mathrm{d}H_s^i \\ &\quad + \int_0^t A^\varepsilon \mathrm{d}s + \int_0^t B^\varepsilon \mathrm{d}B_s + \int_0^t C^\varepsilon \mathrm{d}H_s^i \\ &= \int_0^t D^\varepsilon (x_s^\varepsilon - x_s^* - X_s) \mathrm{d}s + \int_0^t E^\varepsilon (x_s^\varepsilon - x_s^* - X_s) \mathrm{d}B_s \\ &\quad + \int_0^t F^\varepsilon (x_s^\varepsilon - x_s^* - X_s) \mathrm{d}H_s^i, \end{aligned} \quad (33)$$

with $D^\varepsilon = \int_0^1 b_x(s, x_s^* + X_s + \lambda(x_s^\varepsilon - x_s^* - X_s), u_s^\varepsilon) \mathrm{d}\lambda$, $E^\varepsilon = \int_0^1 \sigma_x(s, x_s^* + X_s + \lambda(x_s^\varepsilon - x_s^* - X_s)) \mathrm{d}\lambda$, and $F^\varepsilon = \int_0^1 \sum_{i=1}^{\infty} g_x^i(s, x_s^* + X_s + \lambda(x_s^\varepsilon - x_s^* - X_s)) \mathrm{d}\lambda$.

By Gronwall's inequation, we have

$$\sup_{0\le t\le T} E\left|x_t^\varepsilon - x_t^* - X_t\right|^2 \le C_\varepsilon \varepsilon^2, \quad C_\varepsilon \longrightarrow 0, \text{when } \varepsilon \longrightarrow 0. \quad (34)$$

Next, we prove (26), (27),, (28); it can be easily checked that

$$\begin{aligned} &-\int_t^T f(s, x_s^* + X_s, y_s^* + Y_s, z_s^* + Z_s, r_s^* + R_s, u_s^\varepsilon) \mathrm{d}s \\ &+ \int_t^T (s, z_s^* + Z_s) \mathrm{d}B_s + \int_t^T \sum_{i=1}^{\infty} g^i(s, x_{s-}^* + X_{s-}) \mathrm{d}H_s^i \\ &= \Phi(x_T^*) - y_t^* + \Phi_x(x_T^*) X_T - Y_t - \int_t^T G^\varepsilon \mathrm{d}s. \end{aligned} \quad (35)$$

Here,

$$\begin{aligned} G^\varepsilon = &\int_0^1 (f_x(s, x_s^* + \lambda X_s, y_s^* + \lambda Y_s, z_s^* \\ &+ \lambda Z_s, r_s^* + \lambda R_s, u_s^\varepsilon) - f_x(s)) \mathrm{d}\lambda X_s \\ &+ \int_0^1 \left(f_y(s, x_s^* + \lambda X_s, y_s^* + \lambda Y_s, z_s^* \right. \\ &\left. + \lambda Z_s, r_s^* + \lambda R_s, u_s^\varepsilon) - f_y(s)\right) \mathrm{d}\lambda Y_s \\ &+ \int_0^1 (f_z(s, x_s^* + \lambda X_s, y_s^* + \lambda Y_s, z_s^* \\ &+ \lambda Z_s, r_s^* + \lambda R_s, u_s^\varepsilon) - f_z(s)) \mathrm{d}\lambda Z_s \\ &+ \int_0^1 (f_r(s, x_s^* + \lambda X_s, y_s^* + \lambda Y_s, z_s^* \\ &+ \lambda Z_s, r_s^* + \lambda R_s, u_s^\varepsilon) - f_r(s)) \mathrm{d}\lambda R_s. \end{aligned} \quad (36)$$

Since

$$\begin{aligned} y_t^\varepsilon = &\Phi(x_T^\varepsilon) + \int_t^T f(s, x_s^\varepsilon, y_s^\varepsilon, z_s^\varepsilon, r_s^\varepsilon, u_s^\varepsilon) \mathrm{d}s - \int_t^T z_s \mathrm{d}B_s \\ &- \int_t^T \sum_{i=1}^{\infty} r_s^{i,\varepsilon} \mathrm{d}H_s^i, \end{aligned} \quad (37)$$

then

$$\begin{aligned} y_t^\varepsilon - y_t^* - Y_t = &\Phi(x_T^\varepsilon) - \Phi(x_T^*) - \Phi_x(x_T^*) X_T + \int_t^T [f(s, x_s^\varepsilon, y_s^\varepsilon, z_s^\varepsilon, r_s^\varepsilon, u_s^\varepsilon) - f(s, x_s^* + X_s, y_s^* + Y_s, z_s^* + Z_s, r_s^* + R_s, u_s^\varepsilon] \mathrm{d}s \\ &- \int_t^T (z_s^\varepsilon - z_s^* - Z_s) \mathrm{d}B_s - \int_t^T \sum_{i=1}^{\infty} \left(r_s^{i,\varepsilon} - r_s^{i,*} - R_s\right) \mathrm{d}H_s^i + \int_t^T G^\varepsilon \mathrm{d}s. \end{aligned} \quad (38)$$

Squaring both sides of the equation above, we get

$$E\left|y_t^\varepsilon - y_t^* - Y_t\right|^2 + E\int_t^T (z_s^\varepsilon - z_s^* - Z_s)^2 ds - E\int_t^T \sum_{i=1}^{\infty}\left(r_s^{i,\varepsilon} - r_s^{i,*} - R_s\right)^2 ds$$
$$= E\left\{[f(s, x_s^\varepsilon, y_s^\varepsilon, z_s^\varepsilon, r_s^\varepsilon, u_s^\varepsilon) - f(s, x_s^* + X_s, y_s^* + Y_s, z_s^* + Z_s, r_s^* + R_s, u_s^\varepsilon)]ds + \Phi(x_T^\varepsilon) - \Phi(x_T^*) - \Phi_x(x_T^*)X_T + \int_t^T G^\varepsilon ds\right\}^2. \tag{39}$$

From Lemma 2 and equation (25), we have

$$\sup_{0\le t\le T} E\left(\int_t^T G^\varepsilon ds\right)^2 = o(\varepsilon^2),$$
$$E[\Phi(x_T^\varepsilon) - \Phi(x_T^*) - \Phi_x(x_T^*)X_T]^2 = o(\varepsilon^2). \tag{40}$$

Then, we can get (26), (27), and (28) by applying the iterative method to the above relations. □

Lemma 4 (variational inequality). *Under the conditions that Assumptions 1 and 2 are established, we can get the following variational inequality:*

$$E\gamma_y(y_0^*)Y_0 \ge o(\varepsilon). \tag{41}$$

Proof. From the four estimations in Lemma 3, we have the following estimation:

$$E[\gamma(y_0^\varepsilon) - \gamma(y_0^* + Y_0)] = o(\varepsilon). \tag{42}$$

Therefore,

$$0 \le E[\gamma(y_0^* + Y_0) - \gamma(y_0^*)] + o(\varepsilon) = E\gamma_y(y_0^*)Y_0 + o(\varepsilon). \tag{43}$$

We introduce the following Hamiltonian function $H: [0,T] \times R^n \times R^m \times R^{m\times d} \times l^2(R^m) \times U \times R^n \times R^m \times R^{n\times d} \times l^2(R^n)$ as

$$H(t,x,y,z,r,u,p,q,w,k) = \langle p, b(t,x,u)\rangle + \langle w, \sigma(t,x)\rangle + \langle k, g(t,x)\rangle - \langle q, f(t,x,y,z,r,u)\rangle, \tag{44}$$

and the following adjoint equation

$$\begin{cases} dq_t = H_y(u_t^*)dt + H_z(u_t^*)dB_t + \sum_{i=1}^{\infty} H_r^i(u_t^*)dH_t^i, \\ -dp_t = H_x(u_t^*)dt - w_t dB_t - \sum_{i=1}^{\infty} k_{t-}^i dH_t^i, \\ q_0 = \gamma_y(y_0^*),\ p_T = -\Phi_x(x_T^*)q_T, \quad t\in[0,T], \end{cases} \tag{45}$$

where

$$\begin{aligned} H_x(u_t^*) &= H_x(t, x_t^*, y_t^*, z_t^*, r_t^*, u_t^*, p_t, q_t, w_t, k_t), \\ H_y(u_t^*) &= H_y(t, x_t^*, y_t^*, z_t^*, r_t^*, u_t^*, p_t, q_t, w_t, k_t), \\ H_z(u_t^*) &= H_z(t, x_t^*, y_t^*, z_t^*, r_t^*, u_t^*, p_t, q_t, w_t, k_t), \\ H_r(u_t^*) &= H_r(t, x_t^*, y_t^*, z_t^*, r_t^*, u_t^*, p_t, q_t, w_t, k_t). \end{aligned} \tag{46}$$

It is easily check that adjoint equation (45) has a unique solution quartet $(p_t, q_t, w_t, k_t) \in M^2(0,T; R^{n+m+m\times d}) \times l^2(0,T;R^m)$.

Then, we get the main result of this section. □

Theorem 1. *Let hypothesis, Assumptions 1 and2, hold; u_t^* is an optimal control, and the corresponding optimal state trajectories are $(x_t^*, y_t^*, z_t^*, r_t^*)$; let (p_t, q_t, w_t, k_t) be the solution of adjoint equation (45); and then, for each admissible control $u_t \in U_{ad}[0,T]$, we have*

$$H(t, x_t^*, y_t^*, z_t^*, r_t^*, u_t, p_t, q_t, w_t, k_t) \ge H(t, x_t^*, y_t^*, z_t^*, r_t^*, u_t^*, p_t, q_t, w_t, k_t) \text{ a.s.a.e.}. \tag{47}$$

Proof. Applying Itô's formula to $\langle p, X\rangle$ and $\langle q, Y\rangle$, it follows from (6), (45), and the variational inequality that

$$E\int_0^T [H(t, x_t, y_t, z_t, r_t, u_t^\varepsilon, p_t, q_t, w_t, k_t) - H(t, x_t, y_t, z_t, r_t, u_t^*, p_t, q_t, w_t, k_t)]dt = E\gamma_y(y_0^*)Y_0 \ge o(\varepsilon). \tag{48}$$

By the definition of u_t^ε, we know that, for any $u \in U_{ad}[0,T]$, the following inequation holds:

$$E[H(t, x_t, y_t, z_t, r_t, u_t^\varepsilon, p_t, q_t, w_t, k_t) - H(t, x_t, y_t, z_t, r_t, u_t^*, p_t, q_t, w_t, k_t)] \ge 0. \tag{49}$$

Then, (47) can be easily checked. □

4. Stochastic Control Problem with State Constraints

In this part, we are going to discuss stochastic control problems with state constraints in control system (2). Specifically, the initial state constraints and final state constraints are as follows:

$$\begin{aligned} EG_1(x_T) &= 0, \\ EG_0(y_0) &= 0, \end{aligned} \tag{50}$$

where $G_1 : R^n \longrightarrow R^{n_1}\,(n_1 < n)$, $G_0 : R^m \longrightarrow R^{m_1}\,(m_1 < m)$. Our optimal control problem is to find $u^* \in U_{\mathrm{ad}}$ such that

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in U_{\mathrm{ad}}} J(u(\cdot)), \quad (51)$$

subject to the state constraints (50). In the following, we will apply Ekeland's variational principle to solve this optimal control problem. Firstly, we need the following assumptions.

Assumption 3. Control domain U is assumed to be closed, the mappings in state constraints G_1, G_0 are continuously differentiable, and G_{1x}, G_{0y} are bounded.

For $\forall u_1(\cdot), u_2(\cdot) \in U_{\mathrm{ad}}$, let

$$d(u_1(\cdot), u_2(\cdot)) = \left|\left\{E\left|u_1(\cdot) - u_2(\cdot)\right|^2 > 0; t \in [0,T]\right\}\right|. \quad (52)$$

Same as Section 3, we also assume u_t^* be the optimal control, and the corresponding optimal state trajectories are $(x_t^*, y_t^*, z_t^*, r_t^*)$. In order to solve the constraint problem, we need the following penalty cost functional, for any $\rho > 0$:

$$J_\rho(u) = \left\{\left|E[G_1(x_T)]\right|^2 + \left|E[G_0(y_0)]\right|^2 + \left|J(u) - J(u^*) + \rho\right|^2\right\}^{1/2}. \quad (53)$$

It can be checked that $J_\rho(u) : U_{\mathrm{ad}} \longrightarrow R^1$ is continuous, and for any $u(\cdot) \in U_{ad}$,

$$\begin{gathered} J_\rho(u) \ge 0, J_\rho(u^*) = \rho, \\ J_\rho(u^*) \le \inf_{u(\cdot)\in U_{\mathrm{ad}}} J_\rho(u) + \rho. \end{gathered} \quad (54)$$

It can be obtained by Ekeland's variational principle that there exists $u_t^\rho \in U_{\mathrm{ad}}$ such that

$$\begin{cases} \text{(i) } J_\rho(u^\rho) \le J_\rho(u^*) = \rho, \\ \text{(ii) } d(u^\rho, u^*) \le \sqrt{\rho}, \\ \text{(iii) } J_\rho(v) \ge J_\rho(u^\rho) - \sqrt{\rho}\, d(u^\rho, v), \quad \text{for } \forall v(\cdot) \in U_{\mathrm{ad}}. \end{cases} \quad (55)$$

For fixed ρ and admissible control u_t^ρ, we define the spike variation as follows:

$$u_t^{\rho,\varepsilon} = \begin{cases} v_t, & \tau \le t \le \tau + \varepsilon; \\ u_t^\rho, & \text{otherwise}, \end{cases} \quad (56)$$

for any $\varepsilon > 0$, and it is easy to check from (iii) of (55) that

$$J_\rho(u^{\rho,\varepsilon}) - J_\rho(u^\rho) + \sqrt{\rho}\, d(u^{\rho,\varepsilon}, u^\rho) \ge 0. \quad (57)$$

Let $(x_t^\rho, y_t^\rho, z_t^\rho, r_t^\rho)$ be the trajectories corresponding to u_t^ρ and $(x_t^{\rho,\varepsilon}, y_t^{\rho,\varepsilon}, z_t^{\rho,\varepsilon}, r_t^{\rho,\varepsilon})$ be the trajectories corresponding to $u_t^{\rho,\varepsilon}$. The variational equation we used in this section is the same as the one in Section 3, with $u_t^* = u_t^\rho$ and $(x_t^*, y_t^*, z_t^*, r_t^*) = (x_t^\rho, y_t^\rho, z_t^\rho, r_t^\rho)$. And we also assume that the solution of this variational equation is $(X_t^\rho, Y_t^\rho, Z_t^\rho, R_t^\rho)$. Similar to the approach in Lemmas 2 and 3, it can be shown that

$$\begin{aligned} &\sup\nolimits_{0\le t\le T} E\left|x_t^{\rho,\varepsilon} - x_t^\rho - X_t^\rho\right|^2 \le C_\varepsilon \varepsilon^2, \quad C_\varepsilon \longrightarrow 0, \text{when } \varepsilon \longrightarrow 0, \\ &\sup\nolimits_{0\le t\le T} E\left|y_t^{\rho,\varepsilon} - y_t^\rho - Y_t^\rho\right|^2 \le C_\varepsilon \varepsilon^2, \quad C_\varepsilon \longrightarrow 0, \text{when } \varepsilon \longrightarrow 0, \\ &E\int_0^T \left|z_t^{\rho,\varepsilon} - z_t^\rho - Z_t^\rho\right|^2 ds \le C_\varepsilon \varepsilon^2, \quad C_\varepsilon \longrightarrow 0, \text{when } \varepsilon \longrightarrow 0, \\ &E\int_0^T \left\|r_t^{\rho,\varepsilon} - r_t^\rho - R_t^\rho\right\|^2 ds \le C_\varepsilon \varepsilon^2, \quad C_\varepsilon \longrightarrow 0, \text{when } \varepsilon \longrightarrow 0. \end{aligned} \quad (58)$$

Then, by (57), the following variational inequality holds:

$$\begin{aligned} &J_\rho(u^{\rho,\varepsilon}) - J_\rho(u^\rho) + \sqrt{\rho}\, d(u^{\rho,\varepsilon}, u^\rho) \\ &= \frac{J_\rho^2(u^{\rho,\varepsilon}) - J_\rho^2(u^\rho)}{J_\rho(u^{\rho,\varepsilon}) + J_\rho(u^\rho)} + \varepsilon\sqrt{\rho} \\ &= \langle h_1^{\rho,\varepsilon}, E[G_{1x}(x_T^\rho)]X_T^\rho\rangle + \langle h_0^{\rho,\varepsilon}, E\left[G_{0y}(y_0^\rho)\right]Y_0^\rho\rangle \\ &\quad + h^{\rho,\varepsilon} E\left[\gamma_y(y_0^\rho)Y_0^\rho\right] + \varepsilon\sqrt{\rho} + o(\varepsilon), \end{aligned} \quad (59)$$

where

$$\begin{aligned} h_1^{\rho,\varepsilon} &= \frac{2E[G_1(x_T^\rho)]}{J_\rho(u^{\rho,\varepsilon}) + J_\rho(u^\rho)}, \\ h_0^{\rho,\varepsilon} &= \frac{2E[G_0(y_0^\rho)]}{J_\rho(u^{\rho,\varepsilon}) + J_\rho(u^\rho)}, \\ h^{\rho,\varepsilon} &= \frac{2E[\gamma(y_0^\rho) - \gamma(y_0) + \rho]}{J_\rho(u^{\rho,\varepsilon}) + J_\rho(u^\rho)}. \end{aligned} \quad (60)$$

Now, let $(p_t^{\rho,\varepsilon}, q_t^{\rho,\varepsilon}, w_t^{\rho,\varepsilon}, k_t^{\rho,\varepsilon})$ be the solution of

$$\begin{cases} -dp_t^{\rho,\varepsilon} = \left[(b_x^\rho(t))^\tau p_t^{\rho,\varepsilon} - (f_x^\rho(t))^\tau q_t^{\rho,\varepsilon} + (\sigma_x^\rho(t))^\tau w_t^{\rho,\varepsilon} + \sum_{i=1}^{\infty} (g_x^\rho(t))^\tau k_t^{\rho,\varepsilon})^i\right] dt, \\ -w_t^{\rho,\varepsilon} dB_t - \sum_{i=1}^{\infty} (k_t^{\rho,\varepsilon})^i dH_t^i, \\ dq_t^{\rho,\varepsilon} = (f_y^\rho(t))^\tau q_t^{\rho,\varepsilon} dt + (f_z^\rho(t))^\tau q_t^{\rho,\varepsilon} dB_t + \sum_{i=1}^{\infty} \left(f_r^{\rho,i}(t)\right)^\tau) q_t^{\rho,\varepsilon} dH_t^i, \\ q_0 = -G_{0,y}(y_0^\rho) h_0^{\rho,\varepsilon} + \gamma_y(y_0^\rho) h^{\rho,\varepsilon}, \ p_T = G_{1x}(x_T^\rho) h_1^{\rho,\varepsilon} - \Phi_x(x_T^\rho) q_T^{\rho,\varepsilon}, \quad t \in [0,T]. \end{cases} \quad (61)$$

Here, $b_x^\rho(t) = b_x(t, x_t^\rho, u_t^\rho)$, $\sigma_x^\rho(t) = \sigma_x(t, x_t^\rho)$, $f_y^\rho(t) = f_y(t, x_t^\rho, y_t^\rho, z_t^\rho, r_t^\rho, u_t^\rho)$, etc. Applying Itô's formula to $\langle X_t^\rho, q_t^{\rho,\varepsilon}\rangle + \langle Y_t^\rho, p_t^{\rho,\varepsilon}\rangle$, variational inequality (59) can be rewritten as

$$\begin{aligned} &H(t, x_t^\rho, y_t^\rho, z_t^\rho, r_t^\rho, u_t^{\rho,\varepsilon}, p_t^{\rho,\varepsilon}, q_t^{\rho,\varepsilon}, w_t^{\rho,\varepsilon}, k_t^{\rho,\varepsilon}) \\ &\quad - H(t, x_t^\rho, y_t^\rho, z_t^\rho, r_t^\rho, u_t^\rho, p_t^{\rho,\varepsilon}, q_t^{\rho,\varepsilon}, w_t^{\rho,\varepsilon}, k_t^{\rho,\varepsilon}) \\ &\quad + \varepsilon\sqrt{\rho} + o(\varepsilon) \geq 0, \quad \forall v_t \in U, \text{a.e., a.s.,} \end{aligned} \tag{62}$$

where the Hamiltonian function H is defined as (44). Since

$$\lim_{\varepsilon\longrightarrow 0}\left(\left|h_0^{\rho,\varepsilon}\right|^2 + \left|h_1^{\rho,\varepsilon}\right|^2 + \left|h^{\rho,\varepsilon}\right|^2\right) = 1, \tag{63}$$

there exists a convergent subsequence, still denoted by $(h_0^{\rho,\varepsilon}, h_1^{\rho,\varepsilon}, h^{\rho,\varepsilon})$ such that $(h_0^{\rho,\varepsilon}, h_1^{\rho,\varepsilon}, h^{\rho,\varepsilon}) \longrightarrow (h_0^\rho, h_1^\rho, h^\rho)$ when $\varepsilon \longrightarrow 0$ with $|h_0^\rho|^2 + |h_1^\rho|^2 + |h^\rho|^2 = 1$.

Let $(p_t^\rho, q_t^\rho, w_t^\rho, k_t^\rho)$ be the solution of

$$\begin{cases} -\mathrm{d}p_t^\rho = \left[(b_x^\rho(t))^\tau p_t^\rho - (f_x^\rho(t))^\tau q_t^\rho + (\sigma_x^\rho(t))^\tau w_t^\rho + \sum_{i=1}^{\infty} (g_x^\rho(t))^\tau (k_t^\rho)^i \right] \mathrm{d}t, \\ -w_t^\rho \mathrm{d}B_t - \sum_{i=1}^{\infty} (k_t^\rho)^i \mathrm{d}H_t^i, \\ \mathrm{d}q_t^\rho = \left((f_y^\rho(t))^\tau q_t^\rho \mathrm{d}t + (f_z^\rho(t))^\tau q_t^\rho \mathrm{d}B_t + \sum_{i=1}^{\infty} \left(f_r^{\rho,i}(t)\right)^\tau \right) q_t^\rho \mathrm{d}H_t^i, \\ q_0 = -G_{0,y}(y_0^\rho)h_0^\rho + \gamma_y(y_0^\rho)h^\rho, \; p_T = G_{1x}(x_T^\rho)h_1^\rho - \Phi_x(x_T^\rho)q_T^\rho, \quad t \in [0,T]. \end{cases} \tag{64}$$

From the continuous dependence of the solutions of FBSDEs with Lévy process on the parameters, we can prove that the following convergence holds: $(p_t^{\rho,\varepsilon}, q_t^{\rho,\varepsilon}, w_t^{\rho,\varepsilon}, k_t^{\rho,\varepsilon}) \longrightarrow (p_t^\rho, q_t^\rho, w_t^\rho, k_t^\rho)$; then, in equation (62), it implies that

$$\begin{aligned} &H(t, x_t^\rho, y_t^\rho, z_t^\rho, r_t^\rho, v_t, p_t^\rho, q_t^\rho, w_t^\rho, k_t^\rho) \\ &\quad - H(t, x_t^\rho, y_t^\rho, z_t^\rho, r_t^\rho, u_t^\rho, p_t^\rho, q_t^\rho, w_t^\rho, k_t^\rho) \\ &\quad + \sqrt{\rho} \geq 0, \quad \forall v_t \in U, \text{a.e., a.s..} \end{aligned} \tag{65}$$

Similarly, there exists a convergent subsequence $(h_0^\rho, h_1^\rho, h^\rho)$ such that $(h_0^\rho, h_1^\rho, h^\rho) \longrightarrow (h_0, h_1, h)$ when $\rho \longrightarrow 0$ with $|h_0|^2 + |h_1|^2 + |h|^2 = 1$. Since $u_t^\rho \longrightarrow u_t^*$, as $\rho \longrightarrow 0$, we have $(x_t^\rho, y_t^\rho, z_t^\rho, r_t^\rho) \longrightarrow (x_t^*, y_t^*, z_t^*, r_t^*)$ and $(X_t^\rho, Y_t^\rho, Z_t^\rho, R_t^\rho) \longrightarrow (X_t, Y_t, Z_t, R_t)$, which is the solution of the variational equation as same as (6).

The following adjoint equation is introduced:

$$\begin{cases} -\mathrm{d}p_t = \left[(b_x(t))^\tau p_t - (f_x(t))^\tau q_t + (\sigma_x(t))^\tau w_t + \sum_{i=1}^{\infty} (g_x(t))^\tau k_t)^i \right] \mathrm{d}t, \\ -w_t \mathrm{d}B_t - \sum_{i=1}^{\infty} (k_t)^i \mathrm{d}H_t^i, \\ \mathrm{d}q_t = \left((f_y(t))^\tau q_t \mathrm{d}t + (f_z(t))^\tau q_t \mathrm{d}B_t + \sum_{i=1}^{\infty} \left(f_r^{\rho,i}(t)\right)^\tau \right) q_t \mathrm{d}H_t^i, \\ q_0 = -G_{0,y}(y_0)h_0 + \gamma_y(y_0)h, \; p_T = G_{1x}(x_T)h_1 - \Phi_x(x_T)q_T, \quad t \in [0,T]. \end{cases} \tag{66}$$

Similarly, it can be proved that $(p_t^\rho, q_t^\rho, w_t^\rho, k_t^\rho) \longrightarrow (p_t, q_t, w_t, k_t)$; then, inequation (65) implies

$$\begin{aligned} &H(t, x_t^*, y_t^*, z_t^*, r_t^*, v_t, p_t, q_t, w_t, k_t) \\ &\quad - H(t, x_t^*, y_t^*, z_t^*, r_t^*, u_t^*, p_t, q_t, w_t, k_t) \geq 0, \forall v_t \in U, \text{a.e., a.s..} \end{aligned} \tag{67}$$

Then, we get the following theorem:

Theorem 2. *Let Assumptions 1–3 hold; u_t^* is an optimal control, and the corresponding optimal state trajectories are $(x_t^*, y_t^*, z_t^*, r_t^*)$, and (p_t, q_t, w_t, k_t) is the solution of adjoint equation (66); then, there exists nonzero constant $(h_1, h_0, h) \in (R^{n_1} \times R^{m_1} \times R)$ with $|h_0|^2 + |h_1|^2 + |h|^2 = 1$ such that, for any admissible control $v_t \in U_{ad}[0,T]$, the maximum condition (67) holds.*

This conclusion can be drawn from the above analysis directly.

5. A Financial Example

In this section, we will study the problem of optimal consumption rate selection in the financial market, which will naturally inspire our research to the forward-backward stochastic optimal control problem in Section 3.

Assume the investor's asset process x_t $(t \geq 0)$ in the financial market is described by the following stochastic differential equation with the Teugels martingale:

$$\begin{cases} \mathrm{d}x_t = [\mu_t x_t - C_t]\mathrm{d}t + \sigma_t x_t \mathrm{d}B_t + \sum_{i=1}^{\infty} g_t^i x_t \mathrm{d}H_t^i, \\ x_0 = W > 0,\ t \in [0, T], \end{cases} \tag{68}$$

where μ_t and $\sigma_t \neq 0$ are the expected return and volatility of the value process x_t at time t, respectively, and C_t is the consumption rate process. Assume μ_t, σ_t, g_t, and C_t are all uniformly bounded $\mathcal{F}_t$–measurable random processes. The purpose of investors is to select the optimal consumption strategy C_t^* at time $t \geq 0$, minimizing the following recursive utility:

$$J(C(\cdot)) = E[y_0], \tag{69}$$

where y_t is the following backward stochastic process:

$$\begin{cases} -dy_t = \left[Le^{-rt} \dfrac{C_t^{1-R}}{1-R} - r y_t \right] \mathrm{d}t - z_t \mathrm{d}B_t - \sum_{i=1}^{\infty} r_t^i \mathrm{d}H_t^i, \\ y_T = -x_T > 0, \quad t \in [0, T], \end{cases} \tag{70}$$

with constant $L > 0$, the discount factor $r > 0$, and the Arrow–Pratt measure of risk aversion $R \in (0, 1)$.

If the consumption process C_t is regarded as a control variable, then combining (68) with (70), we encounter the following control system:

$$\begin{cases} \mathrm{d}x_t = [\mu_t x_t - C_t]\mathrm{d}t + \sigma_t x_t \mathrm{d}B_t + \sum_{i=1}^{\infty} g_t^i x_t \mathrm{d}H_t^i, \\ -\mathrm{d}y_t = \left[Le^{-rt} \dfrac{C_t^{1-R}}{1-R} - r y_t \right] \mathrm{d}t - z_t \mathrm{d}B_t - \sum_{i=1}^{\infty} r_t^i \mathrm{d}H_t^i, \\ x_0 = W > 0, y_T = -x_T > 0, \quad t \in [0, T], \end{cases} \tag{71}$$

which is obviously a special case of stochastic control system (2) with $b(t, x, u) = \mu_t x_t - C_t$, $\sigma(t, x) = \sigma_t x_t$, $g(t, x) = g_t^i x_t$, $\Phi_x(x_T) = -X_T$ and $f(t, x, y, z, r, u) = Le^{-rt}(C_t^{1-R}/(1-R)) - r y_t$.

We can check that both Assumptions 1 and 2 are satisfied. Then, we can use our maximum principle (Theorem 1) to solve the above optimization problem. Let C_t^* be an optimal consumption rate and x_t^*, y_t^* be the corresponding wealth process and recursive utility process. In this case, the Hamiltonian function H reduces to

$$\begin{aligned} H(t, x_t^*, y_t^*, C_t, p_t^*, q_t^*, w_t^*, k_t^*) &= \langle p_t^*, \mu_t x_t^* - C_t \rangle \\ &+ \langle w_t^*, \sigma_t x_t^* \rangle + \langle k_t^*, g_t^i x_t^* \rangle - \langle q_t^*, Le^{-rt} \frac{C_t^{1-R}}{1-R} - r y_t^* \rangle, \end{aligned} \tag{72}$$

and $(p_t^*, q_t^*, w_t^*, k_t^*)$ is the solution of the following adjoint equation:

$$\begin{cases} \mathrm{d}q_t = r q_t \mathrm{d}t \\ -\mathrm{d}p_t = \left[\mu_t p_t + \sigma_t w_t + \sum_{i=1}^{\infty} (k_t^i)^{\tau} g_t^i \right] \mathrm{d}t - w_t \mathrm{d}B_t - \sum_{i=1}^{\infty} k_t^i \mathrm{d}H_t^i, \\ q_0 = 1, p_T = q_T, \quad t \in [0, T]. \end{cases} \tag{73}$$

According to the maximum principle (47), we have

$$p_t^* = -Le^{-rt} (C_t^*)^{-R} q_t^*, \tag{74}$$

and the optimal consumption rate

$$C_t^* = \left(-\frac{e^{rt}}{L} \frac{p_t^*}{q_t^*} \right)^{-1/R}. \tag{75}$$

By solving FBSDE (73), we get $q_t^* = e^{rt}$, $p_t^* = e^{rt} e^{\mu(T-t)}$, and $w_t^* = k_t^* = 0$. Then, we get the optimal consumption rate of the investor which is

$$C_t^* = \left(-\frac{e^{rT}}{L} e^{\mu(T-t)} \right)^{-1/R}, \quad t \in [0, T]. \tag{76}$$

6. Conclusions

In this paper, a nonconvex control domain case of the forward-backward stochastic control driven by Lévy process is considered, and we obtain the global stochastic maximum principle for this stochastic control problem. And then, the problem of stochastic control with initial and final state constraints on the state variables is discussed, and a necessary condition about existence of the optimal control is also acquired. A financial example of optimal consumption is discussed to illustrate the application of the stochastic maximum principle.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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