Research Article

Stock Return Uncertainty and Life Insurance

Yang Dong , Hao Wang, and Lihong Zhang

Tsinghua University, School of Economics and Management, Beijing, China

Correspondence should be addressed to Yang Dong; dongy.17@sem.tsinghua.edu.cn

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Knightian uncertainty embedded in stock returns causes rising demand for life insurance, as the uncertainty averse agent seeks alternative investment channels. Life insurance demand of middle-aged agent is more sensitive to the uncertainty. Stock return uncertainty reduces the agent's total wealth and subsequently the propensity of wealthy agent serving as an insurance seller. Rising demand and falling supply of life insurance imply that life insurance is more expensive in the presence of stock return uncertainty. Sensitivity of life insurance demand to the mortality rate and key stock return characteristics also changes with the uncertainty.

1. Introduction

Insurance market and stock market are closely connected in many ways [1–4]. Rational expectations models have been extensively used to study portfolio allocation and life insurance in equilibrium. These models assume that the agent possesses perfect knowledge about the probability law governing the stochastic asset return process. However, the true model is rarely known, so any specified probability law is subject to potential misspecification. Knightian uncertainty (henceforth, uncertainty) arises in the situation where the agent cannot develop a probability distribution to describe potential return misspecification. Stock returns are difficult to forecast [5–7] and are prone to such uncertainty. We in the paper formulate a continuous-time rational expectations model to examine the effects of stock return uncertainty on life insurance. The model admits uncertainty aversion as the agent suspects that stock return in the stochastic process is potentially misspecified. Following Hansen and Sargent [8], Anderson et al. [9], Uppal and Wang [10], and Maenhout [6], we impose an uncertainty penalty to the agent's objective function in reflecting his skeptical and conservative perspective. These years, many studies apply the model uncertainty to different models, for example, the stochastic interest rate [11], the insurer with reinsurance and investment problem [12], and the uncertainty about jump and diffusion risk [13].

We find that optimal insurance demand increases with the level of uncertainty. The agent shifts some of his investment in the stock to life insurance, confirming that the insurance market and the stock market, to some degree, substitute each other. Life insurance is used as a way to circumvent the uncertainty embedded in the stock. This effect is more prominent for the middle-aged agent. Younger agent's demand for life insurance is low in the first place. Elder agent consumes more, repressing demand for life insurance when it is close to the end of his financial planning horizon. As a result, the demand of younger and elder agents for life insurance is less sensitive to stock return uncertainty.

When the agent is endowed with a sufficiently high level of initial wealth, he optimally supplies insurance [14]. By reducing the agent's total wealth, stock return uncertainty decreases the propensity of wealthy agent serving as an insurance seller. The agent would act more conservatively in the insurance market facing stock return uncertainty. Agents would demand more insurance when the supply of insurance falls, implying that insurance premium would increase in equilibrium. We leave it to future research to develop an equilibrium model to explore such implications rigorously.

The sensitivity of insurance demand with respect to the mortality rate may change in the presence of stock return uncertainty. In the absence of uncertainty, an increase in the mortality rate leads to lower insurance demand. However, facing stock return uncertainty, the agent might demand

more for insurance as the mortality rate increases. The rationale is that the mortality rate plays two roles in insurance decision making. On the one hand, it adversely affects the insurance payout ratio, so a higher mortality rate reduces the utility brought by life insurance. On the other hand, the mortality rate affects the probability of the agent obtaining life insurance payment within the financial planning horizon; thereby, the agent with a higher mortality rate is more willing to buy life insurance because there is a greater chance to receive the payment. When stock return uncertainty is low, the first effect dominates, while when the uncertainty is sufficiently high, the second effect is more prominent.

Our discoveries echo the phenomena found in empirical research, especially on the relationship between the stock market and the insurance market. Jawadi et al. [15] find a positive significant long-term relationship between insurance premium and stock price. Lamm-Tennant and Weiss [16] reveal that the insurance premium is significantly and negatively correlated to the stock index. Our findings are supportive of the supplementary relationship between the two markets. Uncertainty about stock returns reduces the stock market's attractiveness and boosts the relative competitiveness of insurance products.

Our work contributes uniquely to the life insurance literature [17–20]. Several works are closely related to ours. Among them, Merton [21] first models the dynamic asset price process and derives Hamilton–Jacobi–Bellman (HJB) equation to solve for optimal controls. Richard [14] combines life insurance and the rational expectations model developed by Merton [21] to investigate the optimal insurance demand problem. Pliska and Ye [22] study life insurance in a setting where the agent's lifetime is unbounded. Kwak and Lim [23], Huang et al. [24], and Pirvu and Zhang [25] examine inflation, stochastic labor income, and mean-reverting Sharpe ratio, respectively, under the Richard [14] framework. Recently, Huang et al. [26] considered stochastic mortality rate. Our work for the first time investigates the externality of stock return uncertainty on life insurance. Methodologically, modeling the dynamic wealth process illustrates age-dependent and wealth-path-contingent decision rules.

The remainder of the paper is organized as follows: Section 2 presents the model. Section 3 solves the general utility model and the CRRA utility model. Section 4 carries out the numerical analysis. Section 5 concludes the paper.

2. Model

This section introduces our model. It first describes the economy, followed by stock return uncertainty and the agent's objective function that admits stock return uncertainty.

2.1. Economy. Consider a simple continuous-time rational expectations model as in Merton [21], which involves one risk-free asset $R(t)$ and one stock $S(t)$. Let $T > 0$ be a finite financial planning horizon, and let $(\Omega, \mathcal{F}, P)$ be the probability space with information filtration $\{\mathcal{F}(t)\}_{t \in [0, T]}$. The prices of the two assets have the following processes:

$$\frac{dR(t)}{R(t)} = r dt, \quad (1)$$

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dz, \quad (2)$$

where α, σ , and r are constants, $\sigma > 0$, and $\alpha > r > 0$. Let $C(t)$ be the consumption rate at time t and $\omega(t)$ the fraction of wealth invested in the stock at time t , respectively. It is assumed that the consumption rate process $C(t)$ is non-negative and $\mathcal{F}(t)$ -progressively measurable, satisfying

$$\int_0^T C(t) dt < \infty, \text{ almost surely (a.s.)}, \quad (3)$$

and, $\omega(t)$ is $\mathcal{F}(t)$ -adapted, satisfying

$$\int_0^T \omega^2(t) dt < \infty, \text{ (a.s.)}. \quad (4)$$

$\omega(t)$ can be negative without short-selling constraint.

Life insurance provides a lump sum payment when the agent deceases. Following Richard [14], we assume that the decease time is a nonnegative random variable τ independent of $z(t)$. Its distribution function and probability density function are given by

$$F(t) = \Pr(\tau < t) = \int_0^t f(u) du, \quad (5)$$

$$f(t) \geq 0, \quad \forall t \in [0, T].$$

We do not require $\int_0^T f(t) dt = 1$, as the agent may still be alive at the end of the financial planning horizon. Let $\bar{F}(t)$ be the survival function at time t , so

$$\bar{F}(t) = \Pr(\tau \geq t) = 1 - F(t). \quad (6)$$

Based on the above expressions, the mortality rate is given by

$$\lambda(t) = \lim_{\Delta t \rightarrow 0} \frac{\Pr(t \leq \tau < t + \Delta t \mid \tau \geq t)}{\Delta t} = \frac{f(t)}{\bar{F}(t)}. \quad (7)$$

Rewrite the mortality rate $\lambda(t)$ and $f(t)$ as

$$\lambda(t) = -\frac{d \ln \bar{F}(t)}{dt}; \quad (8)$$

$$f(t) = \lambda(t) \bar{F}(t) = \lambda(t) \exp \left[- \int_0^t \lambda(u) du \right].$$

The life insurance premium $P(t)$ is paid continuously. In return, when the agent deceases at t , the life insurance pays a lump sum amount of $P(t)/\mu(t)$, where $\mu(t)$ is a continuous and deterministic function of premium-insurance ratio. We assume $\mu(t) = \lambda(t) + \eta(t)$, where $\eta(t)$ represents the security loading.

2.2. Stock Return Uncertainty. Expected stock returns are difficult to estimate [6, 7], so the agent worries that α in equation (2) is potentially misspecified. Uncertainty arises as he cannot come up with a probability to describe such potential misspecification. To solve the problem, the agent

considers a set of alternative models with probability measures P^Q , which are equivalent to the reference measure P^R . As in Anderson et al. [9] and Hansen and Sargent [8], the uncertainty averse agent optimizes his utility based on the worst-case alternative model. According to Girsanov's theorem, the relative entropy between P^Q and P^R is defined as

$$\frac{dP^Q}{dP^R} = \Gamma(t), \quad (9)$$

where

$$\Gamma(t) = \exp \left[- \int_0^t \frac{1}{\sigma} u(s) ds - \frac{1}{2} \int_0^t \frac{1}{\sigma^2} u^2(s) ds \right]. \quad (10)$$

Under Novikov's condition,

$$\mathbb{E}^P \left[\exp \left(\frac{1}{2} \int_0^T \frac{1}{\sigma^2} u^2(s) ds \right) \right] < \infty. \quad (11)$$

$\Gamma(t)$ is the Radon–Nikodym derivative of P^Q with respect to P^R . Uncertainty aversion introduces an adjustment $u(t)$ to the expected stock return. In the alternative model, the stock price process follows

$$\frac{dS(t)}{S(t)} = (\alpha - u(t))dt + \sigma d\tilde{z}. \quad (12)$$

The agent also receives labor income $y(t)$ during the time interval $[0, \min\{T, \tau\}]$. In decision making, the agent simultaneously chooses the fraction of wealth $\omega(t)$ invested in the risky asset, the consumption rate $C(t)$, and the amount of life insurance premium $P(t)$ at time t . His wealth process on $[0, \min\{T, \tau\}]$ satisfies

$$\begin{aligned} dW_t &= (y(t) - C(t) - P(t))dt \\ &\quad + (W_t \omega(t)(\alpha - u(t) - r) + W_t r)dt \\ &\quad + W_t \omega(t) \sigma d\tilde{z}. \end{aligned} \quad (13)$$

2.3. Objective Function. The agent's utility consists of two parts: utility $U(C(t), t)$ from consumption and bequest utility $B(Z(t), t)$ from legacy, where $Z(t)$ denotes the legacy left for future generations:

$$Z(t) = W_t + \frac{P(t)}{\mu(t)}. \quad (14)$$

The uncertainty averse agent optimizes his utility based on the worse-case alternative model by solving the following Max-Min problem:

$$\sup_{\{C, P, \omega\}} \inf_u \mathbb{E} \left[\int_0^T U(C(s), s) ds + B(Z(T), T) + \int_0^T \frac{\phi(s)}{2\sigma^2} u^2(s) ds \right]. \quad (15)$$

The third term of equation (15) is the uncertainty penalty function:

$$\Lambda(t) = \frac{\phi(t)}{2\sigma^2} u^2(t), \quad (16)$$

where $\phi(t)$ is a normalization function to transform the penalty function into the units of utility and $u(t)$ measures the “distance” between P^R and P^Q as the adjustment to α . $u^2(t)/2\sigma^2$ is the corresponding relative entropy. The selection of the special form of $\phi(t)$ should fulfill two purposes: reflecting agent's degree of uncertainty aversion and measuring the level of uncertainty, which can also be regarded as the agent's lack of confidence in the reference model. A larger $\phi(t)$ means the deviation of an alternative model from the reference model would be more heavily penalized, which means the agent highly trusts the reference model [10].

In the presence of stock return uncertainty, the agent's utility function is

$$\begin{aligned} J(W, t; \omega, C, P, u) &= \mathbb{E}_{t, W} \left[\int_t^{T \wedge \tau} \left(U(C(s), s) + \frac{\phi(s)}{2\sigma^2} u^2(s) \right) ds + B(Z(\tau), \tau) \mathbb{I}(\tau \leq T) + L(W_T, T) \mathbb{I}(\tau \geq T) \right] \\ &= \mathbb{E}_{t, W} \left[\int_t^T f(s, t) B(Z(s), s) + \bar{F}(s, t) \left(U(C(s), s) + \frac{\phi(s)}{2\sigma^2} u^2(s) \right) ds + \bar{F}(T, t) L(W_T, T) \right], \end{aligned} \quad (17)$$

where $\mathbb{E}_{t, W}[\cdot]$ denotes the conditional expectation operator under the probability measure P^R ; $T \wedge \tau = \min\{T, \tau\}$. $L(W_T, T)$ represents terminal wealth at time T if the agent survives beyond the time. $U(C(t), t)$ and $L(W_T, T)$ have the same functional form with respect to $C(t)$ and W_T , respectively. $f(s, t) = f(s)/\bar{F}(t)$ is the probability density function of the agent's decease at time s conditional on his

survival at time t ; $\bar{F}(s, t) = \bar{F}(s)/\bar{F}(t)$ is the agent's survival probability at time s conditional on his survival at time t ; and $\mathbb{I}(\tau \leq T)$ is an indicator function as

$$\mathbb{I}(\tau \leq T) = \begin{cases} 1, & \tau \leq T, \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

Substituting equations (7) and (8) into (17) yields

$$J(W, t; \omega, C, P, u) = \int_t^T \lambda(s) e^{-\int_t^s \lambda(v) dv} B(Z(s), s) + e^{-\int_t^s \lambda(v) dv} \left[U(C(s), s) + \frac{\phi(s)}{2\sigma^2} u^2(s) \right] ds + e^{-\int_t^T \lambda(v) dv} L(W_T, T). \quad (19)$$

Let $\mathcal{A} = \{C(t), P(t), \omega(t)\}$; that is, $\mathcal{A}$ is the admissible set of consumption, insurance premium payment, and asset allocation controls. The objective of the agent is to choose the optimal control set $\mathcal{A}^* = \{C^*(t), P^*(t), \omega^*(t)\}$ to

maximize the utility function in equation (19) based on the worst case of uncertainty-induced adjustment $u^*(t)$. Thus, the value function of this problem is

$$V(W, t) = \sup_{\{C, P, \omega\} \in \mathcal{A}} \inf_u J(W, t; \omega, C, P, u) = J(t, W; \omega^*, C^*, P^*, u^*),$$

subject to

$$dW_t = (y(t) - C(t) - P(t))dt + (W_t \omega(t)(\alpha - u(t) - r) + W_t r)dt + W_t \omega(t) \sigma d\tilde{z}; \quad (20)$$

$$Z(t) = W_t + \frac{P(t)}{\mu(t)}.$$

3. Solutions

This section presents the solutions to the model with general utility and then the solutions to the CRRA utility model.

3.1. General Utility. We use the dynamic programming method to obtain the Hamilton–Jacobi–Bellman (HJB) equation. According to the optimality principle, the value function in equation (20) is expressed as

$$V(W, t) = \sup_{\{C, P, \omega\} \in \mathcal{A}} \inf_u \mathbb{E}_{t, W} \left[\int_t^{t+\Delta t} f(s, t) B(Z(s), s) + \bar{F}(s, t) \left(U(C(s), s) + \frac{\phi(s)}{2\sigma^2} u^2(s) \right) ds + e^{-\int_t^{t+\Delta t} \lambda(v) dv} V(W_{t+\Delta t}, t + \Delta t) \right], \quad (21)$$

where $\Delta t \rightarrow 0$ is the instantaneous moment. We have the following approximate relationship:

$$e^{-\int_t^{t+\Delta t} \lambda(v) dv} = 1 - \lambda(t)\Delta t + o(\Delta t^2), \quad (22)$$

where $o(\Delta t^2)$ represents the higher-order infinitesimal of Δt . The relationship between $V(W_t, t)$ and $V(W_{t+\Delta t}, t + \Delta t)$ follows that

$$V(W_{t+\Delta t}, t + \Delta t) = V(W, t) + \mathcal{L}[V(W, t)]\Delta t, \quad (23)$$

where

$\mathcal{L}[V(W, t)] = [\lim_{\Delta t \rightarrow 0} (V(W_{t+\Delta t}, t + \Delta t) - V(W, t))/\Delta t]$, which is also called the Dynkin operator. We substitute equations (22) and (23) into (21). Note $f(t, t) = \lambda(t)$ and $\bar{F}(t, t) = 1$. Dividing both sides of the equation by Δt , we obtain

$$0 = \sup_{\{C, P, \omega\} \in \mathcal{A}} \inf_u \mathcal{L}[V(W, t)] - \lambda(t)V(W, t) + \lambda(t)B(Z(t), t) + U(C(t), t) + \frac{\phi(t)}{2\sigma^2} u^2(t). \quad (24)$$

By Ito's lemma, the Dynkin operator follows that

$$\begin{aligned} \mathcal{L}[V(W, t)] &= V_t + V_W [-C(t) - P(t) + y(t) \\ &\quad + W_t \omega(t)(\alpha - u(t) - r) + W_t r] \\ &\quad + \frac{1}{2} V_{WW} W_t^2 \omega^2(t) \sigma^2, \end{aligned} \quad (25)$$

where V_t and V_W represent the first-order partial derivatives of $V(W, t)$ with respect to t and W , respectively. V_{WW} represents the second-order partial derivative of $V(W, t)$ with respect to W . Then, the HJB equation changes into

$$\begin{aligned} 0 &= \sup_{\{C, P, \omega\} \in \mathcal{A}} \inf_u -\lambda(t)V(W, t) + \lambda(t)B(Z(t), t) \\ &\quad + U(C(t), t) + V_t + V_W [-C(t) - P(t) + y(t) \\ &\quad + W_t \omega(\alpha - u - r) + W_t r] + \frac{1}{2} V_{WW} W_t^2 \omega^2 \sigma^2 + \frac{\phi(t)}{2\sigma^2} u^2(t), \end{aligned} \quad (26)$$

with the boundary condition $V(W_T, T) = L(W_T, T)$. The last item of equation (26) is the uncertainty penalty function $\Lambda(t)$ in equation (16).

We first solve the minimization part of equation (26). Take the first-order condition with respect to $u(t)$ and obtain the worst-case uncertainty adjustment $u^*(t)$ as

$$\begin{aligned} -V_W W_t \omega + \frac{\phi(t)}{\sigma^2} u^*(t) &= 0, \\ u^*(t) &= \frac{1}{\phi(t)} V_W W_t \omega \sigma^2. \end{aligned} \quad (27)$$

Substitute $u^*(t)$ back into equation (26) and take the first-order conditions with respect to $C(t)$, $Z(t)$, and $\omega(t)$, respectively; we drive the optimal controls.

Proposition 1. *For an uncertainty averse agent with a general utility $U(C(t), t)$ and a bequest function $B(Z(t), t)$, the optimal controls are*

$$U_C(C^*(t), t) = V_W; \quad (28)$$

$$\frac{\lambda(t)}{\mu(t)} B_Z(Z^*(t), t) = V_W; \quad (29)$$

$$\omega^*(t) = \frac{V_W(\alpha - r)}{(1/\phi(t))V_W^2 W_t - V_{WW} W_t \sigma^2}. \quad (30)$$

The optimal insurance demand $P^*(t)$ can be obtained with equations (14) and (29). The marginal ratio of consumption utility to bequest utility satisfies

$$\frac{U_C(C^*(t), t)}{B_Z(Z^*(t), t)} = \frac{\lambda(t)}{\mu(t)}. \quad (31)$$

The agent balances the optimal consumption $C^*(t)$ and optimal insurance demand $P^*(t)$ till $C^*(t)$ and the corresponding legacy $Z^*(t) = W_t + (P^*(t)/\mu(t))$ generate the same marginal utility of $\lambda(t)/\mu(t)$. The next section discusses the special case of CRRA utility, under which explicit optimal controls can be derived.

3.2. CRRA Utility. When the agent has a CRRA utility function, $U(C(t), t) = (e^{-\rho t}/\gamma)C^\gamma(t)$; $B(Z(t), t) = (e^{-\rho t}/\gamma)Z^\gamma(t)$; and $L(W_T, T) = (e^{-\rho T}/\gamma)W_T^\gamma$, where $1 - \gamma$ is the risk aversion coefficient, $\gamma < 1$ and $\gamma \neq 0$; ρ is the discount factor and $\rho > 0$. We follow Maenhout [6] to specify the normalization function $\phi(t)$ as

$$\phi(t) = \frac{\gamma}{\theta} V(W_t, t), \quad (32)$$

where θ represents the degree of uncertainty aversion. $\phi(t)$ is decreasing in the degree of uncertainty aversion—when the agent has a high degree of confidence in the reference model (with a small θ), a small deviation from the reference model will lead to a heavy penalty. According to equation (27), the worst-case adjustment $u^*(t)$ in this CRRA utility case is

$$u^*(t) = \frac{\theta}{\gamma V(W_t, t)} V_W W_t \omega \sigma^2. \quad (33)$$

Equation (33) shows that $u^*(t)$ is an increasing function of θ , implying that the more uncertainty averse the agent is, the greater the perceptual adjustment he will make. Substitute equation (33) into (16); we obtain

$$\Lambda(t) = \frac{\gamma u^2(t)}{2\theta\sigma^2} V(W_t, t) = \frac{\theta}{2\gamma V(W_t, t)} V_W^2 W_t^2 \omega^2 \sigma^2. \quad (34)$$

Substitute equations (32) and (33) into (26); we obtain the HJB function as

$$\begin{aligned} 0 &= \max_{\{C, P, \omega\} \in \mathcal{A}} \lambda(t) B(Z(t), t) + U(C(t), t) - \lambda(t) V + V_t \\ &\quad + V_W [-C(t) - P(t) + y(t) + W_t \omega(t)(\alpha - r) + W_t r] \\ &\quad + \frac{1}{2} V_{WW} W_t^2 \sigma^2 \omega^2(t) - \frac{\theta}{2\gamma V(W_t, t)} V_W^2 W_t^2 \omega^2(t) \sigma^2. \end{aligned} \quad (35)$$

We identify and distinguish three types of wealth: the current wealth, the labor wealth, and the total wealth. We will solve the problems above using these definitions. Denote

(i) $W(t)$ as the current wealth—the wealth the agent possesses at time t , whose process is specified in equation (13).

(ii) $b(t)$ as the labor wealth—the present value of the agent's future labor income. The discount rate applied to compute the present value equals the risk-free rate plus the insurance security loading that reflects the insurance market friction. Thus,

$$\begin{aligned} b(t) &= \int_t^{T \wedge \tau} y(s) \exp\left[-\int_t^s r + \eta(v) dv\right] ds \\ &= \int_t^T \bar{F}(s, t) y(s) \exp\left[-\int_t^s r + \eta(v) dv\right] ds \\ &= \int_t^T y(s) \exp\left[-\int_t^s r + \mu(v) dv\right] ds. \end{aligned} \quad (36)$$

(iii) $X(t)$ as the total wealth—the amount of wealth with which the agent uses for decision making. Thus,

$$X(t) = W_t + b(t). \quad (37)$$

At the end of the financial planning horizon, $b(T) = 0$ and $X(T) = W_T$.

The agent's current wealth W_t can be negative, implying that the agent is able to borrow against his future cash flows. However, the total wealth $X(t)$ is always positive as $b(t)$ is always positive, and the amount of borrowing cannot exceed $b(t)$.

We conjecture the objective function as

$$V(W, t) = \frac{e^{-\rho t} a(t)}{\gamma} [W_t + b(t)]^\gamma = \frac{e^{-\rho t} a(t)}{\gamma} X^\gamma(t), \quad (38)$$

where $a(t)$ is a time-varying function that captures the influence of stock return uncertainty. Its functional form is to be determined by the HJB function. Taking the partial derivatives of $V(W, t)$ with respect to W and t , respectively, we obtain

$$\begin{cases} V_W = e^{-\rho t} a(t) X^{\gamma-1}(t); \\ V_{WW} = e^{-\rho t} (\gamma-1) a(t) X^{\gamma-2}(t); \\ V_t = e^{-\rho t} a(t) b'(t) X^{\gamma-1}(t) + \frac{1}{\gamma} (e^{\rho t} a'(t) - \rho e^{-\rho t} a(t)) X^\gamma(t), \end{cases} \quad (39)$$

where $a'(t)$ and $b'(t)$ represent the first derivatives of $a(t)$ and $b(t)$ with respect to t , respectively. In particular, according to equation (36),

$$b'(t) = -\gamma(t) + [r + \mu(t)]b(t). \quad (40)$$

Based on the above results, we have the following proposition.

Proposition 2. For the uncertainty averse agent with CRRA utility $U(C_t, t) = (e^{-\rho t}/\gamma)C^\gamma(t)$ and bequest function $B(Z(t), t) = (e^{-\rho t}/\gamma)Z^\gamma(t)$, where $\gamma < 1$ and $\gamma \neq 0$, the optimal controls are

$$C^*(t) = \left[\frac{1}{a(t)} \right]^{1/(1-\gamma)} X(t), \quad (41)$$

$$\omega^*(t)W_t = \frac{\alpha - r}{(\theta - \gamma + 1)\sigma^2} X(t), \quad (42)$$

$$P^*(t) = \mu(t) \left\{ \left[\left(\frac{\lambda(t)}{\mu(t)a(t)} \right)^{1/(1-\gamma)} - 1 \right] X(t) + b(t) \right\}. \quad (43)$$

The corresponding legacy is

$$Z^*(t) = W_t + \frac{P^*(t)}{\mu(t)} = \left(\frac{\lambda(t)}{\mu(t)a(t)} \right)^{1/(1-\gamma)} X(t), \quad (44)$$

where

$$a(t) = \left[e^{\int_t^T H(v)dv} + \int_t^T K(s) e^{\int_t^s H(v)dv} ds \right]^{1-\gamma}, \quad (45)$$

$$H(t) = -\frac{\rho}{1-\gamma} + \frac{\gamma}{1-\gamma} \left[\mu(t) + r + \frac{1}{2} \frac{(\alpha - r)^2}{(\theta - \gamma + 1)\sigma^2} \right] - \frac{\lambda(t)}{1-\gamma}; \quad (46)$$

$$K(t) = 1 + \left(\frac{\lambda(t)}{\mu^\gamma(t)} \right)^{1/(1-\gamma)}. \quad (47)$$

Moreover, $a(t)$ is strictly positive.

Proof. Using $V(W, t)$ in equation (38) and the first-order conditions in Proposition 1, we obtain $C^*(t)$, $\omega^*(t)W_t$, $Z^*(t)$, and $P^*(t)$ as in equations (41)–(43), respectively.

According to equation (33), the worst-case uncertainty adjustment satisfies

$$u^*(t) = \frac{\theta}{\gamma V(W_t, t)} V_W W_t \omega \sigma^2 = \frac{\theta(\alpha - r)}{\theta + 1 - \gamma}. \quad (48)$$

Substitute $C^*(t)$, $\omega^*(t)W_t$, $Z^*(t)$, $P^*(t)$, and $u^*(t)$ into the HJB function in equation (35); we have the following:

$$\begin{aligned} 0 = & e^{-\rho t} a(t) b'(t) X^{\gamma-1}(t) + \frac{e^{-\rho t}}{\gamma} a'(t) X^\gamma(t) - \frac{\rho e^{-\rho t}}{\gamma} a(t) X^\gamma(t) - \frac{\lambda(t) e^{-\rho t}}{\gamma} a(t) X^\gamma(t) \\ & + \frac{\lambda(t) e^{-\rho t}}{\gamma} \left[\frac{\lambda(t)}{\mu(t)a(t)} \right]^{\gamma/(1-\gamma)} X^\gamma(t) + \frac{e^{-\rho t}}{\gamma} \left[\frac{1}{a(t)} \right]^{\gamma/(1-\gamma)} X^\gamma(t) \\ & + e^{-\rho t} a(t) X^{\gamma-1}(t) \left\{ \frac{(\alpha - r)^2}{(\theta - \gamma + 1)\sigma^2} X(t) + W_t r + \gamma(t) - \left[\frac{1}{a(t)} \right]^{1/(1-\gamma)} X(t) - \mu(t) \left[\left(\frac{\lambda(t)}{\mu(t)a(t)} \right)^{1/(1-\gamma)} X(t) - W_t \right] \right\} \\ & + \frac{1}{2} e^{-\rho t} a(t) X^\gamma(t) \frac{(\alpha - r)^2 (\gamma - 1)}{(\theta - \gamma + 1)^2 \sigma^2} - \frac{1}{2} e^{-\rho t} a(t) X(t) \frac{\theta(\alpha - r)^2}{(\theta + 1 - \gamma)^2 \sigma^2}. \end{aligned} \quad (49)$$

Given $X(t) = W_t + b(t)$, if we express W_t in $X(t)$, the above equation transforms into a homogeneous function of

$e^{-\rho t} X(t)^\gamma$. Eliminating $e^{-\rho t} X(t)^\gamma$ turns equation (49) into an ordinary differential equation with $a(t)$:

$$\left(\frac{1}{\gamma} - 1 \right) \left[1 + \left(\frac{\lambda(t)}{\mu^\gamma(t)} \right)^{1/(1-\gamma)} \right] a^{-(\gamma/(1-\gamma))}(t) + \left[\mu(t) + r - \frac{\rho}{\gamma} - \frac{\lambda(t)}{\gamma} + \frac{1}{2} \frac{(\alpha - r)^2}{(\theta - \gamma + 1)\sigma^2} \right] a(t) + \frac{1}{\gamma} a'(t) = 0. \quad (50)$$

With $H(t)$ and $K(t)$ in equations (46) and (47), we can rewrite equation (50) as

$$(1 - \gamma)K(t)a^{-(\gamma/1-\gamma)}(t) + (1 - \gamma)H(t)a(t) + ar(t) = 0, \quad (51)$$

which is a Bernoulli ordinary differential equation (ODE) that satisfies

$$\frac{da(t)}{dt} = p(t)a(t) + q(t)a(t)^n, \quad (52)$$

where

$$\begin{aligned} p(t) &= (\gamma - 1)H(t); \\ q(t) &= (\gamma - 1)K(t); \\ n &= -\frac{\gamma}{1 - \gamma}. \end{aligned} \quad (53)$$

This ODE has a general solution. The boundary condition that $V(T, W_T) = L(T, X_T)$ implies $a(T) = 1$. Thus,

$$a(t) = \left[e^{\int_t^T H(v)dv} + \int_t^T K(s)e^{\int_t^s H(v)dv} ds \right]. \quad (54)$$

Since $a(t)$ has a solution, $V(W_t, t)$ constitutes a solution to the HJB function. $\square$

Note $u^*(t) = (\theta(\alpha - r)/\theta + 1 - \gamma) > 0$ and note that $u^*(t)$ increases with the uncertainty level θ . The agent suspects that the reference model is potentially misspecified, and he negatively adjusts the expected stock returns in the decision rules. In particular, he perceives a lower expected stock return. Using $a(t)$ expressed in equation (45), we obtain the partial derivative of $a(t)$ with respect to θ :

$$\begin{aligned} \frac{\partial a(t)}{\partial \theta} &= -\frac{\gamma}{2}a^{-(\gamma/1-\gamma)}(t) \frac{(\alpha - r)^2}{(\theta + 1 - \gamma)^2 \sigma^2} \\ &\cdot \left\{ (T - t)e^{\int_t^T H(v)dv} + \int_t^T K(s)(s - t)e^{\int_t^s H(v)dv} ds \right\}. \end{aligned} \quad (55)$$

For the reasonable case of $\gamma < 0$, $(\partial a(t)/\partial \theta) > 0$. The sensitivity of $a(t)$ with respect to θ sheds light on the optimal insurance decision from the perspective of stock return uncertainty. The optimal insurance demand consists of a certainty part and an uncertainty part. The certainty part consists of wealth borrowed against future labor income, $\mu(t)b(t)$. This part has nothing to do with the uncertainty of stock returns. The other part is influenced by the agent's uncertainty aversion and equals $\mu(t)[(\lambda(t)/\mu(t)a(t))^{1/1-\gamma} - 1]X(t)$. Stock return uncertainty not only changes the insurance investment strategy, that is, the proportion of the total wealth assigned to the insurance product $\mu(t)[(\lambda(t)/\mu(t)a(t))^{1/1-\gamma} - 1]$ via $a(t)$, but also decreases the amount of total wealth $X(t)$. The next section examines the impact of stock return uncertainty on the total wealth.

3.3. Wealth Effect. To study the effect of uncertainty embedded in stock returns on insurance investment decisions

at different ages, we consider the path-dependent wealth dynamics. According to equation (37), the process of $X(t)$ that admits stock return uncertainty is

$$\begin{aligned} dX(t) &= dW_t + b(t)dt \\ &= [(r + \mu(t))b(t) - C(t) + P(t) \\ &\quad + W_t \omega(t)(\alpha - u - r) + W_t r]dt + W_t \omega(t)\sigma d\tilde{z}. \end{aligned} \quad (56)$$

We obtain the following proposition.

Proposition 3. *The CRRA agent has the following wealth process:*

$$\begin{aligned} X(t) &= X(0) \left(\frac{a(t)e^{-\rho t}}{a(0)} \right)^{1/1-\gamma} \\ &\cdot \exp \left\{ \left[A \cdot \frac{(\alpha - r)^2}{2(\theta + 1 - \gamma)^2 \sigma^2} + \frac{r}{1 - \gamma} \right] t \right. \\ &\quad \left. + \frac{1}{1 - \gamma} \int_0^t \eta(u)du + \frac{\alpha - r}{(\theta + 1 - \gamma)\sigma} \tilde{z}_t \right\}, \end{aligned} \quad (57)$$

where $A = 1 - \gamma + \gamma\theta/(1 - \gamma)$; the initial wealth $X(0) = W_0 + b(0)$. The expected total wealth at time t is expressed as

$$\begin{aligned} \mathbb{E}(X(t)) &= X(0) \left(\frac{a(t)e^{-\rho t}}{a(0)} \right)^{1/1-\gamma} \\ &\cdot \exp \left\{ \left[(A + 1) \cdot \frac{(\alpha - r)^2}{2(\theta + 1 - \gamma)^2 \sigma^2} + \frac{r}{1 - \gamma} \right] t \right. \\ &\quad \left. + \frac{1}{1 - \gamma} \int_0^t \eta(u)du \right\}. \end{aligned} \quad (58)$$

Proof. Substitute $C^*(t)$, $\omega^*(t)W_t$, $Z^*(t)$, and $u^*(t)$ into equation (56); we have

$$\begin{aligned} \frac{dX(t)}{X(t)} &= \left[\mu(t) + r - \frac{K(t)}{a(t)^{1/1-\gamma}} + \frac{(\alpha - r)^2(1 - \gamma)}{(\theta + 1 - \gamma)^2 \sigma^2} \right] dt \\ &\quad + \frac{\alpha - r}{(\theta - \gamma + 1)\sigma} d\tilde{z}. \end{aligned} \quad (59)$$

Since α , r , and σ are constants, equation (59) is a geometric Brown motion. We conjecture the solution to $X(t)$ as

$$\begin{aligned} X(t) &= X(0) \left(\frac{e^{-\rho t} a(t)}{a(0)} \right)^{1/1-\gamma} \\ &\cdot \exp \left\{ \left[A \cdot \frac{(\alpha - r)^2(1 - \gamma)}{2(1 - \gamma + \theta)^2 \sigma^2} + \frac{r}{1 - \gamma} \right] t \right. \\ &\quad \left. + \frac{1}{1 - \gamma} \int_0^t \eta(u)du + \frac{\alpha - r}{(\theta + 1 - \gamma)\sigma} \tilde{z}_t \right\}, \end{aligned} \quad (60)$$

where A is a constant to be determined. Applying Ito's lemma to $X(t)$ gives

$$\frac{dX(t)}{X(t)} = \left[\frac{1}{(1-\gamma)a(t)} \cdot \frac{\partial a(t)}{\partial t} - \frac{\rho}{1-\gamma} + A \frac{(\alpha-r)^2(1-\gamma)}{2(1-\gamma+\theta)^2\sigma^2} + \frac{r}{1-\gamma} + \frac{1}{1-\gamma} \eta(t) + \frac{(\alpha-r)^2}{2(1-\gamma+\theta)^2\sigma^2} \right] dt + \frac{\alpha-r}{(1-\gamma+\theta)\sigma} d\tilde{z}. \quad (61)$$

Using equation (45), we derive $a(t)$ and

$$\frac{\partial a(t)}{\partial t} = -(1-\gamma)H(t)a(t) - (1-\gamma)a^{-(\gamma/1-\gamma)}(t)K(t). \quad (62)$$

Substituting equation (62) into (61) gives

$$\begin{aligned} \frac{dX(t)}{X(t)} &= \left[-\frac{K(t)}{a(t)^{1/1-\gamma}} - H(t) - \frac{1}{1-\gamma}\rho + A \frac{(\alpha-r)^2(1-\gamma)}{2(\theta+1-\gamma)^2\sigma^2} + \frac{(\alpha-r)^2}{2(\theta+1-\gamma)^2\sigma^2} + \frac{r}{1-\gamma} + \frac{\eta(t)}{1-\gamma} \right] dt + \frac{\alpha-r}{(1-\gamma+\theta)\sigma} d\tilde{z} \\ &= \left[\mu(t) + r - \frac{K(t)}{a^{1/1-\gamma}} + \frac{(\alpha-r)^2}{2(\theta+1-\gamma)^2\sigma^2} \left(-\frac{\gamma}{(1-\gamma)}(\theta+1-\gamma) + A(1-\gamma) + 1 \right) \right] dt + \frac{\alpha-r}{(\theta+1-\gamma)\sigma} d\tilde{z}. \end{aligned} \quad (63)$$

Matching equation (63) to (59) gives

$$\frac{1}{2} \left(-\frac{\gamma}{1-\gamma}(\theta+1-\gamma) + A(1-\gamma) + 1 \right) = 1-\gamma. \quad (64)$$

Thus,

$$A = 1 + \frac{\gamma\theta}{(1-\gamma)^2}. \quad (65)$$

We have proved that the solution to the total wealth satisfies the differential function in equation (56). Expressing the expectation of total wealth $X(t)$ with the standard Brown motion $\tilde{z}_t \sim N(0, t)$ gives

$$\mathbb{E}(X(t)) = X(0) \left(\frac{a(t)e^{-\rho t}}{a(0)} \right)^{1/1-\gamma} \exp \left\{ \left[(A+1) \cdot \frac{(\alpha-r)^2}{2(\theta+1-\gamma)^2\sigma^2} + \frac{r}{1-\gamma} \right] t + \frac{1}{1-\gamma} \int_0^t \eta(u) du \right\}. \quad (66)$$

When $P^*(t)$ is negative, the agent chooses to supply insurance to the market. We examine the impact of stock return uncertainty on his demand/supply of life insurance. Let $P^*(t)$ be zero as the switch point between buying life insurance and selling life insurance; we obtain

$$X(t)|_{P^*(t)=0} = \frac{1}{1-(\lambda(t)/\mu(t)a(t))^{1/1-\gamma}} b(t). \quad (67)$$

Lemma 1. *If the agent's total wealth exceeds a certain threshold, that is, $\mathbb{E}(X(t)) > X(t)|_{P^*(t)=0}$, the agent provides life insurance to the market, that is,*

$$P^*(t) < 0 \Leftrightarrow \mathbb{E}(X(t)) > X(t)|_{P^*(t)=0}, \quad (68)$$

It is sensible that the agent does not leave a legacy $Z^*(t)$ greater than the total wealth $X(t)$; thereby, equation (44) implies

$$\zeta(t) = \left(\frac{\lambda(t)}{\mu(t)a(t)} \right)^{1/1-\gamma} < 1. \quad (69)$$

The decision to demand or supply life insurance depends on the expected total wealth $\mathbb{E}(X(t))$ relative to the labor wealth $b(t)$. Given $\zeta(t) < 1$, $\mathbb{E}(X(t))$ and the threshold $X(t)|_{P^*(t)=0}$ are decreasing in θ . According to Section 4, the magnitude of the decrease in $X(t)|_{P^*(t)=0}$ is much smaller compared to the magnitude of the decrease in $\mathbb{E}(X(t))$, given the same increase in θ . Additional numerical analysis is presented in Section 4.

3.4. Uncertainty-Induced Utility Loss. This section estimates wealth-measured utility loss when the agent follows a suboptimal strategy; that is, he applies the optimal strategy under the uncertainty-free model in the uncertainty aversion model. It provides a yardstick to measure the improvement in utility by undertaking robust optimal decisions. To more

intuitively illustrate the utility loss, we use the indifference curve to transform the implied utility loss into percentage of wealth.

We solve equations (41)–(43) with $\theta = 0$ for the suboptimal decision set $\mathcal{A}^{\text{Sub}}$. The agent exercises the following suboptimal controls:

$$C^{\text{Sub}}(t) = \left[\frac{1}{a^0(t)} \right]^{1/(1-\gamma)} (W_t + b(t)),$$

$$\omega^{\text{Sub}}(t)W_t = \frac{\alpha - r}{(1-\gamma)\sigma^2} (W_t + b(t)),$$

$$P^{\text{Sub}}(t) = \mu(t) \left\{ \left[\left(\frac{\lambda(t)}{\mu(t)a^0(t)} \right)^{1/(1-\gamma)} - 1 \right] X(t) + b(t) \right\}. \quad (70)$$

The suboptimal control functions are used in the HJB function in equation (26), based on which $u(t)$ associated with the worst-case model is attained. The suboptimal objective function is conjectured as

$$V^{\text{Sub}}(W, t) = \frac{a^{\text{Sub}}(t)e^{-\rho t}}{\gamma} [X(t)]^\gamma. \quad (71)$$

Solving the model yields the worst-case adjustment to the expected stock return:

$$u^{\text{Sub}}(t) = \frac{\theta}{\gamma V^{\text{Sub}}(W, t)} V_W^{\text{Sub}} W_t \omega^{\text{Sub}}(t) \sigma^2; \quad (72)$$

$$a^{\text{Sub}}(t) = \left[e^{\int_t^T H^{\text{Sub}}(v) dv} + \int_t^T K(s) e^{\int_s^T H^{\text{Sub}}(v) dv} ds \right]^{1-\gamma}, \quad (73)$$

where $K(t)$ is as in equation (47) and $H(t)$ is expressed as

$$H^{\text{Sub}}(t) = -\frac{\rho}{1-\gamma} + \frac{\gamma}{1-\gamma} \left(\mu(t) + r + \frac{(\alpha - r)^2 (1-\gamma - \theta)}{2(1-\gamma)^2 \sigma^2} \right) - \frac{\lambda(t)}{1-\gamma}. \quad (74)$$

Following Branger and Larsen [13], we transform the loss in utility into the percentage loss in wealth. Denote $\psi(t)$ as the percentage of initial wealth the agent gives up for robust decisions. The loss function can be expressed as

$$V(X(t)(1-\psi(t)), t; C^{\text{Sub}}(t), \omega^{\text{Sub}}(t), P^{\text{Sub}}(t)) \\ = V^{\text{Sub}}(X(t), t; C^{\text{Sub}}(t), \omega^{\text{Sub}}(t), P^{\text{Sub}}(t)). \quad (75)$$

Using the definition of utility loss and equation (75), we compute the percentage loss in wealth as

$$\psi(t) = 1 - \sqrt[\gamma]{\frac{a^{\text{Sub}}(t)}{a(t)}}, \quad (76)$$

where $a(t)$ and $a^{\text{Sub}}(t)$ are given in equations (45) and (73), respectively.

4. Numerical Analysis

This section conducts numerical analysis to examine the qualitative implications of stock return uncertainty. The

TABLE 1: Benchmark parameter values.

Parameter	Value
$\gamma(t)$	$1 \cdot e^{\rho t}$
α	0.1
$\lambda(t)$	$0.005 + 0.001125t$
$\mu(t)$	$0.005 + 0.001125t$
$\eta(t)$	0
r	0.04
σ	0.2
ρ	0.03
$1-\gamma$	4
T	80

benchmark parameter values are given in Table 1. With a slight abuse of notation, we use θ to represent the level of uncertainty and set its values between zero and five.

The parameter values in Table 1 are also used in the previous literature, for example, Pliska and Ye [22] and Kwak et al. [27]. For simplicity, we set security loading $\eta(t) = 0$, so the insurance payout ratio equals $\lambda(t)$. We use a linear mortality rate to study the dynamic change in the optimal controls as the agent ages. Labor income growth is assumed to follow an exponential function, and we normalize the initial labor income to the unit of one. We normalize the initial wealth $W_0 = 10$.

4.1. Wealth. Figure 1 depicts $X(t)$, W_t , and $b(t)$ in the uncertainty-free model. The age-dependent wealth process sheds light on the agent's optimal decision rules at different ages. The agent's total wealth reaches the peak at the age around 65. The agent borrows against his future labor wealth to purchase stock and life insurance. W_t turns out to be negative at ages around 40–70. Figure 2 depicts the change in total wealth for different uncertainty levels at different ages. The total wealth decreases with stock return uncertainty. The pattern is, however, more apparent at an older age than at a younger age. Uncertainty aversion makes the agent allocate less wealth into risky assets as he becomes older. Moreover, the agent has less borrowing power as his future labor wealth diminishes.

4.2. Consumption and Investment. Figure 3 shows that consumption gradually and monotonically increases with the agent's age. The mortality rate is higher at an older age; thereby the agent rationally chooses to consume more and invest less compared to the younger him. However, the agent consumes less than an otherwise uncertainty-neutral agent, which is explained by the notion that the total wealth decreases with stock return uncertainty.

Figure 4 shows that investment in the stock decreases with the level of stock return uncertainty. Intuitively, return uncertainty reduces the attractiveness of stock relative to the life insurance product and the risk-free asset. Uncertainty aversion appears to change the pattern of investment in the stock at different ages. In the absence of uncertainty, the agent first increases his stock investment as he grows at young ages and then reverts to decrease investment in the

FIGURE 1: Total wealth at different ages.

FIGURE 2: Change in total wealth with θ .

FIGURE 3: Optimal consumption.

FIGURE 4: Optimal investment.

stock after a certain age. The peak appears around the age of 65 and drops towards zero as time passes by. This hump-shaped investment pattern, which can be traced to the hump-shaped wealth pattern, is consistent with that in Farhi and Panageas [28]. When $\theta = 5$, there is no investment peak as investment in the stock keeps decreasing as the agent ages. Uncertainty arising from the stock affects the agent's total wealth and subsequently alters his investment behavior. Shrinking wealth due to stock return uncertainty no longer supports increasing stock investment at younger ages, resulting in a monotonic declining pattern.

4.3. Life Insurance. Figure 5 shows that the optimal insurance demand increases with the level of uncertainty at all ages. The result implies that the agent shifts some investment in the stock to life insurance, confirming that the insurance market and the stock market, to some degree, substitute. Life insurance provides a way to hedge and evade the uncertainty embedded in the stock. This effect is more prominent for the middle-aged and elder agent. A young agent's demand for life insurance is low. When it is close to the end of the financial planning horizon, an elder agent consumes more, which represses demand for life insurance. As a result, the demand of younger and elder agents for life insurance is less sensitive to stock return uncertainty.

Figure 6 shows that, under different planning horizons, for example, $T = 60$ and $T = 100$, respectively, the patterns of demand for life insurance with respect to stock return uncertainty are in general the same. However, demand for life insurance is much less under a short planning horizon than under a long planning horizon. The result implies that people would demand more life insurance as they expect to live longer and plan financially for a longer horizon. Moreover, the increase in demand caused by the stock return uncertainty is also more significant within a longer planning horizon.

Lemma 1 shows that the agent becomes an insurance seller when his total wealth $\mathbb{E}(X(t))$ exceeds the threshold $X(t)|_{P^*(t)=0}$. Figure 7 shows that, for an agent at the age of 30 with initial wealth $W_0 = 20$, the agent is an insurance seller as $P^*(30) < 0$ in the absence of uncertainty. The expected total wealth drops sharply in the presence of uncertainty. The value of the threshold $X(t)|_{P^*(t)=0}$ also decreases, but at a much lower rate. When $\theta = 1.2$, the agent switches to buy

FIGURE 5: Demand of insurance at different levels of uncertainty.

FIGURE 6: Demand of insurance for different financial planning horizons. (a) Insurance demand ($T=60$). (b) Insurance demand ($T=60$).

FIGURE 7: Buying vs. selling insurance.

insurance as $\mathbb{E}(X(t))$ falls below the threshold $X(t)|_{P^*(t)=0}$. By reducing the agent's total wealth, the uncertainty decreases the propensity of the agent serving as an insurance seller.

Figure 8 shows that when the agent is endowed with an ultrahigh initial wealth, that is, $W_0 = 100$, this agent is

FIGURE 8: Supply of insurance and θ .

wealthy enough to supply insurance even at a high level of uncertainty. The amount of supply, however, decreases with the level of uncertainty. Regardless of being an insurance buyer or supplier, the agent would be more conservative facing stock return uncertainty. In the insurance market, agents would demand more insurance, while the supply of insurance tends to fall, implying that insurance premium increases in equilibrium. Of course, an equilibrium model is required to explore such implication in a rigorous manner.

FIGURE 9: Insurance demand and α . FIGURE 11: Insurance demand and γ . FIGURE 10: Insurance demand and σ . FIGURE 12: Insurance demand and $\lambda(t)$.

Stock market uncertainty also remarkably affects the relationships between the agent's life insurance decision and other structural factors. Figures 9 and 10 depict life insurance demand with respect to the expected stock return α and stock return volatility σ , respectively, assuming that the agent is 30 years old. The uncertainty tends to reduce the sensitivity of insurance demand with respect to these stock return characteristics. The findings are consistent with the previous result in that the agent tends to reduce investment in the stock when he is skeptical about the expected returns.

Figure 11 depicts life insurance demand with respect to the agent's risk/time preferences, $1 - \gamma$, at different levels of uncertainty. For a CRRA agent, $1 - \gamma$ also captures the time preferences of consumption. An agent with a higher $1 - \gamma$ buys less insurance, as he values more current consumption relative to future consumption. The finding is consistent with those of Kwak and Lim [23] and Huang et al. [24]. Stock return uncertainty reduces the sensitivity of insurance demand to the agent's risk preferences.

Figure 12 shows that, in the absence of uncertainty, an increase in the mortality rate leads to a reduction in insurance demand. Stock return uncertainty could alter such a pattern, causing the agent's demand for insurance to increase with the mortality rate. The rationale is that the mortality rate plays two roles in insurance decision making.

On the one hand, it adversely affects the insurance payout $P(t)/\lambda(t)$. Thus, higher $\lambda(t)$ reduces the utility brought by life insurance. On the other hand, the mortality rate increases the probability of obtaining life insurance compensation within the financial planning horizon—an agent with a higher mortality rate is more willing to buy insurance because there is a higher probability of receiving the insurance payment. When stock return uncertainty is low, the first effect dominates the second effect. However, when the level of uncertainty is sufficiently high, the agent values more the probability of receiving insurance compensation within the limited planning horizon than considering the amount of insurance payment.

4.4. Utility Loss. Figure 13 shows that utility loss due to uncertainty, $\psi(t)$, is increasing in the level of uncertainty, confirming that the agent is willing to give up a certain fraction of wealth for robust decision making. Such impact, however, decreases in age—the utility loss drops as the agent becomes older. A younger agent has a longer future horizon that is subject to stock return uncertain, so his optimal controls are more significantly shaped by the uncertainty.

FIGURE 13: Uncertainty-induced utility loss.

5. Conclusion

This paper formulates a continuous-time rational expectations model to examine the effects of stock return uncertainty on life insurance. The model considers uncertainty aversion as the agent suspects that stock return in the stochastic process is potentially misspecified, and it imposes an uncertainty penalty to the objective function in reflecting his skeptical and conservative perspective. Facing stock return uncertainty, the agent shifts some of the investment in the stock to life insurance, confirming that the insurance market and the stock market are partially supplementary. Life insurance is used as a way to circumvent the uncertainty embedded in the stock. Overall, the agent would behave more conservatively in the insurance market facing stock return uncertainty. Agents would demand more insurance at a falling supply, implying that insurance premium might increase in equilibrium. We leave it to future research to develop an equilibrium model to explore such implications rigorously.

Data Availability

All data generated or analyzed during this study are included in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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