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Research Article

Compound Binomial Model with Batch Markovian Arrival Process

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A compound binomial model with batch Markovian arrival process was studied, and the specific definitions are introduced. We discussed the problem of ruin probabilities. Specially, the recursion formulas of the conditional finite-time ruin probability are obtained and the numerical algorithm of the conditional finite-time nonruin probability is proposed. We also discuss research on the compound binomial model with batch Markovian arrival process and threshold dividend. Recursion formulas of the Gerber–Shiu function and the first discounted dividend value are provided, and the expressions of the total discounted dividend value are obtained and proved. At the last part, some numerical illustrations were presented.

1. Introduction

The compound binomial model is a discrete time analogue of compound Poisson model. In the compound binomial model, the counting process is a binomial process. From the compound binomial model proposed by Gerber [1], a series of papers and books have studied this model (see Gerber [1]; Shiu [2]; Cossette [3]; Wu [4]; Peng et al. [5] and references therein).

As a class of important stochastic point processes, the batch Markovian arrival process (BMAP), proposed by Lucantoni [6], is dense in the class of stationary point processes. BMAP is used to model the stochastic processes in finance, computer, reliability, communication, and inventory conveniently. Particular BMAPs are the batch Poisson arrival process, the Markovian arrival process (MAP), many batch arrival processes with correlated interarrival times and batch sizes, and superpositions of these processes. We note that the MAP, introduced by Neuts [7], includes phase-type (PH) renewal processes and nonrenewal processes such as the Markov modulated Poisson process (MMPP). Like Ahn et al. [8], Eric et al. [9], Artalejo et al. [10], Dong and Liu [11] and many authors have studied the compound Poisson process with MAP.

Inspired by Ahn et al. [8], Badescu et al. [12], Eric et al. [9], Artalejo et al. [10], and Dong and Liu [11], we discuss the compound Binomial model with BMAP. In this model, the counting process is a BMAP, which is a reasonable assumption. For example, an insurance company, which accepts the car insurance policies, might need to deal with several traffic accidents a day. Moreover, in different circumstances, the probability of traffic accident and the claim sizes are of big differences. So it may be more reasonable that the premium rate of car insurance is different in different environments. Therefore, we assume that the premium rate, probability of the claim occurring, and the claim amount are all influenced by the phase process of BMAP. Also, we study the compound binomial model with BMAP and threshold dividend. This study has certain guiding significance in insurance company and shareholders.

This paper is structured as follows: the specific definition of a compound binomial model with BMAP is introduced in Section 2. In Section 3, we discuss the ruin probabilities. Specially, the recursion formulas of the conditional finite-time ruin probability are obtained and the numerical algorithm is proposed. In Section 4, we also discuss research on the compound binomial model with BMAP and threshold dividend. The recursion formulas of the Gerber–Shiu function and the first discounted dividend value are provided, and the explicit expression of the total discounted dividend values are obtained and proved. Finally, we present some numerical examples to illustrate in Section 5.

2. Model

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with filtration $\{\mathscr{F}_t\}$ containing all objects defined in this paper. Assume that $\{\mathscr{F}_t\}$ satisfies the usual conditions, i.e., $\{\mathscr{F}_t\}$ is right-continuous and $\mathbb{P}$-complete. At first, we will introduce the compound binomial model and the batch Markovian arrival process.

2.1. Compound Binomial Model. In the compound binomial model, $\{C(n), n = 0, 1, 2, \ldots\}$ denotes the surplus process of an insurer and is given by

$$C(n) = u + t - S(n), \quad n = 0, 1, 2, \ldots, \tag{1}$$

where the initial surplus u is a nonnegative integer, $S(n)$ is the aggregate claim up to time n, which is described by

$$S(n) = X_1\xi_1 + X_2\xi_2 + \cdots + X_n\xi_n, \tag{2}$$

and $S(0) = 0$. In any time period, the probability with only a claim occurrence is $\theta, 0 < \theta \leq 1$, and the probability with no claim occurrence is $\lambda = 1 - \theta$. We denote by $\xi_n = 1$ the event where a claim occurs in the time period $(n-1, n]$, and we denote by $\xi_n = 0$ the event where no claim occurs in the time period $(n-1, n]$. The occurrences of claims in different time periods are independent events. $X = \{X_n, t = 1, 2, \ldots\}$ denotes the claim amount that probably occurs at time t, and $X_1, X_2, X_3, \ldots$ are mutually independent, identically distributed (i.i.d.), positive integer-valued random variables, which have a common discrete distribution $P(X = k) = f(k), k = 1, 2, \ldots$. Denote $F(k) = P(X \leq k) = \sum_{j=1}^{k} f(j)$ with $F(0) = 0$. And the claim amounts $X = \{X_n, n = 1, 2, \ldots\}$ are independent of $\xi = \{\xi_n, n = 1, 2, \ldots\}$.

2.2. Batch Markovian Arrival Process

Definition 1. Let $E = \{e_1, e_2, \ldots, e_m\} (m \geq 1)$. Given a series of $m \times m$ matrixes $D_0 = (d_{ij}^0)$ and $D_k = (d_{ij}^k) (k \in \mathbb{N}_+)$ which satisfied the following conditions:

(1) $D_k (k \in \mathbb{N}_+)$ are substochastic matrixes

(2) The matrix $I - D_0$ is nonsingular

(3) $P = \sum_{k=0}^{\infty} D_k = (q_{ij})$ is a stochastic matrix, and $q_{ij} = \sum_{k=0}^{\infty} d_{ij}^k$

Then, $(D_0, D_k (k \in \mathbb{N}_+))$ is called the numerical characteristic of discrete-time batch Markovian arrival process.

Proposition 1. *Assume that $(D_0, D_k (k \in \mathbb{N}_+))$ be the numerical characteristic of discrete-time batch Markovian arrival process. Then,*

(1) $P = \sum_{k=0}^{\infty} D_k$ is a $m \times m$ conservative matrix, and each state is sojourned.

(2) Let $E^* = \{(n, j), n \in \mathbb{N}, e_j \in E\}$ and

$$P^* = \begin{pmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ 0 & D_0 & D_1 & D_2 & \cdots \\ 0 & 0 & D_0 & D_1 & \cdots \\ 0 & 0 & 0 & D_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3}$$

Then, P^* is a $E^* \times E^*$ conservative matrix, and each state is sojourned.

(3) P and P^* are both regular.

Definition 2. Given the numerical characteristics of batch Markovian arrival process $(D_0, D_k (k \in \mathbb{N}_+))$, let $X^* = \{X^*(t), t \in \mathbb{N}\}$ be a stochastic process with transition probability matrix P^* and $X^*(t) = (N(t), J(t))$ be a two-dimensional discrete-time batch Markovian process. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T$ be the initial probability distribution vector of X^*, which satisfied $\sum_{j=1}^{m} \alpha_j = 1$. We call X^* as a discrete-time batch Markovian arrival process (DTBMAP); for short, we can denote it as DTBMAP $(\alpha^T, D_0, D_k (k \in \mathbb{N}_+))$. $N = \{N(t), t \in \mathbb{N}\}$ is called the counting process and $J = \{J(t), t \in \mathbb{N}\}$ is called the phase process.

The BMAP is one of the most flexible stochastic processes and is defined as a specific Markov chain (MC). More precisely, the BMAP consists of two different processes with discrete state space. One process represents the dynamics of internal state called phase process, and the other process corresponds to the number of events, i.e., the counting process like a binomial process. The phase process is usually modeled by a MC, and the counting process is modulated by the phase process. In fact, Markov-modulated Bernoulli process and discrete-time platoon arrival process, which are specific and subclasses of BMAP, have been utilized to evaluate the information communication systems based on the queueing analysis and finance. BMAP enables one to capture the realistic assumptions as much as possible and provide solutions that practitioners can implement.

2.3. Modified Model. The model we considered in this paper can be described by

$$U(n) = u + \sum_{k=1}^{n} c(k) - \sum_{k=1}^{N(n)} X(k), \quad n = 0, 1, 2, \ldots, \tag{4}$$

where $N(n)$ is the counting process of DTBMAP $(\alpha^T, D_0, D_k (k \in \mathbb{N}_+))$ with state space $E^* = \{(n, j), n \in \mathbb{N}, j \in E = \{e_1, e_2, e_3, \ldots, e_m\}\}$. $c(k)$ and $X(k)$, representing the size of the kth premium and claim, respectively, are both dependent on the state of the phase process $J(n)$ of DTBMAP $(\alpha^T, D_0, D_k (k \in \mathbb{N}_+))$. That is, given

$J(k) = e_i (\in E)$, $c(k)$ is i.i.d. variable with the common binomial distribution $B(p_i)(p_i \in (0,1], q_i = 1 - p_i)$ and $X(k)$ is i.i.d. positive and integer-valued stochastic series with the mean μ_i and the common distribution $P(X(k) = x) = f_i(x), x = 1, 2, \ldots$. Denote $F_i(k) = P(X(k) \geq x) = \sum_{j=1}^{k} f_i(j)$ with $F_i(0) = 0$. And assume $J(n)$, $N(n), X(n)$, and $c(n)$ are independent.

We should note the following: (1) d_{ii}^0 gives the probability of no state changes without claim arrivals; (2) $d_{ij}^0 (i \neq j)$ gives the probability of state i changes to state j without claim arrivals; (3) $d_{ii}^k (k \in \mathbb{N}_+)$ gives the probability of no state changes with k claims arrival; (4) $d_{ij}^k (i \neq j, k \in \mathbb{N}_+)$ gives the probability of state i changes to state j with k claims arrival. Furthermore, every insurer would want to make a profit. That is, the expected claim size over a single period is strictly inferior to the premium size, i.e.,

$$\sum_{e_i \in E} \sum_{e_j \in E} \sum_{k=0}^{\infty} \alpha_i d_{ij}^k p_j = (1 + \lambda) \sum_{e_i \in E} \sum_{e_j \in E} \sum_{k=1}^{\infty} \alpha_i d_{ij}^k k \mu_j, \tag{5}$$

where $\lambda > 0$ is the safety factor.

Before introducing the main results, we should point out that the DTBMAP is very general. On the one hand, it may represent a renewal process where the interclaim times follow binomial distributions and negative binomial distribution or even discrete-time phase-type distributions. On the other hand, it allows for situations where numbers of claim time and claim size random variables are dependent.

Remark 1. When $E = \{e_1\}, D_0 = p, D_1 = q = 1 - p,$ and $D_k = 0 (k \geq 2, k \in \mathbb{N}_+)$, the compound binomial model with DTBMAP $(\alpha^T, D_0, D_k (k \in \mathbb{N}_+))$ degenerates to the compound binomial model. When $E = \{e_1, e_2, \ldots, e_m\} (m \geq 2)$, $D_0 = \mathrm{diag}(p)_{m\times m}, D_1 = \mathrm{diag}(1 - p)_{m\times m}$, and $D_k = 0_{m\times m} (k \geq 2, k \in \mathbb{N}_+)$, then it is the Markov-modulated compound binomial model, a degenerate case of the compound binomial model with DTBMAP $(\alpha^T, D_0, D_k (k \in \mathbb{N}_+))$.

3. Ruin Probability

3.1. Introduction. We define the time of ruin as

$$\tau = \inf\{k \in \mathbb{N}, U(k) < 0\}. \tag{6}$$

If ruin never occurs, $\tau = \infty$. Also, let us define the conditional finite-time ruin probability as

$$\psi(u, n \mid e_i) = P(\tau \leq n \mid J(0) = e_i)(e_i \in E) \tag{7}$$

and conditional finite-time non-ruin probability as

$$\varphi(u, n \mid e_i) = 1 - \psi(u, n \mid e_i)(e_i \in E). \tag{8}$$

Denote

$$\overrightarrow{\varphi}(u, n) = (\varphi(u, n \mid e_1)\varphi(u, n \mid e_2)\varphi(u, n \mid e_3) \cdots \varphi(u, n \mid e_m))_{1\times m}^T. \tag{9}$$

Obviously, we can see that the unconditional finite-time ruin and nonruin probability, $\psi(u, n)$ and $\varphi(u, n)$, can be derived from the conditional ones with following formulas, respectively:

$$\begin{aligned} \psi(u, n) &= \sum_{e_i \in E} \alpha_i \psi(u, n \mid e_i), \\ \varphi(u, n) &= \sum_{e_i \in E} \alpha_i \varphi(u, n \mid e_i) = \alpha^T \overrightarrow{\varphi}(u, n). \end{aligned} \tag{10}$$

For convenience, we also define the infinite-time ones by simply letting $n \longrightarrow \infty$ in our previous conditional or unconditional finite-time ruin or nonruin probabilities. Thus, if we obtain the conditional finite-time ruin probability, all of ruin probabilities of this model are solved.

3.2. Main Result. For convenience, in the next article, we denote

$$\mathbf{h}^j(x) = \left(h_1^j(x) \;\; h_2^j(x) \;\; h_3^j(x) \;\; \cdots \;\; h_m^j(x) \right)_{1\times m}^T, \tag{11}$$

where $h_i^j(x)(i = 1, 2, \ldots, m)$ is a function of x.

Theorem 1. *In the compound binomial model with DTBMAP $(\alpha^T, D_0, D_k (k \in \mathbb{N}_+))$, the conditional finite-time nonruin probabilities satisfy the following recursive formula:*

$$\begin{aligned} \overrightarrow{\varphi}(u, n+1) &= D_0 \times \mathbf{P} \times \overrightarrow{\varphi}(u+1, n) \\ &\quad + D_0 \times (\mathbf{I} - \mathbf{P}) \times \overrightarrow{\varphi}(u, n) \\ &\quad + \sum_{k=1}^{\infty} D_k \times \mathbf{P} \times \mathbf{h}^1(u.n) \\ &\quad + \sum_{k=1}^{\infty} D_k \times (\mathbf{I} - \mathbf{P}) \times \mathbf{h}^2(u.n), \end{aligned} \tag{12}$$

where $\mathbf{P} = diag(p_1, p_2, \ldots, p_m)$ *and* $I = diag(1, 1, \ldots, 1)$. *Then,*

$$\begin{aligned} h_i^1(u.n) &= \sum_{y=0}^{u} f_i^{*(k)}(y+1)\varphi(u - y, n \mid e_i), \\ h_i^2(u.n) &= \sum_{y=1}^{u} f_1^{*(k)}(y)\varphi(u - y, n \mid e_1), \end{aligned} \tag{13}$$

where $g^{*(k)}(y)$ represents the kth convolution of $g(y)$. And

$$\begin{aligned} \forall e_i \in E, \varphi(0, 1 \mid e_i) &= 1 - \sum_{e_j \in E} d_{ij}^0 - \sum_{e_j \in E} d_{ij}^1 p_j f_j(1), \\ \forall u \in \mathbb{N}, \forall e_i &\in E, \varphi(u, 0 \mid e_i) = 1. \end{aligned} \tag{14}$$

Proof. We can separate some possible cases by conditioning on the r.v.'s $J(1), N(1), c(1)$, and $X(1)$. There are possible cases as follows:

(1) No state changes and no claim arrivals

(2) State i changes to state $j (i \neq j)$ and no claim arrivals

(3) No state changes and $k (k \geq 1)$ claims arrival

(4) State i changes to state $j (i \neq j)$ and $k (k \geq 1)$ claims arrival

Then, the following formula can be easily derived by using the formula of full probability and the Markov property. For all $e_i \in E$, we have

$$\varphi(u,n+1\mid e_i) = \sum_{e_j\in E} d_{ij}^0\left[p_j\varphi\left(u+1,n\mid e_j\right) + q_j\varphi\left(u,n\mid e_j\right)\right] + \sum_{e_j\in E}\sum_{k=1}^{\infty} d_{ij}^k\left[p_j\sum_{y=1}^{u+1} f_j^{*(k)}(y)\varphi\left(u+1-y,n\mid e_j\right) + q_j\sum_{y=1}^{u} f_j^{*(k)}(y)\varphi\left(u-y,n\mid e_j\right)\right]. \quad (15)$$

Thus, (12) is derived when we rewrite (15) into the matrix form.

In order to proof the following theorem, some definitions are required to be introduced. Let $V_i(n) = \sum_{l=1}^{N(n)} I_{\{J(l)=e_i\}}$ be the elapsed time by the phase process $J(n)$ in state e_i over the first k periods where $I_A = 1$ if A is true and $I_A = 0$ if A is false. We also denote $W_i(n)$ the amount of decrease of the surplus process over the first n periods when the phase process $J(n)$ is in state e_i, i.e., $W_i(n) = \sum_{l=1}^{N(n)} (X(l) - c(l)) I_{\{J(l)=e_i\}}$. Denote $W(n)$ as the amount of decrease of the surplus process over the first n periods. Furthermore, $X_i(n)$ and $c_i(n)$ are denoted as the nth premium amount and claim size when the phase process $J(n)$ is in state e_i, respectively.

Proposition 2. *The infinite-time ruin probability tends to 0 as initial surplus u tends to ∞.*

Proof. Obviously, we can see $W(n) = \sum_{e_i\in E} W_i(n)$.

Taking the limit as $n \longrightarrow \infty$ of $W(n)/n$ yields

$$\lim_{n\longrightarrow\infty}\frac{W(n)}{n} = \lim_{n\longrightarrow\infty}\sum_{e_i\in E}\frac{W_i(n)}{n} = \sum_{e_i\in E}\lim_{n\longrightarrow\infty}\frac{W_i(n)}{V_i(n)}\times\frac{V_i(n)}{n}. \quad (16)$$

Since $J(n)$ is irreducible and ergodic, it follows that

$$\lim_{n\longrightarrow\infty}\frac{V_i(n)}{n} = \xi_i, \quad (17)$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ is the stationary distribution of $J(n)$.

We can easily see that for given $J(n) = e_i$, $X(n)$ and $c(n)$ are both i.i.d. and $W_i(n)$ is distributed as $\sum_{l=1}^{V_i(n)} (X_i(l) - c_i(l))$. Because the phase process $J(n)$ is irreducible, $\lim_{n\longrightarrow\infty} V_i(n) = \infty$:

$$\lim_{n\longrightarrow\infty}\frac{W_i(n)}{V_i(n)} = \lim_{n\longrightarrow\infty}\frac{\sum_{l=1}^{V_i(n)}(X_i(l)-c_i(l))}{V_i(n)} = \lim_{n\longrightarrow\infty}\frac{\sum_{l=1}^{n}(X_i(l)-c_i(l))}{n} = \lim_{n\longrightarrow\infty}\frac{L_i(n)}{n}. \quad (18)$$

where $L_i(n) = \sum_{l=1}^{n} (X_i(l) - c_i(l))$. Therefore, $\{L_i(n)\}$ is a random walk for $e_i \in E$ and from the strong law of large numbers, we can find $\forall e_i \in E$:

$$\lim_{n\longrightarrow\infty}\frac{L_i(n)}{n} = E\left[X_i(l) - c_i(l)\right] = \mu_i - p_i. \quad (19)$$

By combining (16) to (19), we can obtain that

$$\lim_{n\longrightarrow\infty}\frac{W(n)}{n} = \sum_{e_i\in E}\alpha_i(\mu_i - p_i). \quad (20)$$

Equation (20) and the safety loading condition imply that $\lim_{n\longrightarrow\infty} W(n) = -\infty$ and thus ensure that $\max_{n\in\mathbb{N}} W(n)$ is finite. Consequently,

$$\lim_{n\longrightarrow\infty}\psi(u) = \lim_{n\longrightarrow\infty} P\left(\max_{n\in\mathbb{N}} W(n) > u\right) = 0. \quad (21)$$

Theorem 2. *In the compound binomial model with DTBMAP* $(\alpha^T, D_0, D_k(k\in\mathbb{N}_+))$ *, the numerical algorithm proposed to obtain the conditional finite-time nonruin probabilities is as follows:*

Fix $\varphi(u\mid e_i) = 1$ *for* $u = n, n+1, n+2, \ldots$ *and* $e_i \in E$.

Find $\varphi(u\mid e_i)$ *for* $u = 0, 1, 2, \ldots, n$ *and* $e_i \in E$ *by solving the following system of* $m\times n$ *equations with* $m\times n$ *unknown parameters:*

$$\varphi(u\mid e_i) = \sum_{e_j\in E} d_{ij}^0\left[p_j\varphi\left(u+1\mid e_j\right) + q_j\varphi\left(u\mid e_j\right)\right] + \sum_{e_j\in E}\sum_{k=1}^{\infty} d_{ij}^k\left[\sum_{y=1}^{u+1} p_j f_j(y)\varphi\left(u+1-y\mid e_j\right) + \sum_{y=1}^{u} q_j f_j(y)\varphi\left(u-y\mid e_j\right)\right]. \quad (22)$$

Proof. First, by conditioning, respectively, on the random variables $J(1), N(1), c(1)$, and $X(1)$, four cases probably occurred. And from the stationarity of the surplus process, we can find

$$\varphi(u \mid e_i) = \sum_{e_j \in E} d_{ij}^0 \left[p_j \varphi(u+1 \mid e_j) + q_j \varphi(u \mid e_j) \right] + \sum_{e_j \in E} \sum_{k=1}^{\infty} d_{ij}^k \left[\sum_{y=1}^{u+1} p_j f_j(y) \varphi(u+1-y \mid e_j) + \sum_{y=1}^{u} q_j f_j(y) \varphi(u-y \mid e_j) \right], \tag{23}$$

for $e_i \in E$ and $u \in \mathbb{N}$. Given that $\varphi(u \mid e_i) = 1$ for $u = n, n+1, n+2, \ldots$ and $e_i \in E$, we must solve the system of $m \times n$ equations-$m \times n$ unknown parameters given by equation (23) for $u = 0, 1, 2, \ldots, n$ and $e_i \in E$.

4. Compound Binomial Model with BMAP and Dividend

In this section, we will embed a threshold dividend strategy in the compound binomial model with BMAP $(\alpha^T, D_0, D_k(k \in \mathbb{N}_+))$. First, we will introduce the specific description of this model.

4.1. Description. Based on the compound binomial model with BMAP $(\alpha^T, D_0, D_k(k \in \mathbb{N}_+))$, we can define the compound binomial model with BMAP $(\alpha^T, D_0, D_k(k \in \mathbb{N}_+))$ and threshold dividend strategy. The surplus process of an insurer is given by

$$\begin{cases} V(n) = \left\{ c(n) - \sum\limits_{k=N(n-1)}^{N(n)} X(k), b \right\}, & n \in \mathbb{N}_+; \\ V(0) = u. \end{cases} \tag{24}$$

where $N(n), c(n)$, and $X(n)$ are entirely the same as the description in model (4). $b(>0)$ is the dividend threshold, i.e., if the surplus of an insurer is greater than b, the exceed part will pay out as dividend to the shareholders and if the surplus of an insurer is smaller than b, nothing is needed to do. And we should point out the assumption that the dividend is paid out after the premium is received and claims are paid out.

Similarly, we define the ruin time of this model as

$$\tau_b = \inf\{k \in \mathbb{N}, V(k) < 0\}. \tag{25}$$

If ruin never occurs, $\tau_b = \infty$.

4.2. Gerber–Shiu Function. The Gerber–Shiu function, also called expected discounted penalty function, was first introduced by Gerber–Shiu [1]. Many papers and books have studied it.

Definition 3. The Gerber–Shiu function is defined by

$$m(u) = E\left[v^{\tau_b} w\left(U(\tau_b - 1), |U(\tau_b)|\right) I_{\{\tau_b < \infty\}} \,\middle|\, U(0) = u \right], \tag{26}$$

where $v \in (0, 1]$ is the discount factor, $w(\cdot,\cdot)$: $\mathbb{N}_+ \times \mathbb{N}_+ \longrightarrow \mathbb{N}$ is a binary function, and $U(\tau_b - 1)$ represents the surplus before ruin. $|U(\tau_b)|$ represents the deficit at ruin, and $I(A)$ is the indicator function of an event A taking value 1 whenever the event A occurs and 0 when it does not.

The Gerber–Shiu function plays an important role in risk theory. When $w(\cdot,\cdot) \equiv 1$ and $v = 1$, the Gerber–Shiu function changes to the ruin probability. When $w(x, y) = x$, it changes to the discounted surplus before ruin time. When $w(x, y) = y$, it changes to the discounted deficit at ruin. Studying on the Gerber–Shiu function can understand this model more deeply and enable to properly handle the operations of an insurance company.

To solve the problem, we denote some auxiliary functions. Denote the conditional Gerber–Shiu function as

$$\begin{aligned} m(u \mid e_i) = E\Big[& v^{\tau_b} w\left(U(\tau_b - 1), |U(\tau_b)|\right) I_{\{\tau_b < \infty\}} \,\Big| \\ & U(0) = u, J(0) = e_i \Big], \end{aligned} \tag{27}$$

and denote

$$\mathbf{m}(u) = \left(m(u \mid e_1) m(u \mid e_2) m(u \mid e_3) \cdots m(u \mid e_m) \right)_{1\times m}^T. \tag{28}$$

We can easily see that

$$m(u) = \sum_{e_i \in E} \alpha_i m(u \mid e_i) = \alpha^T \mathbf{m}(u). \tag{29}$$

Thus, we can solve the Gerber–Shiu function $m(u)$ by solving $m(u \mid e_i)$. Next, we will derive the solution of $m(u \mid e_i)$.

Theorem 3. *In the compound binomial model with DTBMAP $(\alpha^T, D_0, D_k(k \in \mathbb{N}_+))$ and dividend threshold b, the conditional Gerber–Shiu functions satisfy the following recursive formula.*

For $u = 0, 2, \ldots, b-1$,

$$\begin{aligned} \mathbf{m}(u) = {} & v D_0 \times \mathbf{P} \times \mathbf{m}(u+1) + v D_0 \times (\mathbf{I} - \mathbf{P}) \times \mathbf{m}(u) \\ & + v \sum_{k=1}^{\infty} D_k \times \mathbf{P} \times \mathbf{h}^3(u) + v \sum_{k=1}^{\infty} D_k \times (\mathbf{I} - \mathbf{P}) \times \mathbf{h}^4(u), \end{aligned} \tag{30}$$

where

$$h_i^3(u)=\sum_{y=1}^{u} f_i^{*(k)}(y+1)m(u-y\mid e_i)+\sum_{y=1}^{\infty} f_i^{*(k)}(y+1+u)w(u,y),$$
$$h_i^4(u)=\sum_{y=0}^{u} f_1^{*(k)}(y)m(u-y\mid e_1)+\sum_{y=1}^{\infty} f_1^{*(k)}(y+u)w(u,y), \tag{31}$$

and for $u=b,b+1,\ldots,$

$$\mathbf{m}(u)=vD_0\times\mathbf{m}(b)+v\sum_{k=1}^{\infty}D_k\times\mathbf{P}\times\mathbf{h}^5(u)+v\sum_{k=1}^{\infty}D_k\times(\mathbf{I}-\mathbf{P})\times\mathbf{h}^6(u), \tag{32}$$

where

$$h_i^5(u)=\sum_{y=0}^{u-b} f_i^{*(k)}(y+1)m(b\mid e_i)+\sum_{y=u-b+1}^{u} f_i^{*(k)}(y+1)m(u-y\mid e_i)+\sum_{y=1}^{\infty} f_i^{*(k)}(y+1+u)w(u,y),$$
$$h_i^6(u)=\sum_{y=1}^{u-b} f_i^{*(k)}(y)m(b\mid e_i)+\sum_{y=u-b+1}^{u} f_i^{*(k)}(y)m(u-y\mid e_i)+\sum_{y=1}^{\infty} f_i^{*(k)}(y+u)w(u,y). \tag{33}$$

Proof. Similarly, by conditioning on $J(1),N(1),c(1)$, and $X(1)$, we can derive that for $u=b,b+1,\ldots,$

$$\begin{aligned}
m(u\mid e_i)=&\sum_{e_j\in E} d_{ij}^0\left[p_j m(b|e_j)+q_j m(b|e_j)\right]\\
&+\sum_{e_j\in E}\sum_{k=1}^{\infty} d_{ij}^k\Bigg\{\sum_{y=1}^{\infty} p_j f_j^{*(k)}(y)\left[m(b\mid e_j)I_{\{u+1-y>b\}}+m(u+1-y\mid e_j)I_{\{0\le u+1-y\le b\}}+w(u,y-u-1)I_{\{u+1-y<0\}}\right]\\
&+\sum_{y=1}^{\infty} q_j f_j^{*(k)}(y)\left[m(b\mid e_j)I_{\{u-y>b\}}+m(u-y\mid e_j)I_{\{0\le u-y\le b\}}+w(u,y-u)I_{\{u-y<0\}}\right]\Bigg\}\\
=&\sum_{e_j\in E} d_{ij}^0 m(b\mid e_j)+\sum_{e_j\in E}\sum_{k=1}^{\infty} d_{ij}^k p_j\left\{\sum_{y=0}^{u-b} f_j^{*(k)}(y+1)m(b\mid e_j)+\sum_{y=u-b+1}^{u} f_j^{*(k)}(y+1)m(u-y\mid e_j)+\sum_{y=1}^{\infty} f_j^{*(k)}(y+1+u)w(u,y)\right\}\\
&+\sum_{e_j\in E}\sum_{k=1}^{\infty} d_{ij}^k q_j\left\{\sum_{y=1}^{u-b} f_j^{*(k)}(y)m(b\mid e_j)+\sum_{y=u-b+1}^{u} f_j^{*(k)}(y)m(u-y\mid e_j)+\sum_{y=1}^{\infty} f_j^{*(k)}(y+u)w(u,y)\right\}.
\end{aligned} \tag{34}$$

But for $u=0,2,\ldots,b-1,$

$$m(u\mid e_i)=\sum_{e_j\in E} d_{ij}^0\left[p_j m(u+1\mid e_j)+q_j m(u\mid e_j)\right]+\sum_{e_j\in E}\sum_{k=1}^{\infty} d_{ij}^k\left[\sum_{y=1}^{u+1} p_j f_j(y)m(u+1-y\mid e_j)+\sum_{y=1}^{u} q_j f_j(y)m(u-y\mid e_j)\right]. \tag{35}$$

Rewrite the abovementioned equations into matrix, and we can derive the theorem.

Remark 2. When we want to calculate $m(u\mid e_i)$, we can perform the following:

> Choose $u=0,1,2,\ldots,b$; then, we can derive an equation system containing $m(b+1)$ equations with $m(b+1)$ unknown numbers. The solutions of the equation system are just $m(u\mid e_i), e_i\in E, u=0,1,2,\ldots,b$.
>
> $m(u\mid e_i), e_i\in E, u\ge b+1$, can be calculated by (32).

4.3. Discounted Dividend Value. For the risk model with dividend, we are also interested in the expected discounted dividend value of all dividends up to the ruin time in general. Especially in shareholders' standpoint, the dividend value is the aim and the only focus thing. So, we will discuss research on the expected discounted dividend value of all dividends up to the ruin time. Let $\gamma\,(0<\gamma\le 1)$ denote the discounted factor, $B_1(u|e_i)$ denote the expected discounted dividend value of the first dividend under the conditions $V(0)=u$ and $J(0)=e_i$, $B_1(u|e_i)$ denote the expected discounted dividend value of all dividends up to the ruin time under the conditions $V(0)=u$ and $J(0)=e_i$, and $B(u)$ denote the expected discounted dividend value of all dividends up to the ruin time under the conditions $V(0)=u$.

Theorem 4. *In the compound binomial model with DTBMAP $(\alpha^T, D_0, D_k\,(k\in\mathbb{N}_+))$ and dividend threshold b, we have*

(1) $B_1(u|e_i)=\gamma(u-b)$ *for* $e_i\in E$ *and* $u>b$.

(2) *For* $e_i\in E$,

$$B_1(b|e_i)=\gamma\sum_{e_j\in E}d_{ij}^0\left[p_j+q_jB_1(b|e_j)\right]+\gamma\sum_{e_j\in E}\sum_{k=1}^{\infty}d_{ij}^k\left[\sum_{y=1}^{u+1}p_jf_j(y)B_1(u+1-y|e_j)+\sum_{y=1}^{u}q_jf_j(y)B_1(u-y|e_j)\right]. \quad (36)$$

(3) $D_1(u|e_i)$, *for* $e_i\in E$ *and* $u<b$, *satisfied the following recursive formula:*

$$\mathbf{B}_1(u)=\gamma D_0\times\mathbf{P}\times\mathbf{B}_1(u+1)+\gamma D_0\times(\mathbf{I}-\mathbf{P})\times\mathbf{B}_1(u)+\gamma\sum_{k=1}^{\infty}D_k\times\mathbf{P}\times\mathbf{h}^7(u)+\gamma\sum_{k=1}^{\infty}D_k\times(\mathbf{I}-\mathbf{P})\times\mathbf{h}^8(u), \quad (37)$$

where

$$\mathbf{B}_1(u)=\left(B_1(u|e_1)B_1(u|e_2)B_1(u|e_3)\cdots B_1(u|e_m)\right)_{1\times m}^T,$$
$$h_i^7(u)=\sum_{y=1}^{u}f_1^{*(k)}(y+1)B_1(u-y|e_1),$$
$$h_i^8(u)=\sum_{y=0}^{u}f_1^{*(k)}(y)B_1(u-y|e_1). \quad (38)$$

Theorem 5. *In the compound binomial model with DTBMAP $(\alpha^T, D_0, D_k\,(k\in\mathbb{N}_+))$ and dividend threshold b, we have*

(1) $B(u|e_i)=\gamma(u-b)+\gamma B(b|e_i)$ *for* $e_i\in E$ *and* $u>b$.

(2) $B(u|e_i)$, $e_i\in E$, *and* $u\le b$ *satisfied the following expression:*

$$\mathscr{B}=\mathscr{A}^{-1}\mathscr{V}, \quad (39)$$

$$\begin{pmatrix}B(0)\\B(1)\\B(2)\\\vdots\\B(b)\end{pmatrix}=\begin{pmatrix}A_{00}&A_{01}&A_{02}&\cdots&A_{0b}\\A_{10}&A_{11}&A_{12}&\cdots&A_{1b}\\A_{20}&A_{21}&A_{22}&\cdots&A_{2b}\\\vdots&\vdots&\vdots&\ddots&\vdots\\A_{b0}&A_{b1}&A_{b2}&\cdots&A_{bb}\end{pmatrix}^{-1}\begin{pmatrix}v_1(0)\\v_1(1)\\v_1(2)\\\vdots\\v_1(b)\end{pmatrix}, \quad (40)$$

where

$$B(i)=\left(B(i|e_1)B(i|e_2)B(i|e_3)\cdots B(i|e_m)\right)_{1\times m}^T. \quad (41)$$

For $i=0,1,2,\ldots,b-1$,

$$v_1(i)=\left(0\;\;0\;\;0\;\;\cdots\;\;0\right)_{1\times m}^T,$$
$$v_1(b)=\left(-\gamma\sum_{e_j\in E}d_{1j}^0p_j-\gamma\sum_{e_j\in E}d_{2j}^0p_j-\gamma\sum_{e_j\in E}d_{3j}^0p_j\cdots-\gamma\sum_{e_j\in E}d_{mj}^0p_j\right)_{1\times m}^T, \quad (42)$$

and $A_{sl}\,(s=0,1,\ldots,b,\,l=0,1,\ldots,b)$ *is a series of* $m\times m$ *matrixes:*

(1) $s+2\le l$, $A_{sl}=0_{m\times m}$

(2) $s+1=l$, $A_{sl}=(a_{ij})_{m\times m}$,

$$a_{ij}=\gamma d_{ij}^0p_j. \quad (43)$$

(3) $s=l$,

$$s\neq b,\ A_{sl}=(b_{ij})_{m\times m},$$
$$b_{ij}=\begin{cases}\gamma\left(q_id_{ii}^0+p_id_{ii}^1f_i(1)\right)-1, & i=j;\\ \gamma\left(q_jd_{ij}^0+p_jd_{ij}^1f_j(1)\right), & i\neq j.\end{cases}$$
$$s=b,\ A_{bb}=(\tilde{b}_{ij})_{m\times m},$$
$$\tilde{b}_{ij}=\begin{cases}\gamma\left(d_{ii}^0+p_id_{ii}^1f_i(1)\right)-1, & i=j;\\ \gamma\left(d_{ij}^0+p_jd_{ij}^1f_j(1)\right), & i\neq j.\end{cases} \quad (44)$$

(4) $s>l$, $A_{sl}=(c_{ij})_{m\times m}$,

$$c_{ij}=\gamma\sum_{k=1}^{\infty}d_{ij}^k\left(p_jf_j^{*(k)}(s+1-l)+q_jf_j^{*(k)}(s-l)\right). \quad (45)$$

Proof. Similar to the method of the previous theorem, we can obtain a series of equations. Then, we can rewrite the equation system into matrix as follows:

$$\mathscr{A}\mathscr{B} = \mathscr{V}, \tag{46}$$

Hence, to prove this theorem, the most important thing is to prove that $\mathscr{A}$ is nonsingular.

It can be easily seen that since $f_j(x)$ is a probability distribution function, $\forall k = 1, 2, \ldots, m$ and $\forall e_j \in E$,

$$\sum_{x=1}^{\infty} f_j^{*k}(x) = 1. \tag{47}$$

In each r row, we can write $r = sn + l$.

(1) When $s \neq b$, we can see

$$\begin{aligned}
&\sum_{s>l}\sum_{e_j\in E}\sum_{k=1}^{\infty}\gamma d_{rj}^{k}\left(p_j f_j^{*(k)}(s+1-l) + q_j f_j^{*(k)}(s-l)\right) + \sum_{e_j\in E}\gamma\left(q_j d_{rj}^{0} + p_j d_{rj}^{1} f_j(1)\right) + \sum_{e_j\in E}\gamma d_{rj}^{0} p_j \\
&\quad = \sum_{e_j\in E}\left[\sum_{s>l}\sum_{k=1}^{\infty} d_{rj}^{k}\left(p_j f_j^{*(k)}(s+1-l) + q_j f_j^{*(k)}(s-l)\right) + \left(q_j d_{rj}^{0} + p_j d_{rj}^{1} f_j(1)\right) + d_{rj}^{0} p_j\right] \\
&\quad = \gamma\sum_{e_j\in E}\left[\sum_{s>l}\sum_{k=1}^{\infty} d_{rj}^{k}\left(p_j f_j^{*(k)}(s+1-l) + q_j f_j^{*(k)}(s-l)\right) + d_{rj}^{0} + p_j d_{rj}^{1} f_j(1)\right] \\
&\quad \leq \gamma\sum_{e_j\in E}\left[\sum_{k=1}^{\infty}\left[\sum_{s>l} d_{rj}^{k}\left(p_j f_j^{*(k)}(s+1-l) + p_j f_j^{*(k)}(1) + q_j f_j^{*(k)}(s-l)\right)\right] + d_{rj}^{0}\right] \\
&\quad = \gamma\sum_{e_j\in E}\left[\sum_{k=1}^{\infty}\left[\sum_{s>l} d_{rj}^{k}\left(p_j\left(f_j^{*(k)}(s+1-l) + f_j^{*(k)}(1)\right) + q_j f_j^{*(k)}(s-l)\right)\right] + d_{rj}^{0}\right] \\
&\quad \leq \gamma\sum_{e_j\in E}\left[\sum_{k=1}^{\infty}\left[\sum_{s>l} d_{rj}^{k}\left(p_j + q_j\right)\right] + d_{rj}^{0}\right] \\
&\quad = \gamma\sum_{e_j\in E}\sum_{k=0}^{\infty} d_{rj}^{k}.
\end{aligned} \tag{48}$$

Because $P = \sum_{k=0}^{\infty} D_k = (q_{ij})$ is a stochastic matrix, we can derive that

$$\sum_{e_j\in E}\sum_{k=0}^{\infty} d_{rj}^{k} = 1, \forall e_r \in E. \tag{49}$$

So by combining (48) with (49) we can derive

$$\begin{aligned}
&\sum_{s>l}\sum_{e_j\in E}\sum_{k=1}^{\infty}\gamma d_{rj}^{k}\left(p_j f_j^{*(k)}(s+1-l) + q_j f_j^{*(k)}(s-l)\right) \\
&\quad + \sum_{e_j\in E}\gamma\left(q_j d_{rj}^{0} + p_j d_{rj}^{1} f_j(1)\right) + \sum_{e_j\in E}\gamma d_{rj}^{0} p_j \leq \gamma.
\end{aligned} \tag{50}$$

Inequality (50) can be written into

$$\begin{aligned}
&\sum_{s>l}\sum_{e_j\in E}\sum_{k=1}^{\infty}\gamma d_{rj}^{k}\left(p_j f_j^{*(k)}(s+1-l) + q_j f_j^{*(k)}(s-l)\right) \\
&\quad + \sum_{e_j\neq e_r}\gamma\left(q_j d_{rj}^{0} + p_j d_{rj}^{1} f_j(1)\right) + \sum_{e_j\in E}\gamma d_{rj}^{0} p_j \\
&\quad \leq \gamma - \gamma\left(q_r d_{rr}^{0} + p_r d_{rr}^{1} f_r(1)\right) \\
&\quad \leq 1 - \gamma\left(q_r d_{rr}^{0} + p_r d_{rr}^{1} f_r(1)\right) \\
&\quad = \left|\gamma\left(q_r d_{rr}^{0} + p_r d_{rr}^{1} f_r(1)\right) - 1\right|.
\end{aligned} \tag{51}$$

That is,

$$\sum_{e_j\in E} a_{rj} + \sum_{e_j\neq e_r} b_{rj} + \sum_{e_j\in E} c_{rj} \leq \left|b_{rr}\right|. \tag{52}$$

(2) When $s = b$, by using the same method, we can also obtain

$$\begin{aligned}
&\sum_{s>l}\sum_{e_j\in E}\sum_{k=1}^{\infty}\gamma d_{rj}^{k}\left(p_j f_j^{*(k)}(s+1-l) + q_j f_j^{*(k)}(s-l)\right) \\
&\quad + \sum_{e_j\neq e_r}\gamma\left(d_{rj}^{0} + p_j d_{rj}^{1} f_j(1)\right) + \sum_{e_j\in E}\gamma d_{rj}^{0} p_j \\
&\quad \leq \gamma - \gamma\left(d_{rr}^{0} + p_r d_{rr}^{1} f_r(1)\right) \\
&\quad \leq 1 - \gamma\left(d_{rr}^{0} + p_r d_{rr}^{1} f_r(1)\right) \\
&\quad = \left|\gamma\left(d_{rr}^{0} + p_r d_{rr}^{1} f_r(1)\right) - 1\right|.
\end{aligned} \tag{53}$$

That is,

$$\sum_{e_j\in E} a_{rj} + \sum_{e_j\neq e_r} \tilde{b}_{rj} + \sum_{e_j\in E} c_{rj} \leq \left|\tilde{b}_{rr}\right|. \tag{54}$$

Inequality (52) and inequality (54) lead to that the absolute value of diagonal (the rth element in the rth row) is

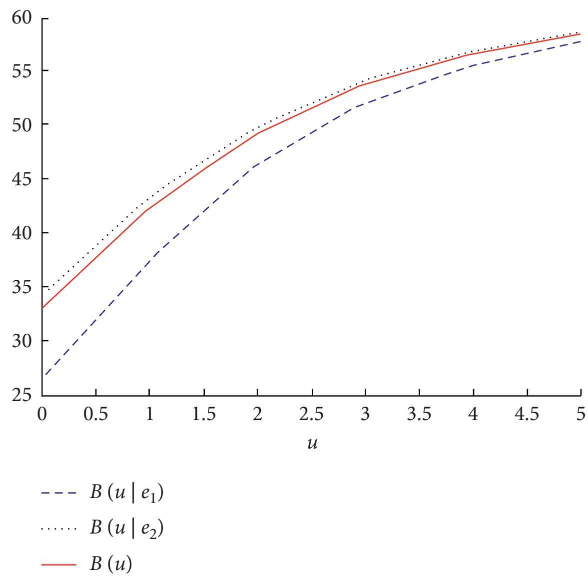

FIGURE 1: Value of all dividends $p_1 = 2/3$ and $p_2 = 3/4$.

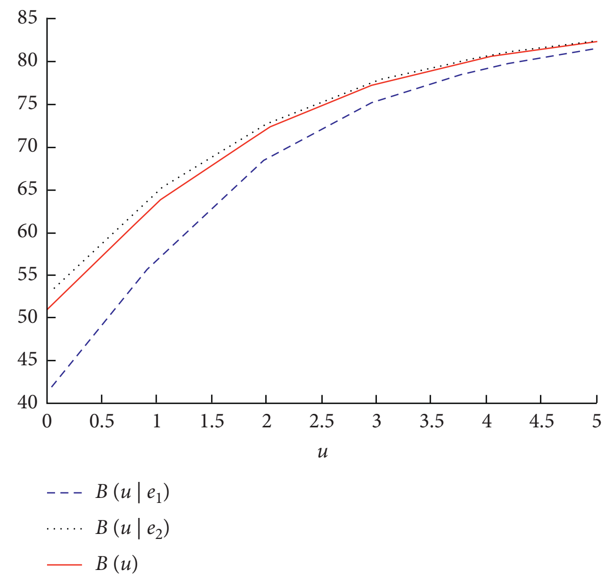

FIGURE 2: Value of all dividends $p_1 = 3/4$ and $p_2 = 3/4$.

greater than the sum of others in this row. For r being a arbitrary, $\mathscr{A}$ is a (row) strictly diagonally dominant matrix. Hence, $\mathscr{A}$ is nonsingular, which leads to the result.

5. Numerical illustration

Example 1. Let $\gamma = 1, b = 5, E = \{e_1, e_2\}$, and $\alpha = (5/6, 1/6)^T$. The matrixes are given as follows:

$$D_0 = \begin{pmatrix} 3/4 & 1/16 \\ 1/2 & 1/18 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1/16 & 1/16 \\ 2/9 & 1/36 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1/32 & 1/32 \\ 1/6 & 1/36 \end{pmatrix}. \tag{55}$$

When $k \geq 3$, we let $D_k = \mathbf{0}_{2\times 2}$. And assume $f_1(1) = 3/4$, $f_1(2) = 1/4$, $f_2(1) = 2/3$, and $f_2(2) = 1/3$. Then, $f_1^{*(2)}(2) = 9/16$, $f_1^{*(2)}(3) = 3/8$, $f_1^{*(2)}(4) = 1/16$, $f_2^{*(2)}(2) = 4/9$, $f_2^{*(2)}(3) = 4/9$, $f_2^{*(2)}(4) = 1/9$, $\mu_1 = 5/4$, and $\mu_2 = 4/3$. From it, we can see that e_1 is a "good" state, but e_2 is a "bad" state. Our example is structured in order to differentiate the "good "state and "bad" state, so we can see the difference between $B(u|e_1)$ and $B(u|e_2)$. When we let $p_1 = 2/3$ and $p_2 = 3/4$, we can obtain the expected dividend value of all dividends up to the ruin time by using Theorem 5. All results are shown in Figure 1.

In order to reflect the effect of each factor, we let $p_1 = 3/4, p_2 = 3/4$; $p_1 = 3/4, p_2 = 4/5$; and $p_1 = 4/5$, $p_2 = 3/4$, respectively, for comparison. The results of the situations $p_1 = 3/4, p_2 = 3/4$; $p_1 = 3/4, p_2 = 4/5$; and $p_1 = 4/5, p_2 = 3/4$ are shown in Figures 2–4, respectively.

Combining all results, we can analyze the following:

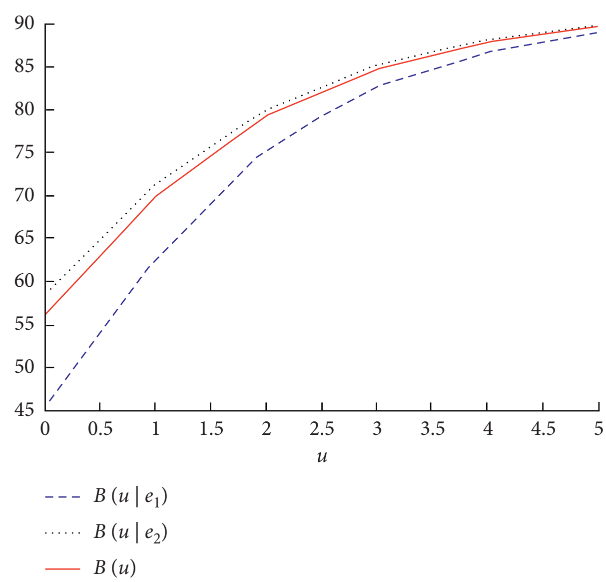

FIGURE 3: Value of all dividends $p_1 = 3/4$ and $p_2 = 4/5$.

(1) $B(u)$ is gradually increased with the increase of initial surplus u

(2) For a given e_i, $B(u \mid e_i)$ is gradually increased with the increase of initial surplus u

(3) When the initial surplus u is more and more big, the difference between $B(u \mid e_1)$ and $B(u \mid e_2)$ is more and more small

(4) For e_1 is a "good" state and e_2 is a "bad" state, $B(u \mid e_1)$ is larger than $B(u \mid e_2)$ when u is equal

(5) $B(u)$ is larger than $B(u \mid e_2)$, but smaller than $B(u \mid e_1)$ when initial surplus u is equal

(6) The safety factor is larger and the expected dividend value of all dividends up to the ruin time is larger when initial surplus u is equal

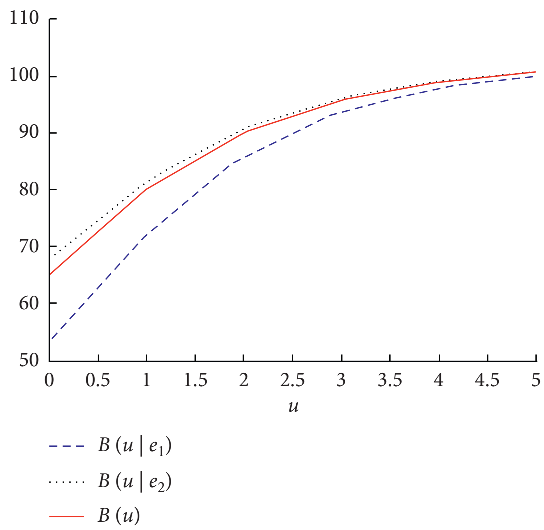

FIGURE 4: Value of all dividends $p_1 = 4/5$ and $p_2 = 3/4$.

Data Availability

The data used to support the findings of this study are currently under embargo while the research findings are commercialized. Requests for data, 6 months after publication of this article, will be considered by the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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