Research Article

Robust Control for Nonlinear Markov Jump Systems with Partially Unknown Transition Probabilities

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This note focuses on the finite horizon H_2/H_∞ control for stochastic nonlinear jump systems with partially unknown transition probabilities. We derive the nonlinear stochastic bounded real lemma and the nonlinear optimal regular result for the considered system at first. A sufficient condition and a necessary condition for the solution of H_2/H_∞ control are, respectively, offered by four cross-coupled Hamilton–Jacobi equations (HJEs). Besides, numerical examples show the effectiveness of the obtained results.

1. Introduction

Control synthesis is one of the important parts of control theory [1–5]. The H_∞ synthesis aims to seek for a suitable controller to suppress the effect of the exogenous disturbance on the dynamic system below a given level [6–9]. However, it is aware that H_∞ control can guarantee the good robustness of the designed system, but cannot optimize the closed-loop to achieve perfect performance. Because of this, the linear quadratic control (H_2 control) is selected to make up the lack in optimization. Combining the two control methods becomes a natural idea to reach the balance. H_2/H_∞ control not only ensures to repress the influence of the disturbance, but also minimizes the energy cost under the disturbance input [10–12]. So far, H_∞ control and H_2/H_∞ control have been paid continuous attention [13–17]. In particular, for stochastic systems, bounded real lemmas in finite and infinite horizon have been derived for linear models by the coupled Riccati equations method [14, 15], and the coupled Hamilton–Jacobi equations method has been performed for nonlinear models [16, 17]. It can be seen that a local solution to the primal nonlinear H_2/H_∞ control exists if its linearized H_2/H_∞ control problem is solvable. However, most of the existing works on

stochastic H_2/H_∞ control are concerned with jump linear systems, while little attention is paid on nonlinear systems with Markov jump.

As we all know, Markov jump systems have been used widely both in theory and in engineering over the past decades [18–24]. The main motivation of research studies is that such models have numerous applications in mechanics, traffic, power, and many other fields in industry and finance. When encountering system failures, sudden environmental changes, and external noise, the structure and parameter factors of dynamics are mutated. The process of state hopping from one mode to another can be marked as Markov jumps. The transition probabilities of a jump process are crucial factors which determine the behavior of a system exactly [25, 26]. Normally, the elements of the transition probability matrix are assumed to be fully known [14, 15]. However, in some practical cases, the transition probabilities may not be fully known, which inspired scholars to study Markov jump systems with partial probability [27–37]. For instance, Zhang and Boukas considered stability and stabilization of Markovian jump systems with partially unknown transition probabilities [27]. In addition, sliding-mode control, H_2, H_∞, and filtering control subject to partially unknown transition probabilities have gained

considerable research interest [29–35]. Nevertheless, to the best of our knowledge, there are no literatures to deal with H_2/H_∞ control for nonlinear jump systems with partially unknown transition probabilities up to now, which is the theoretical significance of this note. More importantly, to get the robust controller, a nonlinear stochastic bounded real lemma is derived.

The remainder of this paper is arranged as follows. The second part provides some useful definitions and lemmas. In the third part, for nonlinear jump systems with partially unknown transition probabilities, a sufficient condition and a necessary condition for the finite horizon H_2/H_∞ control are obtained, respectively. The forth part gives numerical illustrative examples. Conclusions are drawn in the fifth part.

Notations used in this study are as follows. $\mathcal{R}^n$ is the n-dimensional Euclidean space; $A > 0 (A \geq 0)$: A is a positive definite (positive semidefinite) matrix; $E(\cdot)$ is the mathematical expectation; A^T is the transpose of a matrix A; A^{-1} is the inverse of a nonsingular matrix A; $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices; $\|\cdot\|$ is the Euclidean vector norm; $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ is the completed filtration space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., it is right continuous and $\{\mathcal{F}_0\}$ contains all $\mathcal{P}$-null sets; $\mathcal{L}_F^2([0,T], \mathcal{R}^l)$ is the space of all nonanticipative stochastic processes $y(t) \in \mathcal{R}^l$ with respect to an increasing σ-algebra $F_t (t\geq 0)$, which satisfies $E\int_0^T \|y(t)\|^2 dt < \infty$; $\mathscr{C}^{2,1}(U, [0,T])$: the class of all functions $V(x,t)$ are twice continuously differentiable with respect to $x \in U$ and once continuously differentiable with respect to $t \in [0,T]$, except possibly at the point $x = 0$; $I_{n\times n}$ is the identity matrix; $col(x_1, x_2, \ldots, x_n) \coloneqq [x_1, x_2, \ldots, x_n]'$.

2. Preliminaries

Let us think about the following stochastic nonlinear jump systems described by the Itô-type equation:

$$\begin{cases} dx(t) = [f(x,t,\theta_t) + g(x,t,\theta_t)u(t) + h(x,t,\theta_t)v(t)]dt + [l(x,t,\theta_t) + q(x,t,\theta_t)u(t) + s(x,t,\theta_t)v(t)]dw(t), \\ x_0 \in \mathcal{R}^n, \theta_0 \in \mathcal{T}, t \in [0,T], \\ z(t) = col[m(x,t,\theta_t), u(t)] \coloneqq \begin{pmatrix} m(x,t,\theta_t) \\ u(t) \end{pmatrix}, \end{cases} \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $z(t) \in \mathcal{R}^z$, $u(t) \in \mathcal{L}_F^2([0,T], \mathcal{R}^{n_u})$, and $v(t) \in \mathcal{L}_F^2([0,T], \mathcal{R}^{n_v})$ stand for the system state, penalty output, control input, and exogenous disturbance signal, respectively. $\omega(t)$ is an one-dimensional standard wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ with $\mathcal{F}_t = \sigma\{\omega(s): 0 \leq s \leq t\}$. The stochastic mode jump process θ_t is a continuous-time discrete-state Markov process with values in a finite space $\mathcal{T} = \{1, 2, \ldots, N\}$ and is assumed to be independent with $\omega(t)$. The process of transition probabilities are denoted by

$$P\{\theta_{t+\tau} = h \mid \theta_t = r\} = \begin{cases} \pi_{rh}\tau + o(\tau), & r \neq h, \\ 1 + \pi_{rr}\tau + o(\tau), & r = h, \end{cases} \tag{2}$$

where $\tau > 0$, $\lim_{\tau \to 0}(o(\tau)/\tau) = 0$, and $\pi_{rh} \geq 0 (r, h \in \mathcal{T}, r \neq h)$ represent the transition rate from mode r at time t to mode h at time $t + \tau$ and $\pi_{rr} = -\sum_{h=1, r\neq h}^N \pi_{rh}$ for all $r \in \mathcal{T}$. So, the transition probabilities matrix is defined by

$$\prod = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1N} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{N1} & \pi_{N2} & \cdots & \pi_{NN} \end{pmatrix}. \tag{3}$$

In this paper, we suppose that the transition probabilities are partly unknown. For instance, for $N = 4$, the transition rate matrix $\prod$ is given by

$$\prod = \begin{pmatrix} \pi_{11} & ? & \pi_{13} & ? \\ ? & ? & ? & \pi_{24} \\ \pi_{31} & ? & \pi_{33} & ? \\ \pi_{41} & ? & ? & \pi_{44} \end{pmatrix}. \tag{4}$$

In above, we use "?" to sign the unknown element. Furthermore, $\forall r \in \mathcal{T}$, we set $\mathcal{T} = \mathcal{T}_K^r + \mathcal{T}_{UK}^r$, where

$$\begin{aligned} \mathcal{T}_K^r &\triangleq \{h: \pi_{rh} \text{ is known}\}, \\ \mathcal{T}_{UK}^r &\triangleq \{h: \pi_{rh} \text{ is unknown}\}. \end{aligned} \tag{5}$$

In addition, if $\mathcal{T}_K^r \neq \varnothing$, it can be described as $\mathcal{T}_K^r = (\mathcal{K}_1^r, \mathcal{K}_2^r, \ldots, \mathcal{K}_m^r), \forall 1 \leq m \leq N$, where $\mathcal{K}_m^r$ is the mth known element and the exponent $\mathcal{K}_m^r$ is the rth row of matrix $\prod$. Then, set $\pi_K^r \triangleq \sum_{j \in \mathcal{T}_K^r} \pi_{rh}$.

Remark 1. When $\mathcal{T}_K^r = 0$ and $\mathcal{T}_{UK}^r = \mathcal{T}$, the transition rates in the stochastic process are fully unknown. $\mathcal{T}_K^r = \mathcal{T}$ and $\mathcal{T}_{UK}^r = 0$ mean that the transition rates are fully known.

For convenience, we signify $(\cdot)_{\theta_t} \coloneqq (\cdot)(x,t,\theta_t)$ throughout the paper, and all coefficients in (1) are thought to be Borel measurable. Meanwhile, assume that $f(0,t,r) \equiv 0$, $g(0,t,r) \equiv 0$ for $\forall (t,r) \in [0,T] \times \mathcal{T}$.

Now, we introduce the following definitions.

Definition 1. Given $\gamma > 0$, a feedback control law $u = u_T^* \in \mathcal{L}_\mathcal{F}^2([0,T], \mathcal{R}^{n_u})$ is called the finite horizon robust H_2/H_∞ control of system (1), if the following conditions are satisfied:

(i) For any nonzero, $v(t) \in \mathscr{L}_{\mathscr{F}}^2([0,T], \mathscr{R}^{n_u})$, and the trajectory of the resulted closed-loop by (1) starting from $x_0 = 0$ and $\theta_0 = 0$, we always have

$$E\left\{\int_0^T \|m(x,t,\theta_t)\|^2 + \|u_T^*\|^2 dt | \theta_0 = r\right\} \le \gamma^2 E\left\{\int_0^T \|v\|^2 dt | \theta_0 = r\right\}, \quad r \in \mathscr{T}; \tag{6}$$

(ii) When the worst disturbance v_T^* is executed in (1), u_T^* can minimize the quadratic performance $E\left\{\int_0^T [\|m(x,t,\theta_t)\|^2 + \|u\|^2] dt | \theta_0 = r\right\}, r \in \mathscr{T}$.

Definition 2 (see [18]). For each $V(x,t,r) \in C^{2,1}$ $(\mathscr{R}^n \times [0,T] \times \mathscr{T}, \mathscr{R})$, we have an operator $\mathscr{L}V : \mathscr{R}^n \times [0,T] \times \mathscr{T} \longrightarrow \mathscr{R}$ associated with (1) given by

$$\mathscr{L}V(x,t,r) = \frac{\partial V(x,t,r)}{\partial t} + \frac{\partial V'(x,t,r)}{\partial x}[f(x,t,r) + g(x,t,r)u(t) + h(x,t,r)v(t)] + \sum_{h=1}^N \pi_{rh} V_h \tag{7}$$

$$+ \frac{1}{2}[l(x,t,r) + q(x,t,r)u(t) + s(x,t,r)v(t)]^T \times \frac{\partial^2 V(x,t,r)}{\partial x^2}[l(x,t,r) + q(x,t,r)u(t) + s(x,t,r)v(t)].$$

Definition 3 (see [16]). We set two extreme value functions $\widetilde{V}_T^1(x,s,\theta_t) \in \mathscr{R}^n \times [0,T] \times \mathscr{T} \longmapsto \mathscr{R}^-$ and $\widetilde{V}_T^2(x,s,\theta_t) \in \mathscr{R}^n \times [0,T] \times \mathscr{T} \longmapsto \mathscr{R}^+$ related with (1) as follows:

$$\widetilde{V}_T^1(x,s,\theta_t) = \inf_{\substack{v \in \mathscr{L}_{\mathscr{F}}^2([s,T], \mathscr{R}^{n_v}) \\ x(s)=x, u=u_T^*}} J_1^T(u,v),$$

$$\widetilde{V}_T^2(x,s,\theta_t) = \inf_{\substack{u \in \mathscr{L}_{\mathscr{F}}^2([0,T], \mathscr{R}^{n_u}) \\ x(s)=x, v=v_T^*}} J_2^T(u,v), \tag{8}$$

in which the performances are as follows:

$$J_1^T(u,v;x_0,\theta_0) := \gamma^2 \|v\|_{[0,T]}^2 - \|z\|_{[0,T]}^2$$

$$= E\left\{\int_0^T [\gamma^2 \|v\|^2 - \|z\|^2] dt | \theta_0 = r\right\}, \quad r \in \mathscr{T},$$

$$J_2^T(u,v;x_0,\theta_0) := \|z\|_{[0,T]}^2 = E\left\{\int_0^T \|z\|^2 dt | \theta_0 = r\right\}, \quad r \in \mathscr{T}. \tag{9}$$

Remark 2. The following results are obvious:

$$\widetilde{V}_T^1 \le 0, \widetilde{V}_T^1(x,t,\theta_t) = 0 \quad \text{for} \quad \text{all } x \in \mathscr{R}^n;$$

$$\widetilde{V}_T^2 \ge 0, \quad \widetilde{V}_T^2(x,t,\theta_t) = 0 \quad \text{for} \quad \text{all } x \in \mathscr{R}^n. \tag{10}$$

Remark 3. Let the perturbation operator be denoted by $L^T : \mathscr{L}_{\mathscr{F}}^2([0,T], \mathscr{R}^{n_v}) \longrightarrow \mathscr{L}_{\mathscr{F}}^2([0,T], \mathscr{R}^{n_z})$ with the norm

$$\|L^T\| = \sup_{\substack{v \in \mathscr{L}_{\mathscr{F}}^2([0,T], \mathscr{R}^{n_v}) \\ v \ne 0, \theta_0=r, x_0=0}} \frac{\|z(\cdot)\|_{[0,T]}}{\|v(\cdot)\|_{[0,T]}}$$

$$= \sup_{\substack{v \in \mathscr{L}_{\mathscr{F}}^2([0,T], \mathscr{R}^{n_v}) \\ v \ne 0, \theta_0=r, x_0=0}} \frac{E\left\{\int_0^T \|m(x,t,\theta_t)\|^2 dt | \theta_0 = r\right\}^{1/2}}{E\left\{\int_0^T \|v\|^2 dt | \theta_0 = r\right\}^{1/2}}, \quad r \in \mathscr{T}. \tag{11}$$

It can be checked that (6) is equivalent to $\|L^T\| \le \gamma$.

Next, we state the following lemmas, which will be used later.

Lemma 1. *For a given level $\gamma > 0$, with an initial state $x_0 = 0 \in \mathscr{R}^n$ and $\theta_0 \in \mathscr{T}$, think about the following stochastic perturbed system with Markov jumps:*

$$\begin{cases} dx(t) = [f(x,t,\theta_t) + h(x,t,\theta_t)v(t)]dt \\ + [l(x,t,\theta_t) + s(x,t,\theta_t)v(t)]dw(t), \\ z(t) = m(x,t,\theta_t). \end{cases} \tag{12}$$

If there exists $V(x,t,\theta_t) \in C^{2,1}(\mathscr{R}^n \times [0,T] \times \mathscr{T}, \mathscr{R})$ with system (12) and satisfies the following HJE:

$$
\left\{
\begin{aligned}
&\Delta_1\left(V_{\theta_t}\right) := \frac{\partial V_{\theta_t}}{\partial t} + \frac{\partial V_{\theta_t}'}{\partial x} f_{\theta_t} - m_{\theta_t}' m_{\theta_t} + \frac{1}{2} l_{\theta_t}' \frac{\partial^2 V_{\theta_t}}{\partial x^2} l_{\theta_t} + \sum_{h \in \mathcal{T}_K^r} \pi_{rh}\left(V_h - T_r\right) - \frac{1}{4}\left(\frac{\partial V_{\theta_t}'}{\partial x} h_{\theta_t} + l_{\theta_t}' \frac{\partial^2 V_{\theta_t}}{\partial x^2} s_{\theta_t}\right) \\
&\left(\gamma^2 I + \frac{1}{2} s_{\theta_t}' \frac{\partial^2 V_{\theta_t}}{\partial x^2} s_{\theta_t}\right)^{-1}\left(h_{\theta_t}' \frac{\partial V_{\theta_t}}{\partial x} + s_{\theta_t}' \frac{\partial^2 V_{\theta_t}}{\partial x^2} l_{\theta_t}\right) = 0, \gamma^2 I + \frac{1}{2} s_{\theta_t}' \frac{\partial^2 V_{\theta_t}}{\partial x^2} s_{\theta_t} > 0, V_h - T_r \geq 0, \quad h \in \mathcal{T}_{UK}^r, r \neq h, V_h - T_r \geq 0, \\
&h \in \mathcal{T}_{UK}^r, r = h, V\left(x, T, \theta_t\right) = 0.
\end{aligned}
\right.
$$

$$(13)$$

Then, $\|L^T\| \leq \gamma$ holds.

Proof. Noticing that for any symmetric matrix T_r ($r \in \mathcal{T}$), we have $\sum_{h=1}^N \pi_{rh} T_r = 0$. Applying the generalized Itô's formula, one gets

$$
\begin{aligned}
&\mathscr{L} V\left(x, t, \theta_t\right) \\
&= E\left\{\int_0^T \left[\frac{\partial V_{\theta_t}}{\partial t} + \frac{\partial V_{\theta_t}'}{\partial x}\left(f_{\theta_t} + h_{\theta_t} v\right) + \frac{1}{2}\left(l_{\theta_t} + s_{\theta_t} v\right)' \frac{\partial^2 V_{\theta_t}}{\partial x^2}\left(l_{\theta_t} + s_{\theta_t} v\right) + \sum_{h=1}^N \pi_{\theta_t h} V_h\right] dt \mid \theta_0 = r\right\} \\
&= E\left\{\int_0^T \left[\frac{\partial V_{\theta_t}}{\partial t} + \frac{\partial V_{\theta_t}'}{\partial x}\left(f_{\theta_t} + h_{\theta_t} v\right) + \frac{1}{2}\left(l_{\theta_t} + s_{\theta_t} v\right)' \frac{\partial^2 V_{\theta_t}}{\partial x^2}\left(l_{\theta_t} + s_{\theta_t} v\right) + \sum_{h=1}^N \pi_{rh} V_h - \sum_{h=1}^N \pi_{rh} T_r\right] dt \mid \theta_0 = r\right\} \\
&= E\left\{\int_0^T \left[\frac{\partial V_{\theta_t}}{\partial t} + \frac{\partial V_{\theta_t}'}{\partial x}\left(f_{\theta_t} + h_{\theta_t} v\right) + \frac{1}{2}\left(l_{\theta_t} + s_{\theta_t} v\right)' \frac{\partial^2 V_{\theta_t}}{\partial x^2}\left(l_{\theta_t} + s_{\theta_t} v\right) + \sum_{h \in \mathcal{T}_K^r} \pi_{rh}\left(V_h - T_r\right) + \sum_{h \in \mathcal{T}_{UK}^r} \pi_{rh}\left(V_h - T_r\right) dt\right] \mid \theta_0 = r\right\}.
\end{aligned}
$$

$$(14)$$

Then, it can deduced that

$$
\begin{aligned}
&E\left\{\left[V\left(x_T, T, \theta_T\right) - V\left(x_0, 0, \theta_0\right)\right] \mid \theta_0 = r\right\} \\
&= E \int_0^T \mathscr{L}_T V\left(x, t, \theta_t\right) \\
&= E\left\{\int_0^T \left[\frac{\partial V_{\theta_t}}{\partial t} + \frac{\partial V_{\theta_t}'}{\partial x}\left(f_{\theta_t} + h_{\theta_t} v\right) + \frac{1}{2}\left(l_{\theta_t} + s_{\theta_t} v\right)' \frac{\partial^2 V_{\theta_t}}{\partial x^2}\left(l_{\theta_t} + s_{\theta_t} v\right) + \sum_{h=1}^N \pi_{\theta_t h} V\left(x, t, h\right) + \|z\|^2 - \gamma^2 \|v\|^2 - \|z\|^2 + \gamma^2 \|v\|^2\right] dt \mid \theta_0 = r\right\} \\
&= E\left\{\int_0^T \left[\frac{\partial V_{\theta_t}}{\partial t} + \frac{\partial V_{\theta_t}'}{\partial x}\left(f_{\theta_t} + h_{\theta_t} v\right) + \frac{1}{2}\left(l_{\theta_t} + s_{\theta_t} v\right)' \frac{\partial^2 V_{\theta_t}}{\partial x^2}\left(l_{\theta_t} + s_{\theta_t} v\right) + \|z\|^2 - \gamma^2 \|v\|^2 - \|z\|^2 + \gamma^2 \|v\|^2 + \sum_{h \in \mathcal{T}_K^r} \pi_{rh}\left(V_h - T_r\right) \right.\right. \\
&\left.\left. \quad + \sum_{h \in \mathcal{T}_{UK}^r} \pi_{rh}\left(V_h - T_r\right)\right] dt \mid \theta_0 = r\right\} \\
&= E\left\{\int_0^T \left(\|z\|^2 - \gamma^2 \|v\|^2\right) dt + \left\|v - v_T^*\right\|_{\left(\gamma, s, V_{\theta_t}\right)}^2 + \Delta_1\left(V_{\theta_t}\right) + \sum_{h \in \mathcal{T}_{UK}^r} \pi_{rh}\left(V_h - T_r\right) \mid \theta_0 = r\right\}, \quad r \in \mathcal{T},
\end{aligned}
$$

$$(15)$$

where

$$\|z\|^2_{(\gamma,s,V_{\theta_t})} = z'\left(\gamma^2 I + \frac{1}{2}s_{\theta_t}'\frac{\partial^2 V_{\theta_t}}{\partial x^2}s_{\theta_t}\right)z,$$

$$\hat{v}_T^* = -\frac{1}{2}\left(\gamma^2 I + \frac{1}{2}s_{\theta_t}'\frac{\partial^2 V_{\theta_t}}{\partial x^2}s_{\theta_t}\right)^{-1}$$

$$\cdot\left(h_{\theta_t}'\frac{\partial V_{\theta_t}}{\partial x} + s_{\theta_t}'\frac{\partial^2 V_{\theta_t}}{\partial x^2}l_{\theta_t}\right), \qquad (16)$$

$$\pi_{rh} \geq 0\,(\forall r, h \in \mathcal{T}, r \neq h),$$

$$\pi_{rr} = -\sum_{h=1,r\neq h}^{N} \pi_{rh} < 0\,(r \in \mathcal{T}),$$

$$\sum_{h\in\mathcal{T}_{UK}^r} \pi_{rh}(V_h - T_r) \geq 0.$$

$$E\left\{\int_0^T \|z\|^2 dt \,|\theta_0 = r\right\}$$

$$= E\left\{\int_0^T \left\{\gamma^2\|v\|^2 - \|v - v_T^*\|^2_{(\gamma,s,V_{\theta_t})} - \Delta_1(V_{\theta_t}) - \sum_{h\in\mathcal{T}_{UK}^r} \pi_{rh}(V_h - T_r) + E[V(x_T, T, \theta_T) - V(x_0, 0, \theta_0)]\right\}dt \,|\theta_0 = r\right\}$$

$$\leq E\left\{\int_0^T \gamma^2\|v\|^2 dt \,|\theta_0 = r\right\}.$$

$$(17)$$

That is,

$$0 \leq J_1^T(u_T^*, v_T^*; x_0, \theta_0) \leq J_1^T(u_T^*, v; x_0, \theta_0). \qquad (18)$$

This lemma is proved. □

Lemma 2. For a given level $\gamma > 0$, with an initial state $x_0 = 0 \in \mathcal{R}^n$, $\theta_0 \in \mathcal{T}$, think about the following nonlinear stochastic controlled system with Markov jumps:

Inferring from (14)–(16), it reveals that $\hat{v}_T^*$ is the corresponding worst disturbance, which yields that

$$\begin{cases} dx(t) = [f(x,t,\theta_t) + g(x,t,\theta_t)u(t)]dt \\ \quad + [l(x,t,\theta_t) + q(x,t,\theta_t)u(t)]dw(t), \\ z(t) = m(x,t,\theta_t). \end{cases} \qquad (19)$$

If the following HJE

$$\begin{cases} \Delta_2(V_{2\theta_t}) := \frac{\partial V_{2\theta_t}}{\partial t} + \frac{\partial V_{2\theta_t}'}{\partial x}f_{\theta_t} - m_{\theta_t}'m_{\theta_t} + \frac{1}{2}l_{\theta_t}'\frac{\partial^2 V_{2\theta_t}}{\partial x^2}l_{\theta_t} + \sum_{h\in\mathcal{T}_K^r} \pi_{rh}(V_h - T_r) \\ \quad -\frac{1}{4}\left(\frac{\partial V_{2\theta_t}'}{\partial x}g_{\theta_t} + l_{\theta_t}'\frac{\partial^2 V_{2\theta_t}}{\partial x^2}q_{\theta_t}\right)\left(I + \frac{1}{2}q_{\theta_t}'\frac{\partial^2 V_{2\theta_t}}{\partial x^2}q_{\theta_t}\right)^{-1}\left(h_{\theta_t}'\frac{\partial V_{2\theta_t}}{\partial x} + q_{\theta_t}'\frac{\partial^2 V_{2\theta_t}}{\partial x^2}l_{\theta_t}\right) = 0, I \\ \quad + \frac{1}{2}q_{\theta_t}'\frac{\partial^2 V_{2\theta_t}}{\partial x^2}q_{\theta_t} > 0, V_h - T_r \geq 0, \quad h \in \mathcal{T}_{UK}^r, \quad r \neq h, V_h - T_r \leq 0, \quad h \in \mathcal{T}_{UK}^r, \quad r = h, V_2(x, T, \theta_t) = 0. \end{cases} \qquad (20)$$

admits a nonnegative solution $V_2(x,t,\theta_t) \in C^{2,1}(\mathcal{R}^n \times [0,T] \times \mathcal{T}, \mathcal{R})$, then we have

$$J_2^T(u_T^*, v_T^*; x_0, \theta_0) \leq J_2^T(u_T, v_T^*; x_0, \theta_0) = V_2(x_0, 0, r), \quad r \in \mathcal{T},$$

$$(21)$$

with the optimal control

$$\hat{u}_T^* = -\frac{1}{2}\left(I + \frac{1}{2}q_{\theta_t}'\frac{\partial^2 V_{2\theta_t}}{\partial x^2}q_{\theta_t}\right)^{-1}\left(g_{\theta_t}'\frac{\partial V_{2\theta_t}}{\partial x} + q_{\theta_t}'\frac{\partial^2 V_{2\theta_t}}{\partial x^2}l_{\theta_t}\right),$$

$$(22)$$

where

$$J_2^T\left(u_T, v_T^*; x_0, \theta_0\right) := \inf_{u \in \mathscr{L}_F^2\left([0,T],\mathscr{R}^{n_u}\right)} E\left\{\int_0^T \|z\|^2 \mathrm{d}t \mid \theta_0 = r\right\}$$

$$= \inf_{u \in \mathscr{L}_F^2\left([0,T],\mathscr{R}^{n_u}\right)} E\left\{\int_0^T \left\|m_{\theta_t}\right\|^2 + \|u\|^2 \mathrm{d}t \mid \theta_0 = r\right\}. \tag{23}$$

Proof. Taking integration and expectation in $\mathrm{d}V_2\left(x, t, \theta_t\right)$, for any $T > 0$, we get

$$E\int_0^T \mathrm{d}V_2\left(x, t, \theta_t\right)$$

$$= E\left\{\int_0^T L_u V_2 \mathrm{d}t + E\int_0^T \left(l_{\theta_t} + q_{\theta_t} u\right)' \frac{\partial V_{2\theta_t}}{\partial x}\mathrm{d}w(t) \mid \theta_0 = r\right\}$$

$$= E\left\{\int_0^T \left[\frac{\partial V_{2\theta_t}}{\partial t} + \frac{\partial V_{2\theta_t}}{\partial x}' \left(f_{\theta_t} + g_{\theta_t} u\right)\mathrm{d}t + \frac{1}{2}\left(l_{\theta_t} + q_{\theta_t} u\right)' \frac{\partial^2 V_{2\theta_t}}{\partial x^2}\left(l_{\theta_t} + q_{\theta_t} u\right)\right]\mathrm{d}t + \sum_{h=1}^N \pi_{rh} V\left(x, t, h\right) \mid \theta_0 = r\right\}$$

$$= E\left\{\int_0^T \left[\frac{\partial V_{2\theta_t}}{\partial t} + \frac{\partial V_{2\theta_t}}{\partial x}' \left(f_{\theta_t} + g_{\theta_t} u\right) + \frac{1}{2}\left(l_{\theta_t} + q_{\theta_t} u\right)' \frac{\partial^2 V_{2\theta_t}}{\partial x^2}\left(l_{\theta_t} + q_{\theta_t} u\right)\right]\mathrm{d}t + \sum_{h=1}^N \pi_{rh} V_h - \sum_{h=1}^N \pi_{rh} T_r \mid \theta_0 = r\right\} \tag{24}$$

$$= E\left\{\int_0^T \left[\frac{\partial V_{2\theta_t}}{\partial t} + \frac{\partial V_{2\theta_t}}{\partial t}' \left(f_{\theta_t} + g_{\theta_t} u\right)\mathrm{d}t + \frac{1}{2}\left(l_{\theta_t} + q_{\theta_t} u\right)' \frac{\partial^2 V_{2\theta_t}}{\partial x^2}\left(l_{\theta_t} + q_{\theta_t} u\right)\right]\mathrm{d}t + \sum_{h \in \mathscr{T}_K^r} \pi_{rh}\left(V_h - T_r\right)\right.$$

$$\left. + \sum_{h \in \mathscr{T}_{UK}^r} \pi_{rh}\left(V_h - T_r\right) \mid \theta_0 = r\right\}, \quad r \in \mathscr{T}.$$

By Itô's formula and the completing square technique combined with (24), we obtain

$$J_2^T\left(u, v_T^*; x_0, \theta_0\right)$$

$$= E\left\{\int_0^T \|z\|^2 \mathrm{d}t \mid \theta_0 = r\right\}$$

$$= E\left\{\int_0^T \left(\left\|m_{\theta_t}\right\|^2 + \|u\|^2\right)\mathrm{d}t \mid \theta_0 = r\right\}$$

$$= E\left\{\int_0^T \left(m_{\theta_t}' m_{\theta_t} + u'u\right)\mathrm{d}t + \mathscr{L}V_2\left(x, t, \theta_t\right) + V_2\left(x_0, 0, \theta_0\right) - E V_2\left(x, t, \theta_t\right) \mid \theta_0 = r\right\}$$

$$= E\left\{\int_0^T \left[\frac{\partial V_{2\theta_t}}{\partial t} + l_{\theta_t}' \frac{\partial V_{2\theta_t}}{\partial x}' \left(f_{\theta_t} + g_{\theta_t} u\right) + \frac{1}{2}\left(l_{\theta_t} + q_{\theta_t} u\right)' \frac{\partial^2 V_{2\theta_t}}{\partial x^2}\left(l_{\theta_t} + q_{\theta_t} u\right)\right]\mathrm{d}t + \sum_{h=1}^N \pi_{rh} V\left(x, t, h\right) + m_{\theta_t}' m_{\theta_t} + u'u\mathrm{d}t\right.$$

$$\left. + V_2\left(x_0, 0, \theta_0\right) - E V\left(x, t, \theta_t\right) \mid \theta_0 = r\right\}$$

$$= E\left\{\int_0^T \left[\frac{\partial V_{2\theta_t}}{\partial t} + l_{\theta_t}' \frac{\partial V_{2\theta_t}}{\partial x}' f_{\theta_t} + \frac{1}{2} l_{\theta_t}^T \frac{\partial^2 V_{2\theta_t}}{\partial x^2} l_{\theta_t} + m_{\theta_t}' m_{\theta_t} + \sum_{h \in \mathscr{T}_K^r} \pi_{rh}\left(V_h - T_r\right) + \sum_{h \in \mathscr{T}_{UK}^r} \pi_{rh}\left(V_h - T_r\right)\right.\right.$$

$$\left. - \frac{1}{4}\left(\frac{\partial^2 V_{2\theta_t}}{\partial x^2}' g_{\theta_t} + l_{\theta_t}' \frac{\partial^2 V_{2\theta_t}}{\partial x^2} q_{\theta_t}\right)\left(I + \frac{1}{2} q_{\theta_t}' \frac{\partial^2 V_{2\theta_t}}{\partial x^2} q_{\theta_t}\right)^{-1}\left(g_{\theta_t}' \frac{\partial V_{2\theta_t}}{\partial x} + q_{\theta_t}' \frac{\partial^2 V_{2\theta_t}}{\partial x^2} l_{\theta_t}\right) + \left(u - \widehat{u}_T^*\right)'\left(I + \frac{1}{2} q_{\theta_t}' \frac{\partial^2 V_{2\theta_t}}{\partial x^2} q_{\theta_t}\right)\left(u - \widehat{u}_T^*\right)\right]\mathrm{d}t$$

$$\left. + V_2\left(x_0, 0, \theta_0\right) - E V\left(x, t, \theta_t\right) \mid \theta_0 = r\right\}$$

$$= E\int_0^T \left\|u - \widehat{u}_T^*\right\|^2 \mathrm{d}t + \triangle_2\left(V_{2\theta_t}\right) + \sum_{h \in \mathscr{T}_{UK}^r} \pi_{rh}\left(V_h - T_r\right) + V_2\left(x_0, 0, r\right) - E V\left(x, t, \theta_t\right), \quad r \in \mathscr{T}, \tag{25}$$

in which the inequality $J_2^T(u_T^*, v_T^*; x_0, \theta_0) \le J_2^T(u_T, v_T^*;$ $x_0, \theta_0) = V_2(x_0, 0, r)$ holds since that $V_2(x, T, \theta_t) = 0$, $\triangle_2(V_{2\theta_t}) = 0$, and $\sum_{h \in \mathcal{T}_{UK}^r} \pi_{rh}(V_h - T_r) \ge 0$ ($r = h$ or $r \ne h$). So, from (25), this lemma is obvious with the optimal control (22). □

3. Main Results

In this part, a sufficient condition comes down to Theorem 1 for the solvability of the H_2/H_∞ control with nonlinear jump systems (1).

Theorem 1. *For a given $\gamma > 0$, think about the following four cross-coupled HJEs:*

$$
\begin{cases}
\triangle_3(V_{1\theta_t}) := \dfrac{\partial V_{1\theta_t}}{\partial t} + \dfrac{\partial V_{1\theta_t}'}{\partial x} f_{\theta_t} - \widetilde{m}_{\theta_t}' \widetilde{m}_{\theta_t} + \dfrac{1}{2} \widetilde{l}_{\theta_t}' \dfrac{\partial^2 V_{\theta_t}}{\partial x^2} \widetilde{l}_{\theta_t} + \sum_{h \in \mathcal{T}_K^r} \pi_{rh}(V_h - T_r) - \dfrac{1}{4} \left(\dfrac{\partial V_{1\theta_t}'}{\partial x} h_{\theta_t} + \widetilde{l}_{\theta_t}' \dfrac{\partial^2 V_{1\theta_t}}{\partial x^2} s_{\theta_t} \right) \left(\gamma^2 I + \dfrac{1}{2} s_{\theta_t}' \dfrac{\partial^2 V_{\theta_t}}{\partial x^2} s_{\theta_t} \right)^{-1} \\[2mm]
\left(h_{\theta_t}' \dfrac{\partial V_{\theta_t}}{\partial x} + s_{\theta_t}' \dfrac{\partial^2 V_{\theta_t}}{\partial x^2} \widetilde{l}_{\theta_t} \right) = 0, \gamma^2 I + \dfrac{1}{2} s_{\theta_t}' \dfrac{\partial^2 V_{1\theta_t}}{\partial x^2} s_{\theta_t} > 0, V_h - T_r \ge 0, \quad h \in \mathcal{T}_{UK}^r, r \ne h, V_h - T_r \le 0, \quad h \in \mathcal{T}_{UK}^r, r = h, V_1(x, T, \theta_t) = 0,
\end{cases}
$$
(26)

$$
K_1 = -\dfrac{1}{2} \left(\gamma^2 I + \dfrac{1}{2} s_{\theta_t}' \dfrac{\partial^2 V_{1\theta_t}}{\partial x^2} s_{\theta_t} \right)^{-1} \left(h_{\theta_t}' \dfrac{\partial V_{1\theta_t}}{\partial x} + s_{\theta_t}' \dfrac{\partial^2 V_{1\theta_t}}{\partial x^2} \widetilde{l}_{\theta_t} \right),
$$
(27)

$$
\begin{cases}
\triangle_4(V_{2\theta_t}) := \dfrac{\partial V_{2\theta_t}}{\partial t} + \dfrac{\partial V_{2\theta_t}'}{\partial x} \widetilde{f}_{\theta_t} - m_{\theta_t}' m_{\theta_t} + \dfrac{1}{2} \widetilde{l}_{1\theta_t}' \dfrac{\partial^2 V_{\theta_t}}{\partial x^2} \widetilde{l}_{1\theta_t} + \sum_{h \in \mathcal{T}_K^r} \pi_{rh}(V_h - T_r) - \dfrac{1}{4} \left(\dfrac{\partial V_{2\theta_t}'}{\partial x} g_{\theta_t} + \widetilde{l}_{1\theta_t}' \dfrac{\partial^2 V_{2\theta_t}}{\partial x^2} s_{\theta_t} \right) \\[2mm]
\left(I + \dfrac{1}{2} q_{\theta_t}' \dfrac{\partial^2 V_{2\theta_t}}{\partial x^2} q_{\theta_t} \right)^{-1} \left(h_{\theta_t}' \dfrac{\partial V_{2\theta_t}}{\partial x} + q_{\theta_t}' \dfrac{\partial^2 V_{2\theta_t}}{\partial x^2} \widetilde{l}_{1\theta_t} \right) = 0, I + \dfrac{1}{2} q_{\theta_t}' \dfrac{\partial^2 V_{2\theta_t}}{\partial x^2} q_{\theta_t} > 0, V_h - T_r \ge 0, \quad h \in \mathcal{T}_{UK}^r, r \ne h, V_h - T_r \le 0, \\[2mm]
h \in \mathcal{T}_{UK}^r, r = h, V_1(x, T, \theta_t) = 0,
\end{cases}
$$
(28)

$$
K_2 = -\dfrac{1}{2} \left(I + \dfrac{1}{2} q_{\theta_t}' \dfrac{\partial^2 V_{2\theta_t}}{\partial x^2} q_{\theta_t} \right)^{-1} \left(g_{\theta_t}' \dfrac{\partial V_{2\theta_t}}{\partial x} + q_{\theta_t}' \dfrac{\partial^2 V_{2\theta_t}}{\partial x^2} \widetilde{l}_{1\theta_t} \right).
$$
(29)

If there exist solutions $(V_{1\theta_t}, V_{2\theta_t})$ with $V_{1\theta_t} \le 0$ and $V_{2\theta_t} \ge 0$ for (26)–(29), then the finite horizon H_2/H_∞ control of nonlinear Markov jumps has a pair of solutions (u_T^*, v_T^*) with $u_T^* = K_2 x$, $v_T^* = K_1 x$, and $J_2^T(u_T^*, v_T^*; x_0, \theta_0) = V_2(x_0, 0, r)$.

Proof. Notice the following transformations in (26)–(29):

$$
\begin{aligned}
\widetilde{f}_{\theta_t} &= f_{\theta_t} + g_{\theta_t} K_2, \\
\widetilde{l}_{\theta_t} &= l_{\theta_t} + q_{\theta_t} K_2, \\
\widetilde{m}_{\theta_t} &= col(m_{\theta_t}, K_2), \\
\widetilde{f}_{1\theta_t} &= f_{\theta_t} + h_{\theta_t} K_1, \\
\widetilde{l}_{1\theta_t} &= l_{\theta_t} + s_{\theta_t} K_1.
\end{aligned}
$$
(30)

Substituting $u = K_2 x$ with K_2 defined by (29) into (1), we have

$$
\begin{cases}
dx(t) = \left[\widetilde{f}(x, t, \theta_t) + h(x, t, \theta_t) v(t) \right] dt \\
\quad + \left[\widetilde{l}(x, t, \theta_t) + s(x, t, \theta_t) v(t) \right] dw(t), \\
z(t) = (m_{\theta_t}, K_2).
\end{cases}
$$
(31)

Applying Lemma 1 to system (31), we conclude

$$
E \left\{ \int_0^T \|m_{\theta_t}\|^2 + \|u_T^*\|^2 dt | \theta_0 = r \right\} \le \gamma^2 E \left\{ \int_0^T \|v\|^2 dt | \theta_0 = r \right\}, \quad r \in \mathcal{T},
$$
(32)

and $v = K_1 x$ is the worst case disturbance. In the meantime, implementing $v = v_T^* = K_1 x$ into (1), one yields that

$$
\begin{cases}
dx(t) = \left[\widetilde{f}_1(x, t, \theta_t) + g(x, t, \theta_t) u(t) \right] dt \\
\quad + \left[\widetilde{l}_1(x, t, \theta_t) + q(x, t, \theta_t) u(t) \right] dw(t), \\
x_0 \in \mathcal{R}^n, \theta_0 \in \mathcal{T}, t \in [0, T].
\end{cases}
$$
(33)

Minimizing $J_2^T(u, v_T^*; x_0, \theta_0)$ under the constraint of (33) is a standard nonlinear quadratic optimal problem. By applying the Lemma 2, $J_2^T(u, v_T^*; x_0, \theta_0)$ achieves its

minimum at $u_T^* = K_2 x$, and $J_2^T(u_T^*, v_T^*; x_0, \theta_0) = V_2(x_0, 0, r)$. By Definition 1, this theorem is proved.

Theorem 2 offers a necessary condition for the H_2/H_∞ control of system (1). □

Theorem 2. *For a given* $\gamma > 0$, $\widetilde{V}_{T\theta_t}^1, \widetilde{V}_{T\theta_t}^2 \in \mathscr{C}^{2,1} \in \mathscr{R}^n \times [0, T] \times \mathscr{T}$, *think about system (1). If the finite horizon* H_2/H_∞ *of nonlinear stochastic jump systems has solutions* $(u_T^* = K_2 x, v_T^* = K_1 x) \in \mathscr{L}_F^2([0, T], \mathscr{R}^{n_u}) \times ([0, T], \mathscr{R}^{n_u})$ *satisfying the following terms:*

$$
\begin{cases}
\gamma^2 I + \dfrac{1}{2} s_{\theta_t}{}' \dfrac{\partial^2 \widetilde{V}_{T\theta_t}^1}{\partial x^2} s_{\theta_t} > 0, & V_T^1(x, t, \theta_t) = 0, \\[3mm]
I + \dfrac{1}{2} q_{\theta_t}{}' \dfrac{\partial^2 \widetilde{V}_{T\theta_t}^2}{\partial x^2} q_{\theta_t} > 0, & V_T^2(x, t, \theta_t) = 0.
\end{cases}
\tag{34}
$$

$$
\begin{cases}
\mathscr{L}_{u=u_T^*, v=0} \widetilde{V}_T^1 - m_{\vartheta_t}{}' m_{\theta_t} - \lVert K_2 \rVert^2 - \displaystyle\sum_{h \in \mathscr{T}_K^r} \pi_{rh}(V_h - T_r) \\[3mm]
\quad - \dfrac{1}{4} \left(\dfrac{\partial \widetilde{V}_{T\theta_t}^1}{\partial x} h_{\theta_t} + \widetilde{l}_{\theta_t}{}' \dfrac{\partial^2 \widetilde{V}_{T\theta_t}^1}{\partial x^2} s_{\theta_t} \right) \left(\gamma^2 I + \dfrac{1}{2} s_{\theta_t}{}' \dfrac{\partial^2 \widetilde{V}_{T\theta_t}^1}{\partial x^2} s_{\theta_t} \right)^{-1} \left(h_{\theta_t}{}' \dfrac{\partial \widetilde{V}_{T\theta_t}^1}{\partial x} + s_{\theta_t}{}' \dfrac{\partial^2 \widetilde{V}_{T\theta_t}^1}{\partial x^2} \widetilde{l}_{\theta_t} \right) = 0, \\[3mm]
\gamma^2 I + \dfrac{1}{2} s_{\theta_t}{}' \dfrac{\partial^2 \widetilde{V}_{T\theta_t}^1}{\partial x^2} s_{\theta_t} > 0, \\[3mm]
V_h - T_r \geqslant 0, \quad h \in \mathscr{T}_{UK}^r, \; r \neq h, \\[3mm]
V_h - T_r \leqslant 0, \quad h \in \mathscr{T}_{UK}^r, \; r = h, \\[3mm]
\widetilde{V}_T^1(x, T, \theta_t) = 0.
\end{cases}
\tag{35}
$$

Notice the inequality (15), apparently,

$$
\begin{aligned}
E &\left\{ \int_0^T \left(\gamma^2 \lVert v \rVert^2 - \lVert z \rVert^2 \right) \mathrm{d}t \middle| \theta_0 = r \right\} \\
&= V(0, 0, \theta_0) - E V(x, t, \theta_t) + E \int_0^T \lVert v - v_T^* \rVert^2_{\left(\gamma, s, \widetilde{V}_{T\theta_t}^1 \right)} \mathrm{d}t + \Delta_3 \left(V_{1\theta_t} \right) + \sum_{h \in \mathscr{T}_{UK}^r} \pi_{rh}(V_h - T_r).
\end{aligned}
\tag{36}
$$

This, together with (35) and Definition 3, for each $T > 0$

$$
\begin{aligned}
J_1^T(u_T^*, v; x_0, \theta_0) = \widetilde{V}_T^1(x_0, 0, \theta_0) + E \int_0^T \lVert v - \widetilde{v}_T^* \rVert^2_{\left(\gamma, s, \widetilde{V}_{T\theta_t}^1 \right)} \mathrm{d}t \\
+ \sum_{h \in \mathscr{T}_{UK}^r} \pi_{rh}(V_h - T_r).
\end{aligned}
\tag{37}
$$

From (37), we see that $J_1^T(u_T^*, v_T^*; x_0, \theta_0) = \min J_1^T(u_T^*, v; x_0, \theta_0)$, and

$$
\text{Зen, } \widetilde{V}_{T\theta_t}^1 \leq 0 \text{ and } \widetilde{V}_{T\theta_t}^2 \geq 0 \text{ are the solutions of the four cross-coupled HJEs (26)–(29).}
$$

Proof. Substituting $u = u_T^* = K_2 x$ into (1), one reads (31). Hence, (6) holds because (u_T^*, v_T^*) solves the H_2/H_∞ control. By Definition 3, Lemma 1, and Lemma 4.1 of [17], we assert that $\widetilde{V}_{T\theta_t}^1 \leq 0$ solves the following HJE:

$$
\widetilde{v}_T^* = -\dfrac{1}{2} \left(\gamma^2 I + \dfrac{1}{2} s_{\theta_t}{}' \dfrac{\partial^2 \widetilde{V}_{T\theta_t}^1}{\partial x^2} s_{\theta_t} \right)^{-1} \left(h_{\theta_t}{}' \dfrac{\partial \widetilde{V}_{T\theta_t}^1}{\partial x} + s_{\theta_t}{}' \dfrac{\partial^2 \widetilde{V}_{T\theta_t}^1}{\partial x^2} \widetilde{l}_{\theta_t} \right),
\tag{38}
$$

is the worst perturbance. So, $v_T^* = \widetilde{v}_T^* = K_1 x$. Then, substituting $v = v_T^* = K_1 x$ into (1), (33) can be obtained. Owing to minimizing $\widetilde{V}_T^2(x, s, \theta_t) = \inf_{u \in \mathscr{L}_{\mathscr{F}}^2([0, T], \mathscr{R}^{n_u}) x(s) = x, v = v_T^*} J_2^T(u, v)$ under the constraint of (33), we can infer that $u_T^* = K_2 x$ is the optimal solution. Next,

considering the stochastic dynamic programming principle, we can certify that $(\widetilde{V}^2_{T\theta_t}, u^*_T)$ solves the following HJE:

$$-\frac{\partial \widetilde{V}^2_{T\theta_t}}{\partial t} + \max_{u \in \mathscr{S}^2_F([0,T],\mathscr{R}^{n_u})} H\left(x, t, \theta_t, u, -\frac{\partial \widetilde{V}^2_{T\theta_t}}{\partial x}, -\frac{\partial^2 \widetilde{V}^2_{T\theta_t}}{\partial x^2}\right) = 0,$$

(39)

that is,

$$-\frac{\partial \widetilde{V}^2_{T\theta_t}}{\partial t} + H\left(x, t, \theta_t, u^*_T, -\frac{\partial \widetilde{V}^2_{T\theta_t}}{\partial x}, -\frac{\partial^2 \widetilde{V}^2_{T\theta_t}}{\partial x^2}\right) = 0.$$

(40)

In this step, the generalized Hamiltonian function is prescribed as follows:

$$H\left(x, t, \theta_t, u, -\frac{\partial \widetilde{V}^2_{T\theta_t}}{\partial x}, -\frac{\partial^2 \widetilde{V}^2_{T\theta_t}}{\partial x^2}\right)$$

$$:= -\|u\|^2 - \|m_{\theta_t}\|^2 - \frac{\partial \widetilde{V}^2_{T\theta_t}}{\partial x}\left(\widetilde{f}_{1\theta_t} + g_{\theta_t}u\right) - \sum_{h=1}^{N} \pi_{rh}V(x,t,h)$$

$$-\frac{1}{2}\left(\widetilde{l}_{1\theta_t} + q_{\theta_t}u\right)'\frac{\partial^2 \widetilde{V}^2_{T\theta_t}}{\partial x^2}\left(\widetilde{l}_{1\theta_t} + q_{\theta_t}u\right) = -\Delta_4\left(\widetilde{V}^2_{T\theta_t}\right) + \frac{\partial \widetilde{V}^2_{T\theta_t}}{\partial t} - \|u - u^*_T\|^2_{\left(1,q,\widetilde{V}^2_{T\theta_t}\right)} - \sum_{h \in \mathscr{T}^r_{UK}} \pi_{rh}(V_h - T_r),$$

(41)

with

$$\widetilde{u}^*_T = -\frac{1}{2}\left(I + \frac{1}{2}q'_{\theta_t}\frac{\partial^2 \widetilde{V}^2_{T\theta_t}}{\partial x^2}q_{\theta_t}\right)^{-1}\left(g'_{\theta_t}\frac{\partial \widetilde{V}^2_{T\theta_t}}{\partial x} + q'_{\theta_t}\frac{\partial^2 \widetilde{V}^2_{T\theta_t}}{\partial x^2}\widetilde{l}_{1\theta_t}\right).$$

(42)

From (41), we get $u^*_T = \widetilde{u}^*_T = K_2 x$. Substituting the above u^*_T into (40) and considering Definition 3, it tests that $\widetilde{V}^2_{T\theta_t} \geq 0$ solves the following HJE:

$$\begin{cases} \dfrac{\partial \widetilde{V}^2_{T\theta_t}}{\partial t} + \dfrac{\partial \widetilde{V}^2_{T\theta_t}}{\partial x}\widetilde{f}_{1\theta_t} + m'_{\theta_t}m_{\theta_t} + \dfrac{1}{2}\widetilde{l}_{1\theta_t}\dfrac{\partial^2 \widetilde{V}^2_{T\theta_t}}{\partial x^2}\widetilde{l}_{1\theta_t} + \displaystyle\sum_{h \in \mathscr{T}^r_K} \pi_{rh}(V_h - T_r) \\[4mm] \quad -\dfrac{1}{4}\left(\dfrac{\partial \widetilde{V}^2_{T\theta_t}}{\partial x}g_{\theta_t} + \widetilde{l}'_{1\theta_t}\dfrac{\partial^2 \widetilde{V}^2_{T\theta_t}}{\partial x^2}q_{\theta_t}\right)\left(I + \dfrac{1}{2}q'_{\theta_t}\dfrac{\partial^2 \widetilde{V}^2_{T\theta_t}}{\partial x^2}q_{\theta_t}\right)^{-1}\left(g'_{\theta_t}\dfrac{\partial \widetilde{V}^2_{T\theta_t}}{\partial x} + q'_{\theta_t}\dfrac{\partial^2 \widetilde{V}^2_{T\theta_t}}{\partial x^2}\widetilde{l}_{1\theta_t}\right) = 0, \\[4mm] I + \dfrac{1}{2}q'_{\theta_t}\dfrac{\partial^2 \widetilde{V}^2_{T\theta_t}}{\partial x^2}q_{\theta_t} > 0, \\[4mm] V_h - T_r \geq 0, \quad h \in \mathscr{T}^r_{UK}, r \neq h, \\[4mm] V_h - T_r \leq 0, \quad h \in \mathscr{T}^r_{UK}, r = h, \\[4mm] \widetilde{V}^2_T(x, T, \theta_t) = 0. \end{cases}$$

(43)

Combining (35) and (43), the desired result therefore is obtained. □

Remark 4. It should be noted that HJEs (26)–(29) are hard to solve in general. To get the analytic expression of the controller, Takagi–Sugeno fuzzy model is constantly used, which can approximate nonlinear system effectively. The

method to solve HJEs (26)–(29) is worth to make further studies.

4. Examples

In this part, we will give two examples to illustrate the usefulness of above results.

Example 1. Think about the following one-dimensional Markov jump systems with three modes:

$$
\begin{cases}
\mathrm{d}x(t) = \left(-\dfrac{3}{2}x^3 + xu(t) + \sqrt{2}\,xv(t)\right)\mathrm{d}t + x^2\mathrm{d}w(t), \\[2mm]
z(t) = col\left(\dfrac{1}{2}x, u(t)\right), \quad \theta_0 \in \mathcal{T},
\end{cases}
\tag{44}
$$

$$
\begin{cases}
\mathrm{d}x(t) = \left(\left(-\dfrac{1}{2}x^3 - \sqrt{2}\,x^2 - 8x\right) + \sqrt{2}\,u(t) + 4v(t)\right)\mathrm{d}t \\[2mm]
\quad + (x^2 + \sqrt{2}\,x)\mathrm{d}w(t), \\[2mm]
z(t) = col\left(\dfrac{\sqrt{5}}{2}x, u(t)\right), \quad \theta_0 \in \mathcal{T},
\end{cases}
\tag{45}
$$

$$
\begin{cases}
\mathrm{d}x(t) = \left(\left(-\dfrac{3}{2}x^3 - \sqrt{6}\,x^2 - \dfrac{25}{2}x\right) + 2u(t) + 2\sqrt{7}\,v(t)\right)\mathrm{d}t \\[2mm]
\quad + (\sqrt{3}\,x^2 + \sqrt{2}\,x)\mathrm{d}w(t), \\[2mm]
z(t) = col\left(\dfrac{\sqrt{5}}{2}x, u(t)\right), \quad \theta_0 \in \mathcal{T}.
\end{cases}
\tag{46}
$$

We assume that the elements of the transition probabilities matrix ($\prod_1$) are fully known:

$$
\prod{}_1 = \begin{pmatrix}
-\left(\dfrac{1}{2}\right) & \dfrac{1}{4} & \dfrac{1}{4} \\[3mm]
\dfrac{1}{2} & -1 & \dfrac{1}{2} \\[3mm]
\dfrac{1}{2} & \dfrac{1}{2} & -1
\end{pmatrix}.
\tag{47}
$$

Set $V_{rh}(x) = p_{rh}x^2$ ($r = 1, 2, h = 1, 2, 3$). For given $\gamma = 1$, the corresponding HJEs have solutions with $p_{11} = -1$, $p_{12} = -(1/2)$, $p_{13} = -(1/2)$, $p_{21} = 2$, $p_{22} = (3/2)$, and $p_{23} = (3/2)$. According to Theorem 1, the H_2/H_∞ controller of (44)–(46) can be chosen as $v_T^*(x, t, 1) = \sqrt{2}x^2$, $v_T^*(x, t, 2) = 2x$, $v_T^*(x, t, 3) = \sqrt{7}x\,u_T^*(x, t, 1) = -2x^2$, $u_T^*(x, t, 2) = -(3/\sqrt{2})x$, and $u_T^*(x, t, 3) = -3x$.

Example 2. Think about the following one-dimensional Markov jump systems with three modes:

$$
\begin{cases}
\mathrm{d}x(t) = \left(-\dfrac{3}{2}x^3 + \dfrac{1}{4}u(t) + \dfrac{1}{2}v(t)\right)\mathrm{d}t + \sqrt{3}\,x^2\mathrm{d}w(t), \\[2mm]
z(t) = col\left(\dfrac{1}{2}x, u(t)\right), \quad \theta_0 \in \mathcal{T},
\end{cases}
\tag{48}
$$

$$
\begin{cases}
\mathrm{d}x(t) = \left(\left(-\dfrac{1}{2}x^3 - \sqrt{2}\,x^2 - 8x\right) + \dfrac{\sqrt{19}}{3}u(t) + \sqrt{\dfrac{50}{3}}\,v(t)\right)\mathrm{d}t + (x^2 + \sqrt{2}\,x)\mathrm{d}w(t), \\[2mm]
z(t) = col\left(\dfrac{\sqrt{5}}{2}x, u(t)\right), \quad \theta_0 \in \mathcal{T},
\end{cases}
\tag{49}
$$

$$
\begin{cases}
\mathrm{d}x(t) = \left(\left(-\dfrac{3}{2}x^3 - \sqrt{6}\,x^2 - \dfrac{25}{2}x\right) + \dfrac{\sqrt{37}}{3}u(t) + \sqrt{\dfrac{86}{3}}\,v(t)\right)\mathrm{d}t + (\sqrt{3}\,x^2 + \sqrt{2}\,x)\mathrm{d}w(t), \\[2mm]
z(t) = col\left(\dfrac{\sqrt{5}}{2}x, u(t)\right), \quad \theta_0 \in \mathcal{T}.
\end{cases}
\tag{50}
$$

The elements of the transition probability matrix ($\prod_2$) are supposed to be partially unknown:

$$
\prod{}_2 = \begin{pmatrix}
-\left(\dfrac{1}{2}\right) & ? & ? \\[3mm]
\dfrac{1}{2} & -1 & ? \\[3mm]
\dfrac{1}{2} & ? & -1
\end{pmatrix},
\tag{51}
$$

where "?" represents the unaccessible element. Set $V_{rh}(x) = p_{rh}x^2$ ($r = 1, 2, h = 1, 2, 3$) and $\gamma = 1$. It can be found that $p_{11} = -1$, $p_{12} = -(1/2)$, $p_{13} = -(1/2)$, $p_{21} = 2$, $p_{22} = (3/2)$, and $p_{23} = (3/2)$ solving the corresponding HJEs. According to Theorem 1, the H_2/H_∞ controller of (48)–(50) can be given by $v_T^*(x, t, 1) = (1/2)x$, $v_T^*(x, t, 2) = (5/\sqrt{6})x$, $v_T^*(x, t, 3) = (\sqrt{43/6})x$, $u_T^*(x, t, 1) = -(1/2)x$, $u_T^*(x, t, 2) = -(\sqrt{19}/2)x$, and $u_T^*(x, t, 3) = -(\sqrt{37}/2)x$.

5. Conclusions

This note dealt with the finite horizon H_2/H_∞ control for stochastic nonlinear Markov jump systems with partially unknown transition probabilities. Based on the four cross-

coupled Hamilton–Jacobi equations, a sufficient condition and a necessary condition for the existence of H_2/H_∞ control are respectively drawn, which can be regarded as the generalization of [16] to nonlinear jump models. The validity of the results has been demonstrated by two numerical examples.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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