Research Article

The Existence of Zero Waves for Nonsimplified Chromatography System

Maozhou Lin , Lihui Guo , and Yinsong Bai

College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, China

Correspondence should be addressed to Lihui Guo; lihguo@126.com

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In this paper, we mainly consider Riemann problem for the widely used nonsimplified chromatography system with initial data consisting of three pieces of constant states. Through phase plane analysis, the solutions of the nonsimplified chromatography system are established. When the different initial data tend to -1 from the right side, the existence of zero shock wave, zero delta shock wave, and zero rarefaction wave is obtained via analyzing its wave interaction. Finally, the correctness of the main conclusions is verified by numerical simulation, and the numerical results are in good agreement with the theoretical solutions of several experimental cases.

1. Introduction

Chromatography plays an important role in modern industry. It is used to separate two chemical components especially by chemists and engineers. The research of the nonlinear chromatography system is, in particular, a significant part of the theory of chromatography. Mazzotti [1–3] captured the delta shock wave for local equilibrium model (1) by numerical and experimental methods:

$$\begin{cases} u_t + \left(u + \frac{au}{1-u+v} \right)_x = 0, \\ v_t + \left(v + \frac{bv}{1-u+v} \right)_x = 0, \end{cases} \quad (1)$$

where $b > a > 0$ are constants. u and v , which describe concentrations of two chemical species, are positive function with $1 - u + v > 0$. The complexity of system (1) is due to hyperbolicity in the region $a(1+v) + b(1-u)^2 - 4ab(1-u+v) > 0$ and ellipticity in the rest of the phase plane.

In order to make the deep research for (1), one could consider the following chromatography system:

$$\begin{cases} u_t + \left(\frac{u}{1-u+v} \right)_x = 0, \\ v_t + \left(\frac{v}{1-u+v} \right)_x = 0. \end{cases} \quad (2)$$

In 2013, Wang [4] had found the Riemann solution of (2) contains delta shock wave with Dirac delta function support on both u and v . It is clear that (2) belongs to the Temple class [5, 6]; in other words, the shock curves and rarefaction curves are connected in the phase plane. One can refer to [7–13].

Recently, Ambrosio et al. [14] introduced the change of variables:

$$\begin{aligned} p &= v - u, \\ q &= v + u, \end{aligned} \quad (3)$$

and derived the following simplified chromatography system:

$$\begin{cases} p_t + \left(\frac{p}{1+p} \right)_x = 0, \\ q_t + \left(\frac{q}{1+p} \right)_x = 0. \end{cases} \quad (4)$$

They obtain the well-posedness. In 2011, Sun [15] discovered that solutions of (4) and (6) contain delta shock wave in case of $p_- = 0 < p_+$ via self-similar vanishing viscosity approach. In 2015, Shen [16] considered the asymptotic behaviors of solutions of the perturbed Riemann problem (4) and (6) near the singular curve. For other related results, one can refer to [17–24].

The zero waves describe the equilibrium of separation; under this state, there is no exchange of kinetic between two chemical species. Zero waves play an important role in analyzing the chromatography systems [1, 2, 23]. Moreover, it plays a key role in the study of traffic flow, production flow, and the supply chain with limited buffers [25–29].

At the same time, solving partial differential equations is always an interesting and difficult task. In recent decades, many kinds of analytical methods have been proposed. There are also some numerical methods, which are powerful treatments in solving nonlinear equations. Such as numerical methods of two-phase flow equations are proposed and developed by Zeidan et al. [30–34], which provide perfect approaches for the study of wave structure and our future studies. In this paper, we construct a solution of (2) with Riemann initial data:

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & -\infty < x < 0, \\ (u_m, v_m), & 0 < x < 1, \\ (u_+, v_+), & 1 < x < +\infty, \end{cases} \quad (5)$$

and analyse the interaction of various waves. We find interesting phenomena: when the differences $v_- - u_-$ and $v_m - u_m$ in (5) tend to $(-1)^+$, the zero waves occur. Now, we give our main results.

Theorem 1. *According to different zero waves produced, we divide the solution of (2) and (5) into three types:*

- (1) *When $-1 < v_- - u_- < v_m - u_m < 0$, on the one hand, the solution performs as a zero shock wave if $v_+ - u_+ < 0$ and $(v_- - u_-) \rightarrow (-1)^+$. On the other hand, its solution performs as a zero shock wave in the interval $[0, 1]$ and a zero delta shock wave in the interval $(1, \infty)$ if $v_+ - u_+ > 0$ and $(v_- - u_-) \rightarrow (-1)^+$.*
- (2) *When $-1 < v_- - u_- < 0 < v_m - u_m$ and $v_- - u_- < v_+ - u_+$, the solution performs as a zero delta shock wave when $(v_- - u_-) \rightarrow (-1)^+$.*
- (3) *When $-1 < v_m - u_m < 0$ and $v_m - u_m < \min\{v_- - u_-, v_+ - u_+\}$, a zero rarefaction wave is alive in the interval $[0, 1]$ when $(v_m - u_m) \rightarrow (-1)^+$.*

This paper is divided into seven sections. In Section 2, we review the Riemann problem (2) and (6) and discover the zero waves appear when the differences $v_- - u_-$ and $v_+ - u_+$ in (5) tend to $(-1)^+$. In Section 3, we construct solution of Riemann problem (2) and (5), and then we obtain zero shock wave by studying the interaction of waves. In Section 4 and Section 5, we obtain, by analogy, zero delta shock wave and zero rarefaction wave of system (2) and (5). In Section 6, we

verify the presentation of zero waves by numerical simulations. Finally, conclusions are drawn in Section 7.

2. Zero Waves in the Limit of Riemann Solutions to (2)

In this section, we review the solution to system (2) with initial data:

$$(u, v)(x, 0) = \begin{cases} (u_-, v_-), & x < 0, \\ (u_+, v_+), & x > 0, \end{cases} \quad (6)$$

where $u_{\pm}$ and $v_{\pm}$ are positive constants. In light of [4], we know characteristics satisfy $\lambda_1 = 1/(1 - u + v)^2$ and $\lambda_2 = 1/(1 - u + v)$. System (2) is strictly hyperbolic if $-1 < v - u < 0$ or $v - u > 0$ and nonstrictly hyperbolic if $v - u = 0$. The characteristics λ_1 and λ_2 are genuinely nonlinear and linearly degenerate, respectively. Therefore, the associated waves are rarefaction waves R or shock waves S for λ_1 and contact discontinuities J for λ_2 :

$$\begin{aligned} R: & \begin{cases} \frac{u}{u_-} = \frac{v}{v_-}, & v + u < v_- - u_-, \\ \lambda = \frac{1}{(1 - u + v)^2}, \end{cases} \\ S: & \begin{cases} \frac{u}{u_-} = \frac{v}{v_-}, & v + u > v_- - u_-, \\ \omega = \frac{1}{(1 - u + v)(1 - u_- + v_-)}. \end{cases} \\ J: & \begin{cases} v - u = v_- - u_-, \\ \tau = \frac{1}{1 - u + v} = \frac{1}{1 - u_- + v_-}. \end{cases} \end{aligned} \quad (7)$$

Using the phase plane analysis method (see Figure 1), Riemann solutions of (2) are constructed as follows:

- (1) If $0 < v_- - u_- < v_+ - u_+$, then the Riemann solution is $\overleftarrow{S} + J$:

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & \xi < \omega, \\ \left(\frac{(v_+ - u_+)u_-}{v_- - u_-}, \frac{(v_+ - u_+)v_-}{v_- - u_-} \right), & \omega < \xi < \tau, \\ (u_+, v_+), & \xi > \tau, \end{cases} \quad (8)$$

in which the propagation speeds of $\overleftarrow{S}$ and J are $\omega = 1/((1 - u_- + v_-)(1 - u_+ + v_+))$ and $\tau = 1/(1 - u_+ + v_+)$, respectively. In particular, if $v_+ - u_+ = 0$, then $\omega = \tau = 1$, and the line of $\overleftarrow{S}$ is identical with J . Therefore, there exists the composite wave $\overleftarrow{S}J$.

- (2) If $0 < v_+ - u_+ < v_- - u_-$, then solution is $\overleftarrow{R} + J$:

FIGURE 1: Phase plane.

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & \xi < \lambda_1(u_-, v_-), \\ \frac{u}{u_-} = \frac{v}{v_-}, & \lambda_1(u_-, v_-) < \xi < \lambda_1(u_+, v_+), \\ \left(\frac{(v_+ - u_+)u_-}{v_- - u_-}, \frac{(v_+ - u_+)v_-}{v_- - u_-} \right), & \lambda_1(u_-, v_-) < \xi < \tau, \\ (u_+, v_+), & \xi > \tau. \end{cases} \quad (9)$$

If $v_+ - u_+ = 0$, then a composite wave $\overleftarrow{R}J$ forms.

(3) If $-1 < v_- - u_- < v_+ - u_+ < 0$, then the Riemann solution is $J + \overrightarrow{S}$:

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & \xi < \tau, \\ \left(\frac{(v_- - u_-)u_+}{v_+ - u_+}, \frac{(v_- - u_-)v_+}{v_+ - u_+} \right), & \tau < \xi < \omega, \\ (u_+, v_+), & \xi > \omega. \end{cases} \quad (10)$$

in which the propagation speeds of J and S are $\tau = 1/(1 - u_- + v_-)$ and $\omega = 1/((1 - u_- + v_-)(1 - u_+ + v_+))$, respectively.

We deduce from (10) that

$$\begin{aligned} \lim_{(v_- - u_-) \rightarrow (-1)^+} \tau &= \lim_{(v_- - u_-) \rightarrow (-1)^+} \frac{1}{1 - u_- + v_-} = +\infty, \\ \lim_{(v_- - u_-) \rightarrow (-1)^+} \omega &= \lim_{(v_- - u_-) \rightarrow (-1)^+} \frac{1}{(1 - u_- + v_-)(1 - u_+ + v_+)} = +\infty. \end{aligned} \quad (11)$$

This fact in turn implies that both the propagation speeds of J and $\vec{S}$ tend to $+\infty$. In other words, both J and $\vec{S}$ coincide with the positive x -axis and the middle state $((v_- - u_-)u_+ / (v_+ - u_+), (v_- - u_-)v_+ /$

$(v_+ - u_+))$ disappears. Thus, zero shock wave [28] appears in this situation.

- (4) If $-1 < v_+ - u_+ < 0 < v_- - u_-$, then the solution is $J + \vec{R}$:

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & \xi < \tau, \\ \left(\frac{(v_- - u_-)u_+}{v_+ - u_+}, \frac{(v_- - u_-)v_+}{v_+ - u_+} \right), & \tau < \xi < \lambda_1(u_-, v_-), \\ \frac{u}{u_+} = \frac{v}{v_+}, & \lambda_1(u_-, v_-) < \xi < \lambda_1(u_+, v_+), \\ (u_+, v_+), & \xi > \lambda_1(u_+, v_+). \end{cases} \quad (12)$$

in which $\tau = 1/(1 - u_- + v_-)$. Since $(v_+ - u_+) \rightarrow (-1)^+$, then

$$\lim_{(v_+ - u_+) \rightarrow (-1)^+} \lambda_1(u_+, v_+) = \lim_{(v_+ - u_+) \rightarrow (-1)^+} \frac{1}{(1 - u_+ + v_+)^2} = +\infty. \quad (13)$$

Equality (13) implies the propagation speed of the wave front of $\vec{R}$ tends to $+\infty$. Then, zero rarefaction wave forms.

- (5) If $-1 < v_+ - u_+ < v_- - u_- < 0$, then the Riemann solution is $\overleftarrow{R} + \vec{R}$:

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & \xi < \lambda_1(u_-, v_-), \\ \frac{u}{u_-} = \frac{v}{v_-}, & \lambda_1(u_-, v_-) < \xi < 1, \\ \frac{u}{u_+} = \frac{v}{v_+}, & 1 < \xi < \lambda_1(u_+, v_+), \\ (u_+, v_+), & \xi > \lambda_1(u_+, v_+). \end{cases} \quad (14)$$

The line is $x = t$ both the wave front of $\overleftarrow{R}$ and the wave back of $\vec{R}$. As $(v_+ - u_+) \rightarrow (-1)^+$, the zero rarefaction wave also appears in this condition.

- (6) If $-1 < v_- - u_- < 0 < v_+ - u_+$, Wang [4] constructed delta shock wave:

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & x < \sigma t, \\ (\beta(t), \beta(t))\delta(x - \sigma t), & x = \sigma t, \\ (u_+, v_+), & x > \sigma t, \end{cases} \quad (15)$$

where

$$\sigma = \frac{1}{(1 - u_- + v_-)(1 - u_+ + v_+)}, \quad (16)$$

$$\beta(t) = \frac{(u_- v_+ - u_+ v_-)t}{(1 - u_- + v_-)(1 - u_+ + v_+)}.$$

He checked that the measure solution (15) with (16) satisfies the generalized Rankine-Hugoniot condition:

$$\begin{cases} \frac{dx}{dt} = \sigma, \\ \frac{d\beta(t)}{dt} = -\sigma[u] + \left[\frac{u}{1 - u + v} \right], \\ \frac{d\beta(t)}{dt} = -\sigma[v] + \left[\frac{v}{1 - u + v} \right]. \end{cases} \quad (17)$$

To ensure the uniqueness of delta shock wave solution, entropy condition should be satisfied provided $-1 < v_- - u_- < 0 < v_+ - u_+$,

$$\lambda_1(u_+, v_+) < \lambda_2(u_+, v_+) < \sigma < \lambda_2(u_-, v_-) < \lambda_1(u_-, v_-), \quad (18)$$

namely,

$$\frac{1}{(1 - u_+ + v_+)^2} < \frac{1}{1 - u_+ + v_+} < \sigma < \frac{1}{1 - u_- + v_-} < \frac{1}{(1 - u_- + v_-)^2}. \quad (19)$$

That is to say, all the characteristics are incoming from both sides.

For the measure solution (15) with (16), we deduce

$$\begin{aligned}
\lim_{(v_- - u_-) \rightarrow (-1)^+} (x_1, t_1) &= \lim_{(v_- - u_-) \rightarrow (-1)^+} (x_2, t_2) \\
&= \lim_{(v_- - u_-) \rightarrow (-1)^+} (x_3, t_3) = (1, 0).
\end{aligned} \tag{30}$$

Based on the above discussion, as $(v_- - u_-) \rightarrow (-1)^+$, the solution of (2) and (5) has the zero shock wave starting from the point $(0, 0)$.

Case 2. $-1 < v_- - u_- < v_m - u_m < v_+ - u_+ < 0$.

In this case, for $-1 < v_- - u_- < v_m - u_m < v_+ - u_+ < 0$ (see Figure 3), from Case 1, it is clear to see that the interaction point (x_1, t_1) of $\vec{S}_1$ and J_2 is given by (24), and the states (u_1, v_1) , (u_2, v_2) , and (u_3, v_3) satisfy (23) and (25), respectively.

Since $\omega_3 = 1/((1 - u_- + v_-)(1 - u_m + v_m)) > \omega_2 = 1/((1 - u_m + v_m)(1 - u_+ + v_+))$, for $-1 < v_- - u_- < v_+ - u_+ < 0$, then $\vec{S}_3$ overtakes the $\vec{S}_2$ in finite time and the interaction point is as follows:

$$\begin{aligned}
(x_2, t_2) &= \left(\frac{1 - u_+ + v_+}{v_+ - u_+ - (v_- - u_-)}, \right. \\
&\quad \left. \frac{(1 - u_- + v_-)(1 - u_m + v_m)(1 - u_+ + v_+)}{v_+ - u_+ - (v_- - u_-)} \right).
\end{aligned} \tag{31}$$

In view of (24) and (31), we see

$$\lim_{(v_- - u_-) \rightarrow (-1)^+} (x_1, t_1) = \lim_{(v_- - u_-) \rightarrow (-1)^+} (x_2, t_2) = (1, 0). \tag{32}$$

When $(v_- - u_-) \rightarrow (-1)^+$, the propagation speeds of all shock waves and contact discontinuities tend to $+\infty$. Thus, there is a zero shock wave starting from $(0, 0)$ in the limit $(v_- - u_-) \rightarrow (-1)^+$.

Case 3. $-1 < v_- - u_- < v_m - u_m < 0 < v_+ - u_+$.

Our plan is to show that a delta shock wave δS_1 emits from the point $(1, 0)$ as $-1 < v_- - u_- < v_m - u_m < 0 < v_+ - u_+$ (see Figure 4). The state (u_{12}, v_1) is given by (23).

Owing to $\omega_1 > \sigma_1$, then $\vec{S}_1$ overtakes δS_1 at the point

$$\begin{aligned}
(x_1, t_1) &= \left(\frac{1 - u_+ + v_+}{v_+ - u_+ - (v_- - u_-)}, \right. \\
&\quad \left. \frac{(1 - u_- + v_-)(1 - u_m + v_m)(1 - u_+ + v_+)}{v_+ - u_+ - (v_- - u_-)} \right).
\end{aligned} \tag{33}$$

It is easy to see that a new delta shock δS_2 forms, with $t > t_1$, $-1 < v_- - u_- < v_m - u_m < 0 < v_+ - u_+$. The propagation

FIGURE 3: When $-1 < v_- - u_- < v_m - u_m < v_+ - u_+ < 0$.

FIGURE 4: When $-1 < v_- - u_- < v_m - u_m < 0 < v_+ - u_+$.

speed of δS_2 is $\sigma_2 = 1/((1 - u_- + v_-)(1 - u_+ + v_+))$. Notice that propagation speed of contact discontinuity J is τ_1 and $\tau_1 > \sigma_2$, and J_1 and δS_2 interact at the point

$$\begin{aligned}
(x_2, t_2) &= \left(\frac{(v_+ - u_+ - (v_m - u_m))(1 - u_+ + v_+)}{(v_+ - u_+ - (v_- - u_-))(v_+ - u_+)}, \right. \\
&\quad \left. \frac{(v_+ - u_+ - (v_m - u_m))(1 - u_- + v_-)(1 - u_+ + v_+)}{(v_+ - u_+ - (v_- - u_-))(v_+ - u_+)} \right).
\end{aligned} \tag{34}$$

For $t > t_2$, there is a new delta shock δS_3 with same propagation speed of δS_2 . It follows from (16) that the strengths of the all delta shock waves satisfy the following equation:

$$\begin{aligned}
\beta_1(t) &= \frac{u_m v_+ - u_+ v_m}{(1 - u_m + v_m)(1 - u_+ + v_+)} t, & 0 \leq t \leq t_1, \\
\beta_2(t) &= \beta_1(t_1) + \frac{u_1 v_+ - u_+ v_1}{(1 - u_1 + v_1)(1 - u_+ + v_+)} (t - t_1), & t_1 \leq t \leq t_2, \\
\beta_3(t) &= \beta_2(t_2) + \frac{u_- v_+ - u_+ v_-}{(1 - u_- + v_-)(1 - u_+ + v_+)} (t - t_2), & t \geq t_2.
\end{aligned} \tag{35}$$

We then combine (33) and (34) to discover

$$\begin{aligned} \lim_{(v_- - u_-) \rightarrow (-1)^+} (x_1, t_1) &= \lim_{(v_- - u_-) \rightarrow (-1)^+} (x_2, t_2) = (1, 0), \\ \lim_{(v_- - u_-) \rightarrow (-1)^+} \beta_1(t) &= \lim_{(v_- - u_-) \rightarrow (-1)^+} \beta_2(t) \\ &= \lim_{(v_- - u_-) \rightarrow (-1)^+} \beta_3(t) = \infty. \end{aligned} \quad (36)$$

Based on the above analysis, all the waves overlap with the positive x -axis when $(v_- - u_-) \rightarrow (-1)^+$. Hence, at the interval $[0, 1]$ and the point $(1, 0)$, zero shock wave and zero delta shock wave occur, respectively.

The above discussions verify the type (1) of Theorem 1.

4. Analysis of Zero Delta Shock Wave

In this section, we mainly consider zero delta shock wave interacts with other waves. Taking sufficiently ε together with $-1 < v_- - u_- < -1 + \varepsilon < 0 < v_m - u_m$ and $v_- - u_- \rightarrow (-1)^+$, the Riemann solution of (2) has a zero delta shock wave at the point $(0, 0)$.

Case 4. $-1 < v_- - u_- < 0 < v_+ - u_+ < v_m - u_m$.

In this case, we are concentrated on the interaction of δS with $\bar{R} + J$ (see Figure 5). The state between $\bar{R}$ and J is $(u_1, v_1) = ((v_+ - u_+)u_m / (v_m - u_m), (v_+ - u_+)v_m / (v_m - u_m))$. The propagation speeds of δS_1 and the wave back of $\bar{R}$ are $\sigma_1 = 1 / ((1 - u_- + v_-)(1 - u_m + v_m))$ and $\lambda_1(u_m, v_m) = 1 / (1 - u_m + v_m)^2$. If $-1 < v_- - u_- < 0 < v_m - u_m$, then the δS will meet $\bar{R}$ at point

$$(x_1, t_1) = \left(\frac{1 - u_m + v_m}{v_m - u_m - (v_- - u_-)}, \frac{(1 - u_- + v_-)(1 - u_m + v_m)^2}{v_m - u_m - (v_- - u_-)} \right). \quad (37)$$

The strength of δS_1 is given by

$$\beta_1(t) = \frac{u_- v_m - u_m v_-}{(1 - u_- + v_-)(1 - u_m + v_m)} t. \quad (38)$$

By substituting (37) into (38), we can obtain the following equation:

$$\beta_1(t_1) = \frac{(u_- v_m - u_m v_-)(1 - u_m + v_m)}{v_m - u_m - (v_- - u_-)}. \quad (39)$$

After the time t_1 , there is δS_2 which has the same expression of (28).

Next, we compute the strength of δS_2 , by virtue of splitting the delta function method introduced by Nedeljkov and Oberguggeenberger in [35]. This method has also been used in [36, 37]. Now, we construct the delta shock wave $x = x(t)$ which satisfies the following equation:

FIGURE 5: When $-1 < v_- - u_- < 0 < v_+ - u_+ < v_m - u_m$.

$$v(x, t) = \begin{cases} v_-, & x < x(t), \\ \left(\sqrt{\frac{t}{x-1}} - 1 \right) \frac{v_m}{v_m - u_m}, & x > x(t) \end{cases} \quad (40)$$

$$+ \beta_2^-(t) \delta^-(x - x(t)) + \beta_2^+(t) \delta^+(x - x(t)),$$

$$u(x, t) = \begin{cases} u_-, & x < x(t), \\ \left(\sqrt{\frac{t}{x-1}} - 1 \right) \frac{u_m}{v_m - u_m}, & x > x(t), \end{cases} \quad (41)$$

$$+ \beta_2^-(t) \delta^-(x - x(t)) + \beta_2^+(t) \delta^+(x - x(t)),$$

in which $\beta_2(t) \delta(x - x(t)) = \beta_2^-(t) \delta^-(x - x(t)) + \beta_2^+(t) \delta^+(x - x(t))$ is a split delta function and $\beta_2(t) = \beta_2^-(t) + \beta_2^+(t)$ denotes the strength of δS_2 . Furthermore, we see from (40) and (41) that

$$\begin{aligned} v_t &= \frac{v_m}{v_m - u_m} \cdot \frac{H}{2\sqrt{(x-1)t}} - \left(\left(\sqrt{\frac{t}{x-1}} - 1 \right) \frac{v_m}{v_m - u_m} - v_- \right) \\ &\cdot \delta \sigma_2 + \left(\beta_2^-(t) + \beta_2^+(t) \right) \delta - \sigma_2 \left(\beta_2^-(t) + \beta_2^+(t) \right) \delta', \end{aligned} \quad (42)$$

$$\begin{aligned} \left(\frac{v}{1 - u + v} \right)_x &= - \frac{v_m}{v_m - u_m} \cdot \frac{H}{2\sqrt{(x-1)t}} \\ &+ \left(\frac{v_m}{v_m - u_m} - \frac{v_m}{v_m - u_m} \cdot \sqrt{\frac{x-1}{t}} - \frac{v_-}{1 - u_- + v_-} \right) \delta \\ &+ \left(\beta_2^-(t) + \beta_2^+(t) \right) \delta', \end{aligned} \quad (43)$$

where H , δ , and δ' are the functions of $x - x(t)$; besides, $x(t)$ is given by (28). We substitute (42) and (43) into the first equation of (2) and compare the coefficients of δ and δ' , we obtain

$$\begin{aligned}
& -\sigma_2 \left(\left(\sqrt{\frac{t}{x-1}} - 1 \right) \frac{v_m}{v_m - u_m} - v_- \right) + \beta_2^-(t) + \beta_2^+(t) \\
& + \frac{v_m}{v_m - u_m} \left(1 - \sqrt{\frac{x-1}{t}} \right) - \frac{v_-}{1 - u_- + v_-} = 0,
\end{aligned} \quad (44)$$

$$-\sigma_2(\beta_2^+(t) + \beta_2^-(t)) + \beta_2^+(t) + \beta_2^-(t) = 0. \quad (45)$$

According to (28), we have

$$\sigma_2 = \frac{1}{(1 - u_- + v_-)(1 - u + v)} = \frac{1}{1 - u_- + v_-} \cdot \sqrt{\frac{x-1}{t}}. \quad (46)$$

Replacing σ_2 by (46) in (44), we conclude that

$$\begin{aligned}
\beta_2^-(t) + \beta_2^+(t) &= \left(\frac{v_- - u_-}{1 - u_- + v_-} + \sqrt{\frac{v_m - u_m - (v_- - u_-)}{(1 - u_- + v_-)t}} \right) \\
&\cdot \frac{u_- v_m - v_- u_m}{(1 - u_- + v_-)(v_m - u_m)}.
\end{aligned} \quad (47)$$

Integrating (47) with respect to t , recall the initial condition (39), thus

$$\begin{aligned}
\beta_2(t) &= \frac{u_- v_m - v_- u_m}{(1 - u_- + v_-)(v_m - u_m)} \\
&\cdot \left(\frac{v_- - u_-}{1 - u_- + v_-} t + 2 \sqrt{\frac{v_m - u_m - (v_- - u_-)}{1 - u_- + v_-}} t - 1 \right. \\
&\left. + u_m - v_m \right).
\end{aligned} \quad (48)$$

Since $-1 < v_- - u_- < 0 < v_+ - u_+$, then the delta shock wave δS_2 will cross the whole $\overleftarrow{R}$ at the point

$$\begin{aligned}
(x_2, t_2) &= \left(1 + \frac{(v_m - u_m - v_- + u_-)(1 - u_- + v_-)}{(v_+ - u_+ - v_- + u_-)^2}, \right. \\
&\left. \frac{(v_m - u_m - v_- + u_-)(1 - u_- + v_-)(1 - u_+ + v_+)}{(v_+ - u_+ - v_- + u_-)^2} \right).
\end{aligned} \quad (49)$$

After t_2 , there is a new delta shock with δS_3 whose propagation velocity and strength, respectively, are given by

$$\sigma_3 = \frac{1}{(1 - u_- + v_-)(1 - u_+ + v_+)}, \quad (50)$$

$$\beta_3(t) = \beta_2(t_2) + \frac{v_+ - u_+}{v_m - u_m} \cdot \frac{u_- v_m - u_m v_-}{(1 - u_- + v_-)(1 - u_+ + v_+)} (t - t_2), \quad (51)$$

where $\beta_2(t_2)$ can be calculated by (48).

The δS_3 interacts with J at point (x_3, t_3) which is resolved by the following equation:

$$\begin{cases} x_3 - 1 = \frac{t_3}{1 - u_+ + v_+}, \\ x_3 - x_2 = \frac{t_3 - t_2}{(1 - u_- + v_-)(1 - u_+ + v_+)}. \end{cases} \quad (52)$$

Recalling (49), the following is clear:

$$\begin{aligned}
(x_3, t_3) &= \left(1 + \frac{(v_m - u_m - v_- + u_-)(1 - u_- + v_-)}{(v_+ - u_+ - v_- + u_-)(u_- - v_-)}, \right. \\
&\left. \frac{(v_m - u_m - v_- + u_-)(1 - u_+ + v_+)(1 - u_- + v_-)}{(v_+ - u_+ - v_- + u_-)(u_- - v_-)} \right).
\end{aligned} \quad (53)$$

After t_3 , there is a new delta shock wave δS_4 which has the same speed with δS_3 . The strength of δS_4 is given by

$$\beta_4(t) = \beta_3(t_3) + \frac{u_- v_+ - u_+ v_-}{(1 - u_- + v_-)(1 - u_+ + v_+)} (t - t_3), \quad t > t_3. \quad (54)$$

Passing to limits as $(v_- - u_-) \rightarrow (-1)^+$ in (26), (49), and (53), we derive

$$\begin{aligned}
\lim_{(v_- - u_-) \rightarrow (-1)^+} (x_1, t_1) &= \lim_{(v_- - u_-) \rightarrow (-1)^+} (x_2, t_2) \\
&= \lim_{(v_- - u_-) \rightarrow (-1)^+} (x_2, t_2) = (1, 0).
\end{aligned} \quad (55)$$

Since $(v_- - u_-) \rightarrow (-1)^+$, then all the propagation speeds and strength of each delta shock wave tend to infinite. Thus, the solution of this case has a zero delta shock wave for $(v_- - u_-) \rightarrow (-1)^+$.

Case 5. $-1 < v_- - u_- < v_+ - u_+ < 0 < v_m - u_m$.

Based on the analysis of Case 4, δS_1 overtakes $\overleftarrow{R}$ (see Figure 6). The intersection point (x_1, t_1) and the strength of delta shock wave at t_1 are given by (26) and (39), respectively. The δS_2 satisfies (28) and its strength is given by (48). The intersection point of the curve of δS_2 and forward wave of $\overleftarrow{R}$ (or back wave of $\overrightarrow{R}$) can be calculated by the following equation:

$$\begin{cases} \sqrt{x_2 - 1} = \frac{\sqrt{t_2}}{1 - u_- + v_-} - \sqrt{\frac{v_m - u_m - (v_- - u_-)}{1 - u_- + v_-}}, \\ x_2 - 1 = t_2, \end{cases} \quad (56)$$

It follows from (56) that

$$\begin{aligned}
(x_2, t_2) &= \left(1 + \frac{(v_m - u_m - v_- + u_-)(1 - u_- + v_-)}{(u_- - v_-)^2}, \right. \\
&\left. \frac{(v_m - u_m - v_- + u_-)(1 - u_- + v_-)}{(u_- - v_-)^2} \right).
\end{aligned} \quad (57)$$

FIGURE 6: When $-1 < v_- - u_- < v_+ - u_+ < 0 < v_m - u_m$.

In view of (48) and (57), we conclude

$$\beta_2(t_2) = \frac{u_- v_m - v_- u_m}{u_- - v_-}. \quad (58)$$

It is easy to see that the state (u, v) of $\vec{R}$ satisfies $u/u_+ = v/v_+$. At the point (x_2, t_2) , a new local Riemann problem containing a Dirac delta measure is formulated as

$$v|_{t=t_2} = \begin{cases} v_-, & x < x_2 \\ \theta, & x > x_2 \end{cases} + \beta_2(t_2) \delta\left(x - x_2 - \frac{t - t_2}{1 - u_- + v_-}\right), \quad (59)$$

$$u|_{t=t_2} = \left\{ \begin{array}{ll} u_-, & x < x_2 \\ \frac{\theta u_+}{v_+}, & x > x_2 \end{array} \right\} + \beta_2(t_2) \delta \left(x - x_2 - \frac{t - t_2}{1 - u_- + v_-} \right), \quad (60)$$

here we assume that $\theta \rightarrow 0$ and satisfies $(x-1)/t = 1/(1+\theta)^2$.

At the time t_2 , the initial data become (59) and (60), we construct the solution with the following form:

$$v = \left\{ \begin{array}{ll} v_-, & x < x_2, \\ \frac{v_- - u_-}{v_+ - u_+} v_+, & x_2 < x < x(t), \\ \theta, & x > x(t), \end{array} \right\} \quad (61)$$

$$\begin{aligned}
& + \beta_2(t_2) \delta \left(x - x_2 - \frac{t - t_2}{1 - u_- + v_-} \right), \\
u = & \left\{ \begin{array}{ll} u_-, & x < x_2, \\ \frac{v_- - u_-}{v_+ - u_+} u_+, & x_2 < x < x(t), \\ \frac{\theta u_+}{v_+}, & x > x(t), \end{array} \right\} \quad (62) \\
& + \beta_2(t_2) \delta \left(x - x_2 - \frac{t - t_2}{1 - u_- + v_-} \right),
\end{aligned}$$

where $x(t) = x_2 + t - t_2 / ((1 - u_- + v_-)(1 - (\theta u_+ / v_+) + \theta))$ given nearby (x_2, t_2) for sufficiently small θ denotes shock wave curve of S_1 .

Next, we clarify equalities (61) and (62) satisfy equations (2) with initial data (59) and (60) in the weak sense. Combining (61) and (62), we deduce

$$v_t = \left(\frac{v_- - u_-}{v_+ - u_+} v_+ \right) \delta \cdot \left(-\frac{1}{1 - u_- + v_-} \right) \quad (63)$$

$$+ \beta_2(t_2) \delta' \left(-\frac{1}{1 - u_- + v_-} \right),$$

$$\begin{aligned} \left(\frac{v}{1-u+v}\right)_x = & \left(\frac{(v_- - u_-)v_+}{(1-u_-+v_-)(v_+ - u_+)} - \frac{v_-}{1-u_-+v_-}\right)\delta \\ & + \frac{\beta_2(t_2)}{1-u_-+v_-}\delta', \end{aligned} \quad (64)$$

where δ and δ' are the functions of $x - x_2 - (t - t_2)/(1 - u_- + v_-)$. In light of (63) and (64), we thus deduce that

$$v_t + \left(\frac{v}{1-u+v} \right)_x = 0. \quad (65)$$

Similarly, we claim that

$$u_t + \left(\frac{u}{1 - u + v} \right)_x = 0. \quad (66)$$

Since contact discontinuity line $x = x_2 + (t - t_2)/(1 - u_- + v_-)$ with a Dirac delta function, then there is a delta contact discontinuity δJ in [35]. After t_2 , δS_2 splits into δJ and S_1 which are connected by intermediate state $(u_1, v_1) = ((v_- - u_-/v_+ - u_+)v_+, (v_- - u_-/v_+ - u_+)u_+)$. The δJ continues to move forwards with the constant speed $\tau = 1/(1 - u_- + v_-)$ and the strength $\beta_2(t_2)$ is given by (58). Besides, S_1 continues to cross the $\overline{R}$ with the initial speed $\omega_1 = 1/(1 - u_- + v_-)$. During the process, the state of the left-hand side is (u_1, v_1) and the right-hand side is $(u, v) = ((\sqrt{(t/x - 1)} - 1)u_+/v_+ - u_+, (\sqrt{(t/x - 1)} - 1)v_+/v_+ - u_+)$. In the physics plane, the curve of S_1 satisfies the following equation:

$$\sqrt{x-1} = \frac{\sqrt{t}}{1-u_-+v_-} - \sqrt{\frac{v_m-u_m-v_-+u_-}{1-u_-+v_-}}. \quad (67)$$

According to $-1 \leq v_- - u_- < v_+ - u_+ < 0$, the S_1 can penetrate the whole $\overrightarrow{R}$ completely at the point (x_3, t_3) , which is

$$(x_3, t_3) = \left(1 + \frac{(v_m - u_m - v_- + u_-)(1 - u_- + v_-)}{(v_+ - u_+ - v_- + u_-)^2}, \right. \\ \left. \frac{(v_m - u_m - v_- + u_-)(1 - u_- + v_-)(1 - u_+ + v_+)^2}{(v_+ - u_+ - v_- + u_-)^2} \right). \quad (68)$$

After t_3 , the shock wave is denoted by S_2 which has the propagation speed $\omega_2 = 1/((1 - u_- + v_-)(1 - u_+ + v_+))$.

As $(v_-, u_-) \rightarrow (-1)^+$, it is clear to see that all intersection points tend to $(1, 0)$ and all speeds of waves tend to

infinite. Thus, a zero delta shock will appear in the interval $[0, 1]$. For $x > 1$, the zero delta shock wave also emerges since the superposition of a zero delta contact discontinuity and a zero shock wave.

Case 6. $-1 < v_- - u_- < 0 < v_m - u_m < v_+ - u_+$.

In this case, we consider the interaction of δS with $\overleftarrow{S} + J$, and the middle state of $\overleftarrow{S}$ and J is $(u_1, v_1) = ((v_+ - u_+)u_m / (v_m - u_m), (v_+ - u_+)v_m / (v_m - u_m))$. It is clear to see that δS_1 catches up $\overleftarrow{S}$ in finite time, and the intersection point (x_1, t_1) can be seen in (33) (see Figure 7). Like as before, due to $-1 < v_- - u_- < 0 < v_+ - u_+$, after the interaction of δS_1 and $\overleftarrow{S}$, there is a new delta shock wave δS_2 whose propagation speed is $\sigma_2 = 1 / ((1 - u_- + v_-)(1 - u_+ + v_+))$.

For $\sigma_2 > \tau_1 = 1 / (1 - u_+ + v_+)$, δS_2 interacts with J_1 at the point (x_2, t_2) which satisfies the following equation:

$$\begin{cases} x_2 - x_1 = \frac{t_2 - t_1}{(1 - u_- + v_-)(1 - u_+ + v_+)}, \\ x_2 - 1 = \frac{t_2}{1 - u_+ + v_+}. \end{cases} \quad (69)$$

Together with (33), we deduce that

$$(x_2, t_2) = \left(\frac{(v_- - u_- - v_m + u_m)(1 - u_- + v_-)}{(v_+ - u_+ - v_- + u_-)(v_- - u_-)} + 1, \frac{(v_- - u_- - v_m + u_m)(1 - u_- + v_-)(1 - u_+ + v_+)}{(v_+ - u_+ - v_- + u_-)(v_- - u_-)} \right). \quad (70)$$

When the delta shock wave passes through J_1 , it continues to move forward without change of direction. The strengths of delta shock waves are given by

$$\beta_1(t) = \frac{u_- v_m - u_m v_-}{(1 - u_- + v_-)(1 - u_m + v_m)} t, \quad 0 \leq t \leq t_1, \quad (71)$$

$$\beta_2(t) = \beta_1(t_1) + \frac{u_- v_1 - u_1 v_-}{(1 - u_- + v_-)(1 - u_1 + v_1)} (t - t_1), \quad t_1 \leq t \leq t_2, \quad (72)$$

$$\beta_3(t) = \beta_2(t_2) + \frac{u_- v_+ - u_+ v_-}{(1 - u_- + v_-)(1 - u_+ + v_+)} (t - t_2), \quad t \geq t_2, \quad (73)$$

in which t_1 and t_2 can be obtained by (33) and (71).

If $(v_- - u_-) \rightarrow (-1)^+$, then the propagation speeds and strengths of all delta shock tend to infinite. Therefore, there is a zero delta shock in the positive x -axis as $(v_- - u_-) \rightarrow (-1)^+$.

In this section, we mainly testify the type (2) of Theorem 1.

5. Analysis of Zero Rarefaction Wave

In this section, we are interested in the zero rarefaction wave interacting with the other waves. As $-1 < v_m - u_m < -1 + \varepsilon$, for sufficiently small ε and $(v_m - u_m) \rightarrow (-1)^+$, there is a zero rarefaction wave at the point $(0, 0)$.

FIGURE 7: When $-1 < v_- - u_- < 0 < v_m - u_m < v_+ - u_+$.

According to $v_- - u_- < 0$ or not, we separate our discussion into two parts. To be more precise, if $v_- - u_- < 0$ and $-1 < v_m - u_m < -1 + \varepsilon$, for sufficiently small ε , then the Riemann solution of (2) at the point $(0, 0)$ is $J + \overrightarrow{R}$. Otherwise, the Riemann solution is $\overleftarrow{R} + \overrightarrow{R}$ at the point $(0, 0)$. Firstly, we study the $v_- - u_- > 0$ and then consider the situation $v_- - u_- < 0$.

Case 7. $-1 < v_m - u_m < v_- - u_- < v_+ - u_+ < 0$.

When $-1 < v_m - u_m < v_- - u_- < v_+ - u_+ < 0$ (see Figure 8), the middle state satisfies $(u_1, v_1) = ((v_- - u_-)/v_m - u_m)u_m, (v_- - u_-)/v_m - u_m)$ and $(u_2, v_2) = ((v_m - u_m)/v_+ - u_+)u_+, (v_m - u_m)/v_+ - u_+)$, respectively. The propagation speeds of J_2 and the wave front of $\overrightarrow{R}_1$ are $\tau_2 = 1 / (1 - u_m + v_m)$ and $\lambda_2^1(u_m, v_m) = 1 / (1 - u_m + v_m)^2$, respectively.

It is easy to see that $\overrightarrow{R}$ overtakes J_2 in finite time, and the interaction point is

$$(x_1, t_1) = \left(\frac{1}{u_m - v_m}, \frac{(1 - u_m + v_m)^2}{u_m - v_m} \right). \quad (74)$$

Based on the above discussions, we derive that J_2 begins to penetrate $\overrightarrow{R}_1$ at point (x_1, t_1) . During the process of penetration, there is a new contract discontinuity J_3 satisfying the following equation:

$$\sqrt{x} = \sqrt{t} + \sqrt{u_m - v_m}. \quad (75)$$

When J_3 crosses the whole $\overrightarrow{R}_1$, there is a new wave J_4 whose propagation speed is same with J_1 .

Besides, we denote $\overrightarrow{R}_2$ after $\overrightarrow{R}_1$ penetrates J_3 . The $\overrightarrow{R}_2$ has the same propagation speed with $\overrightarrow{R}_1$. The shock $\overrightarrow{S}_2$ will penetrate $\overrightarrow{R}_2$, and the new shock $\overrightarrow{S}_2$ appears after t_2 . The wave $\overrightarrow{S}_2$ penetrates whole $\overrightarrow{R}_2$ at the point (x_3, t_3) :

$$(x_3, t_3) = \left(\frac{(v_+ - u_+ - v_m + u_m)(1 - u_+ + v_+)}{(v_+ - u_+ - v_- + u_-)^2}, \frac{(v_+ - u_+ - v_m + u_m)(1 - u_+ + v_+)(1 - u_- + v_-)^2}{(v_+ - u_+ - v_- + u_-)^2} \right). \quad (76)$$

$$\beta_2'(t) = \beta_2'^-(t) + \beta_2'^+(t) = \frac{v_+ u_m - v_m u_+}{(1 - u_+ + v_+)(v_m - u_m)} \cdot \left(\frac{v_+ - u_+}{1 - u_+ + v_+} - \sqrt{\frac{v_+ - u_+ - v_m + u_m}{(1 - u_+ + v_+)t}} \right). \quad (85)$$

Recalling (78), we deduce that

$$\beta_2(t) = \frac{v_+ u_m - v_m u_+}{(1 - u_+ + v_+)(v_m - u_m)} \cdot \left(\frac{v_+ - u_+}{1 - u_+ + v_+} t - 2 \sqrt{\frac{v_+ - u_+ - v_m + u_m}{1 - u_+ + v_+} t} - 1 + u_m - v_m \right). \quad (86)$$

After the time t_2 , δS_2 is decomposed into a shock wave S_1 and a delta contact discontinuity δJ with the middle state $(u_1, v_1) = ((v_+ - u_+/v_- - u_-)u_+, (v_+ - u_+/v_- - u_-)v_+)$ between them. Consequently, δJ continues to move forwards with the constant speed $1/(1 - u_+ + v_+)$ and the invariant strength $\beta_2(t_2)$. On the other hand, S_1 continues to penetrate $\overleftarrow{R}$ with a varying propagation speed, which satisfies the following equation:

$$\begin{cases} \frac{dx}{dt} = \frac{1}{1 - u_+ + v_+} \sqrt{\frac{x}{t}}, \\ (x_2, t_2) = \left(\frac{(v_+ - u_+ - v_m + u_m)(1 - u_+ + v_+)}{v_+ - u_+}, \right. \\ \left. \frac{(v_+ - u_+ - v_m + u_m)(1 - u_+ + v_+)}{v_+ - u_+} \right), \end{cases} \quad (87)$$

thus

$$\sqrt{x} = \frac{\sqrt{t}}{1 - u_+ + v_+} + \sqrt{\frac{(v_+ - u_+ - v_m + u_m)(v_+ - u_+)}{1 - u_+ + v_+}}. \quad (88)$$

For $-1 < 0 < v_+ - u_+ < v_- - u_-$, S_1 is able to penetrate whole $\overleftarrow{R}$.

For $(v_m - u_m) \rightarrow (-1)^+$, it is clear to see that zero rarefaction wave only exists in the interval $[0, 1]$ since the delta shock wave begins to penetrate the rarefaction wave from the point $(1, 0)$.

Case 11. $-1 < v_m - u_m < 0 < v_+ - u_+ < v_- - u_-$.

In this case, it is easy to find similar to Case 10, and S_1 cannot penetrate $\overleftarrow{R}$ completely and has the asymptote line $x = t/(1 - u_+ + v_+)^2$.

Case 12. $-1 < v_m - u_m < v_+ - u_+ < 0 < v_- - u_-$.

As in this case, we are concerned with the interaction of $\overleftarrow{R} + \overrightarrow{R}$ and $J + S$. In the beginning, the situation is similar to Case 7. More precisely, J has asymptote $x = t$ and do not able to enter the region of $\overleftarrow{R}$. On the other hand, $\overrightarrow{R}$ intersects

with J denoted $\overrightarrow{R}_1$, and the shock wave S cannot penetrate whole $\overrightarrow{R}_1$ and finally has the asymptote $x = t/(1 - u_+ + v_+)^2$.

Therefore, we have proved the type (3) of Theorem 1.

6. Numerical Simulations

In this section, we present some representative numerical simulations for the formation of zeros waves, mentioned in this paper, when the differences of initial data tend to -1 from the right side. In this paper, we employ the upwind scheme to validate our main results.

The following numerical simulations correspond to the zero shock wave of Case 1 (Figure 2). We take the initial data as follows:

$$\begin{aligned} u_- &= 1.9, v_- = 1, \\ u_m &= 3.5, \\ v_m &= 3, \\ u_+ &= 2.7, \\ v_+ &= 2, \end{aligned} \quad (89)$$

$$\begin{aligned} u_- &= 1.99, \\ v_- &= 1, \\ u_m &= 3.5, \\ v_m &= 3, \\ u_+ &= 2.7, \\ v_+ &= 2. \end{aligned} \quad (90)$$

This implies that the $v_- - u_-$ from -0.9 (Figure 10) tends to -0.99 (Figure 11). It can be clearly seen from the numerical results that when $v_- - u_-$ approaches -0.99 from -0.9 , u , v and the region of negative states all increase. Namely, the velocity of shock wave increases. That is to say, if $-1 < v_- - u_- < v_m - u_m < 0$, $v_+ - u_+ < 0$ and if $(v_- - u_-) \rightarrow (-1)^+$, then the velocity tends to ∞ . There is a zero shock wave.

The numerical simulations (Figures 12 and 13) correspond to the zero delta shock wave of Case 4 (Figure 5) with initial data as follows:

$$\begin{aligned} u_- &= 1.9, \\ v_- &= 1, \\ u_m &= 1, \\ v_m &= 3, \end{aligned} \quad (91)$$

$$\begin{aligned} u_+ &= 50, \\ v_+ &= 51, \\ u_- &= 1.99, \\ v_- &= 1, \\ u_m &= 1, \\ v_m &= 3, \\ u_+ &= 50, \\ v_+ &= 51. \end{aligned} \quad (92)$$

FIGURE 10: (a) u and (b) v for $v_- - u_- = 0.9$ with $v_- = 1$, $u_- = 1.9$, $v_m = 3.5$, $u_m = 3$, $v_+ = 2$, and $u_+ = 2.7$.

FIGURE 11: (a) u and (b) v for $v_- - u_- = 0.99$ with $v_- = 1$, $u_- = 1.99$, $v_m = 3.5$, $u_m = 3$, $v_+ = 2$, and $u_+ = 2.7$.

FIGURE 12: (a) u and (b) v for $v_- - u_- = 0.9$ with $u_- = 1.9$, $v_- = 1$, $u_m = 1$, $v_m = 3$, $u_+ = 50$, and $v_+ = 51$.

That is to say, $v_- - u_-$ from -0.9 (Figure 12) tends to -0.99 (Figure 13). As can be seen from the following numerical results, the negative state region and the values of u and v are getting more larger. As $v_- - u_-$ tends to -1 , the shock velocity and u , v approach ∞ . In other words, when

$-1 < v_- - u_- < 0 < v_m - u_m$, $v_- - u_- < v_+ - u_+$, and $(v_- - u_-) \rightarrow (-1)^+$, there is a zero delta shock wave.

The numerical simulations (Figures 14 and 15) correspond to the zero rarefaction waves of Case 7 (Figure 8) with the initial data as follows:

FIGURE 13: (a) u and (b) v for $v_- - u_- = 0.99$ with $u_- = 1.99$, $v_- = 1$, $u_m = 1$, $v_m = 3$, $u_+ = 50$, and $v_+ = 51$.

FIGURE 14: (a) u and (b) v for $v_m - u_m = 0.9$ with $u_- = 1.5$, $v_- = 1$, $u_m = 1.9$, $v_m = 1$, $u_+ = 2.3$, and $v_+ = 2$.

FIGURE 15: (a) u and (b) v for $v_m - u_m = 0.99$ with $u_- = 1.5$, $v_- = 1$, $u_m = 1.99$, $v_m = 1$, $u_+ = 2.3$, and $v_+ = 2$.

$$\begin{aligned}
 u_- &= 1.5, \\
 v_- &= 1, \\
 u_m &= 1.9, \\
 v_m &= 1, \\
 u_+ &= 2.3, \\
 v_+ &= 2,
 \end{aligned}
 \tag{93}$$

$$\begin{aligned}
 u_- &= 1.5, \\
 v_- &= 1, \\
 u_m &= 1.99, \\
 v_m &= 1, \\
 u_+ &= 2.3, \\
 v_+ &= 2.
 \end{aligned}
 \tag{94}$$

This implies that the $v_m - u_m$ from -0.9 (Figure 14) tends to -0.99 (Figure 15). In order to better explain the generation of zero rarefaction wave on the interval $[0, 1]$, we subdivide $[-1, 3]$ on the space. It can be seen from the following numerical simulations that in a very short period of time, when $v_- - u_-$ approaches -0.99 from -0.9 , the intermediate state $(u_m, v_m) = (1.9, 1)$ disappears. That is to say, the wave velocity of rarefaction wave $\vec{R}_1$ increases. This is very consistent with our conclusion when $-1 < v_m - u_m < 0$ and $v_m - u_m < \min\{v_- - u_-, v_+ - u_+\}$, a zero rarefaction wave is alive in the interval $[0, 1]$ when $(v_m - u_m) \rightarrow (-1)^+$.

7. Conclusions

In this paper, we mainly study the zero waves of the non-simplified chromatographic equation. From the analysis of Section 3, one can know when $-1 < v_- - u_- < v_m - u_m < 0$, on the one hand, the solution performs as a zero shock wave if $v_+ - u_+ < 0$ and $(v_- - u_-) \rightarrow (-1)^+$. On the other hand, its solution performs as a zero shock wave in the interval $[0, 1]$ and a zero delta shock wave in the interval $(1, \infty)$ if $v_+ - u_+ > 0$ and $(v_- - u_-) \rightarrow (-1)^+$. From the analysis of Section 4, when $-1 < v_- - u_- < 0 < v_m - u_m$ and $v_- - u_- < v_+ - u_+$, the solution performs as a zero delta shock wave when $(v_- - u_-) \rightarrow (-1)^+$. From the analysis of Section 5, when $-1 < v_m - u_m < 0$ and $v_m - u_m < \min\{v_- - u_-, v_+ - u_+\}$, a zero rarefaction wave is alive in the interval $[0, 1]$ when $(v_m - u_m) \rightarrow (-1)^+$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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