Research Article

A Highly Efficient and Accurate Finite Iterative Method for Solving Linear Two-Dimensional Fredholm Fuzzy Integral Equations of the Second Kind Using Triangular Functions

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This work introduces a computational method for solving the linear two-dimensional fuzzy Fredholm integral equation of the second form (2D-FFIE-2) based on triangular basis functions. We have used the parametric form of fuzzy functions and transformed a 2D-FFIE-2 with three variables in crisp case to a linear Fredholm integral equation of the second kind. First, a method based on the use of two m -sets of orthogonal functions of triangular form is implemented on the integral equation under study to be changed to coupled algebraic equation system. In order to solve these two schemes, a finite iterative algorithm is then applied to evaluate the coefficients that provided the approximate solution of the integral problems. Three examples are given to clarify the efficiency and accuracy of the method. The obtained numerical results are compared with other direct and exact solutions.

1. Introduction

Several methods have been developed to estimate the solution of integral equation systems [1–3]. Many simple functions are used to approximate the solution of integral equations, such as orthogonal bases dependent on wavelets [4]. In addition, Maleknejad and Mirzaee developed the rationalized Haar functions [5] to approximate the solutions of the Fredholm linear integral equation method. In addition, second-type Fredholm integral equations are solved using direct triangular functions method as seen in [6] and using iterative algorithm-hybrid triangular functions method presented by Ramadan and Ali [7] where this hybrid method treats Fredholm integral equation of one dimension. More recently, Ramadan et al. [8] implemented such hybrid method to tackle system of two linear Fredholm integral equations of one dimension.

Furthermore, Maleknejad et al. [9] suggested by block pulse functions a numerical solution of the integral second-type equation.

It is explained using a series of orthogonal triangular functions, derived from the series of block pulses. Nevertheless, the fuzzy integral equations (FIEs) are required to solve and research a wide number of problems in various applied mathematics subjects, such as connection to physics, spatial, medical, and biology. FIEs therefore require approximate numerical solutions, as they are typically difficult to analytically solve. This thesis introduces a methodology used by the triangular functions (TFs) to solve the fuzzy linear FIE method of the second kind. In various implementation problems, certain parameters are typically represented by a fuzzy number rather than a crisp state, which involves the creation of mathematical models and computational algorithms to handle and solve the general fuzzy integral equations. A general method for solving the fuzzy Fredholm second-type integral equation is proposed in [10]. Recently, numerical methods have been developed to solve linear fuzzy Fredholm integral equation of the second kind in one-dimensional space (1D-FFIE-2) and two-dimensional space (2D-FFIE-2). Also, Fredholm fuzzy integral equations

of the second kind are solved using the triangular functions [11], and numerical solution of linear Fredholm fuzzy equation of the second kind by block pulse functions is considered in [12]. Barkhordary et al. and Ramadan et al. [13, 14] presented a numerical technique for solving the fuzzy Fredholm integral equation of second kind. Numerical solution of two-dimensional fuzzy Fredholm integral equations of the second kind is presented via direct method using triangular functions [15]. Nouriani et al. [16] proposed a quadrature iterative method for solving the two-dimensional fuzzy Fredholm integral equations. Ezzati and Ziari [17], Hengamian Asl and Saberi-Nadjafi [18], and Bica and Popescu [19] illustrated a solution of the two-dimensional fuzzy Fredholm integral equations. A modified homotopy perturbation method for solving the two-dimensional fuzzy Fredholm integral equation is detailed in [20]. A two-dimensional nonlinear Volterra–Fredholm fuzzy integral equation is solved by using the Adomian decomposition method [21] and fuzzy bivariate triangular functions [22].

The aim of paper is to generalize the work proposed in [7] and [8] of these basis orthogonal triangular functions on $(0, 1)$ to solve two-dimensional fuzzy Fredholm integral equations.

$$\tilde{u}(x, y) = \tilde{f}(x, y) + \lambda \int_a^b \int_c^d k(x, y, s, t) \tilde{u}(s, t) ds dt. \quad (1)$$

Section 2 presents some definitions and properties of the orthogonal triangular functions (TFs) (1D-TFs and 2D-TFs). Also, it expands functions by TFs. In Section 3, the definitions and properties of fuzzy function are given while a finite iterative algorithm is presented to solve coupled system of matrix equations in Section 4. The two-dimensional fuzzy integral equation is demonstrated and explained in Section 5 while the suggested method and the proposed iterative algorithm are detailed in Section 6. The illustrative examples and numerical results obtained are presented and discussed in Section 8.

2. Review of Triangular Functions (TFs)

2.1. Triangular Functions (TFs) of One Dimension

Definition 1. Two m -sets of triangular functions (TFs) are defined over the interval $[0, T]$ [5]:

$$T1_i(t) = \begin{cases} 1 - \frac{t - ih}{h}, & ih \leq t < (i+1)h, \\ 0, & \text{o.w.}, \end{cases} \quad (2)$$

$$T2_i(t) = \begin{cases} \frac{t - ih}{h}, & ih \leq t < (i+1)h, \\ 0, & \text{o.w.}, \end{cases}$$

where $i = 0, 1, \dots, m-1$; m has a positive integer value; $h = (T/m)$; $T1_i$ is the i th left-handed triangular function; and $T2_i$ is the i th right-handed triangular function.

Assuming $T = 1$, the TFs are defined over $[0, 1)$ and $h = (1/m)$. Based on this definition, it is clear that TFs are disjoint, orthogonal, and complete [5]. Therefore, one may write

$$\int_0^1 T1_i(t)T1_j(t)dt = \int_0^1 T2_i(t)T2_j(t)dt = \begin{cases} \frac{h}{3}, & i = j, \\ 0, & i \neq j, \end{cases} \quad (3)$$

$$\int_0^1 T1_i(t)T2_j(t)dt = \int_0^1 T2_i(t)T1_j(t)dt = \begin{cases} \frac{h}{6}, & i = j, \\ 0, & i \neq j. \end{cases} \quad (4)$$

The first m terms in the left-hand triangular functions and in the right-hand triangular functions can be written concisely in m -vectors format as

$$T1(t) = [T1_0(t), T1_1(t), \dots, T1_{m-1}(t)]^T, \quad (5)$$

$$T2(t) = [T2_0(t), T2_1(t), \dots, T2_{m-1}(t)]^T,$$

where $T1(t)$ and $T2(t)$ are called left-handed triangular function (LHTF) vector and right-handed triangular function (RHTF) vector, respectively. The product of two TF vectors yields the following properties:

$$T1(t)T1^T(t) \cong \begin{pmatrix} T1_0(t) & 0 & \dots & 0 \\ 0 & T1_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T1_{m-1}(t) \end{pmatrix},$$

$$T2(t)T2^T(t) \cong \begin{pmatrix} T2_0(t) & 0 & \dots & 0 \\ 0 & T2_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T2_{m-1}(t) \end{pmatrix}, \quad (6)$$

$$T1(t)T2^T(t) \cong 0,$$

$$T2(t)T1^T(t) \cong 0,$$

where 0 is the zero $m \times m$ matrix. Also,

$$\int_0^1 T1(t)T1^T(t)dt = \int_0^1 T2(t)T2^T(t)dt \cong \frac{h}{3}I, \quad (7)$$

$$\int_0^1 T1(t)T2^T(t)dt = \int_0^1 T2(t)T1^T(t)dt \cong \frac{h}{6}I,$$

in which I is an $m \times m$ identity matrix.

2.2. Two-Dimensional Triangular Functions and Their Properties [15]. An $(m_1 \times m_2)$ -set of 2D-TFs on the region $(\Omega = [0, 1] \times [0, 1])$ is defined by

$$\begin{aligned}
T_{i,j}^{1,1}(s,t) &= \begin{cases} \left(1 - \frac{s - ih_1}{h_1}\right) \left(1 - \frac{t - jh_2}{h_2}\right), & ih_1 \leq s \leq (i+1)h_1, \\ & jh_2 \leq t \leq (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases} \\
T_{i,j}^{1,2}(s,t) &= \begin{cases} \left(1 - \frac{s - ih_1}{h_1}\right) \left(\frac{t - jh_2}{h_2}\right), & ih_1 \leq s \leq (i+1)h_1, \\ & jh_2 \leq t \leq (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases} \\
T_{i,j}^{2,1}(s,t) &= \begin{cases} \left(\frac{s - ih_1}{h_1}\right) \left(1 - \frac{t - jh_2}{h_2}\right), & ih_1 \leq s \leq (i+1)h_1, \\ & jh_2 \leq t \leq (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases} \\
T_{i,j}^{2,2}(s,t) &= \begin{cases} \left(\frac{s - ih_1}{h_1}\right) \left(\frac{t - jh_2}{h_2}\right), & ih_1 \leq s \leq (i+1)h_1, \\ & jh_2 \leq t \leq (j+1)h_2, \\ 0, & \text{otherwise,} \end{cases} \quad (8)
\end{aligned}$$

where: $i = 0, 1, 2, \dots, m_1 - 1; j = 0, 1, 2, \dots, m_2 - 1;$
 $h_1 = (1/m_1); h_2 = (1/m_2);$ and m_1 and m_2 are arbitrary positive integers. Therefore,

$$\begin{cases} T_{i,j}^{1,1}(s,t) = T1_i(s) \cdot T1_j(t), \\ T_{i,j}^{1,2}(s,t) = T1_i(s) \cdot T2_j(t), \\ T_{i,j}^{2,1}(s,t) = T2_i(s) \cdot T1_j(t), \\ T_{i,j}^{2,2}(s,t) = T2_i(s) \cdot T2_j(t). \end{cases} \quad (9)$$

Furthermore,

$$T_{i,j}^{1,1}(s,t) + T_{i,j}^{1,2}(s,t) + T_{i,j}^{2,1}(s,t) + T_{i,j}^{2,2}(s,t) = \phi_{i,j}(s,t), \quad (10)$$

where $\phi_{i,j}(s,t)$ is the $\{im_2 + j + 1\}$ th block pulse function defined on $ih_1 \leq s \leq (i+1)h_1$ and $jh_2 \leq t \leq (j+1)h_2$ as

$$\phi_{i,j}(s,t) = \begin{cases} 1, & ih_1 \leq s \leq (i+1)h_1, \\ & jh_2 \leq t \leq (j+1)h_2, \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

Each of the sets $\{T_{i,j}^{1,1}(s,t), T_{i,j}^{1,2}(s,t), T_{i,j}^{2,1}(s,t)\}$ and $\{T_{i,j}^{2,2}(s,t)\}$ is obviously disjoint:

$$T_{i_1 j_1}^{p_1 q_1}(s,t) \cdot T_{i_2 j_2}^{p_2 q_2}(s,t) \approx \begin{cases} T_{i_1 j_1}^{p_1 q_1}(s,t), & p_1 = p_2, q_1 = q_2, \\ & i_1 = i_2, j_1 = j_2, \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

For $p, q \in \{1, 2\}, i_1, i_2 = 0, 1, 2, \dots, m_1 - 1$ and $j_1, j_2 = 0, 1, 2, \dots, m_2 - 1.$

Also, the 2D-TFs are orthogonal, that is,

$$\int_0^1 \int_0^1 T_{i_1 j_1}^{p_1 q_1}(s,t) \cdot T_{i_2 j_2}^{p_2 q_2}(s,t) ds dt = \Delta_{p_1 p_2} \delta_{i_1 i_2} \cdot \Delta_{q_1 q_2} \delta_{j_1 j_2}, \quad (13)$$

where δ denotes the Kronecker delta function and

$$\Delta_{\alpha\beta} = \begin{cases} \frac{h}{3}, & \alpha = \beta \in \{1, 2\}, \\ \frac{h}{6}, & \alpha \neq \beta. \end{cases} \quad (14)$$

On the other hand, if

$$\begin{aligned}
T11(s,t) &= [T_{0,0}^{1,1}(s,t), \dots, T_{i, m_2-1}^{1,1}(s,t), T_{1,0}^{1,1}(s,t), \dots, T_{m_1-1, m_2-1}^{1,1}(s,t)]^T, \\
T12(s,t) &= [T_{0,0}^{1,2}(s,t), \dots, T_{i, m_2-1}^{1,2}(s,t), T_{1,0}^{1,2}(s,t), \dots, T_{m_1-1, m_2-1}^{1,2}(s,t)]^T, \\
T21(s,t) &= [T_{0,0}^{2,1}(s,t), \dots, T_{i, m_2-1}^{2,1}(s,t), T_{1,0}^{2,1}(s,t), \dots, T_{m_1-1, m_2-1}^{2,1}(s,t)]^T, \\
T22(s,t) &= [T_{0,0}^{2,2}(s,t), \dots, T_{i, m_2-1}^{2,2}(s,t), T_{1,0}^{2,2}(s,t), \dots, T_{m_1-1, m_2-1}^{2,2}(s,t)]^T, \end{aligned} \quad (15)$$

then $T(s,t)$, the 2D-TF vector, can be defined as

$$T(s, t) = \begin{bmatrix} T11(s, t) \\ T12(s, t) \\ T21(s, t) \\ T22(s, t) \end{bmatrix}_{4m_1 m_2 \times 1},$$

$$T11 \cdot T11^T \simeq \begin{bmatrix} T_{0,0}^{1,1} & 0 & \cdots & 0 \\ 0 & T_{0,1}^{1,1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & T_{m_1-1, m_2-1}^{1,1} \end{bmatrix} = \text{diag}(T11),$$

$$T11 \cdot T12^T \simeq 0_{m_1 m_2 \times m_1 m_2},$$

$$T11 \cdot T21^T \simeq 0_{m_1 m_2 \times m_1 m_2},$$

$$T11 \cdot T22^T \simeq 0_{m_1 m_2 \times m_1 m_2}. \quad (16)$$

These relations are also satisfied for $T12(s, t)$, $T21(s, t)$, $T22(s, t)$, similarly. Hence,

$$T \cdot T^T \simeq \begin{bmatrix} \text{diag}(T11) & 0_{m_1 m_2 \times m_1 m_2} & 0_{m_1 m_2 \times m_1 m_2} & 0_{m_1 m_2 \times m_1 m_2} \\ 0_{m_1 m_2 \times m_1 m_2} & \text{diag}(T12) & 0_{m_1 m_2 \times m_1 m_2} & 0_{m_1 m_2 \times m_1 m_2} \\ 0_{m_1 m_2 \times m_1 m_2} & 0_{m_1 m_2 \times m_1 m_2} & \text{diag}(T21) & 0_{m_1 m_2 \times m_1 m_2} \\ 0_{m_1 m_2 \times m_1 m_2} & 0_{m_1 m_2 \times m_1 m_2} & 0_{m_1 m_2 \times m_1 m_2} & \text{diag}(T22) \end{bmatrix}, \quad (17)$$

$$\text{or } T(s, t) \cdot T^T(s, t) \simeq \text{diag}(T(s, t)).$$

Finally by the orthogonality of $T11$, we have

$$\begin{aligned} \int_0^1 \int_0^1 T11(s, t)^T T11(s, t) ds dt &= \frac{h_1}{3} I_{m_1 \times m_1} \otimes \frac{h_2}{3} I_{m_2 \times m_2}, \\ \int_0^1 \int_0^1 T11(s, t)^T T12(s, t) ds dt &= \frac{h_1}{3} I_{m_1 \times m_1} \otimes \frac{h_2}{6} I_{m_2 \times m_2}, \\ \int_0^1 \int_0^1 T11(s, t)^T T21(s, t) ds dt &= \frac{h_1}{6} I_{m_1 \times m_1} \otimes \frac{h_2}{3} I_{m_2 \times m_2}, \\ \int_0^1 \int_0^1 T11(s, t)^T T22(s, t) ds dt &= \frac{h_1}{6} I_{m_1 \times m_1} \otimes \frac{h_2}{6} I_{m_2 \times m_2}, \end{aligned} \quad (18)$$

where $\otimes$ denotes the Kronecker product defined for two arbitrary matrices P and Q as

$$P \otimes Q = P_{i,j} Q. \quad (19)$$

The same equations are implied for $T12(s, t)$, $T21(s, t)$, and $T22(s, t)$, by similar computations. Hence, we can carry out double integration of $T(s, t)$:

$$\int_0^1 \int_0^1 T(s, t)^T T(s, t) ds dt = D, \quad (20)$$

where D is $4m_1 m_2 \times 4m_1 m_2$ matrix as follows:

$$D = \begin{bmatrix} \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 \\ \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 \\ \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 \\ \frac{h_1}{6} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{6} I_1 \otimes \frac{h_2}{3} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{6} I_2 & \frac{h_1}{3} I_1 \otimes \frac{h_2}{3} I_2 \end{bmatrix}, \quad (21)$$

where $I_1 = I_{m_1 \times m_1}$ and $I_2 = I_{m_2 \times m_2}$.

2.3. Function Expansion with 1D-TFs and 2D-TFs. The expansion of functions using triangular functions occurs in four situations.

- (1) The expansion of function $f(t)$ over $[0, 1)$ with respect to 1D-TFs is compactly written as

$$f(t) \simeq \sum_{i=0}^{m-1} c_i T1_i(t) + \sum_{i=0}^{m-1} d_i T2_i(t) = c^T T1(t) + d^T T2(t), \quad (22)$$

where we may put $c_i = f(ih)$ and $d_i = f((i+1)h)$ for $i = 0, 1, \dots, m-1$.

- (2) The expansion of the function $f(s, t)$ defined over Ω ($[0, 1] \times [0, 1]$) by 2D-TFs is as follows:

$$f(s, t) = \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} c_{i,j} T_{i,j}^{1,1}(s, t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} d_{i,j} T_{i,j}^{1,2}(s, t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} e_{i,j} T_{i,j}^{2,1}(s, t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} l_{i,j} T_{i,j}^{2,2}(s, t) \quad (23)$$

$$f(s, t) = C^T T11(s, t) + D^T T12(s, t) + E^T T21(s, t) + L^T T22(s, t),$$

where F is a $4m_1m_2$ vector given by

$$F = [C^T \ D^T \ E^T \ L^T]^T, \quad (24)$$

and $T(s, t)$ is defined in equation (21). The 2D-TF coefficients in C, D, E , and L can be computed by sampling the function $f(s, t)$ at grid points s_i and t_j such that $s_i = ih_1$ and $t_j = jh_2$, for various i and j . So, we have

$$\begin{aligned} C_k &= c_{i,j} = f(s_i, t_j), \\ D_k &= d_{i,j} = f(s_i, t_{j+1}), \\ E_k &= e_{i,j} = f(s_{i+1}, t_j), \\ L_k &= l_{i,j} = f(s_{i+1}, t_{j+1}), \end{aligned} \quad (25)$$

where $k = im_2 + j$ and $i = 0, 1, 2, \dots, m_1 - 1$, $j = 0, 1, 2, \dots, m_2 - 1$. The $4m_1m_2$ vector F is called the 2D-TF coefficient vector.

- (3) The expansion of the function $f(s, t, r)$ of three variables on $(\Omega \times [0, 1])$ with respect to 2D-TFs and 1D-TFs is as follows:

$$f(s, t, r) = T^T(s, t) \cdot F \cdot T(r), \quad (26)$$

where $T(s, t)$ and $T(r)$ are 2D-TF vector and 1D-TF vector of dimension $4m_1m_2$ and $2m_3$, respectively, and F is a $(4m_1m_2 \times 2m_3)$ 2D-TF coefficient matrix. This matrix can be represented as

$$F = \begin{bmatrix} F11 & F12 \\ F21 & F22 \\ F31 & F32 \\ F41 & F42 \end{bmatrix}, \quad (27)$$

where each block of F is an $(m_1m_2 \times m_3)$ -matrix that can be computed by sampling the function $f(s, t, r)$ at grid points (s_i, t_j, r_k) such that

$$\begin{aligned} s_i &= ih_1, \quad i = 0, 1, \dots, m_1 - 1, \quad h_1 = \frac{1}{m_1}, \\ t_j &= jh_2, \quad j = 0, 1, \dots, m_2 - 1, \quad h_2 = \frac{1}{m_2}, \\ r_k &= kh_3, \quad k = 0, 1, \dots, m_3 - 1, \quad h_3 = \frac{1}{m_3}. \end{aligned} \quad (28)$$

Let $l = im_2 + j$; then,

$$\begin{aligned} F11_{l,k} &= f(s_i, t_j, r_k), \\ F12_{l,k} &= f(s_i, t_j, r_{k+1}), \\ F21_{l,k} &= f(s_i, t_{j+1}, r_k), \\ F22_{l,k} &= f(s_i, t_{j+1}, r_{k+1}), \\ F21_{l,k} &= f(s_i, t_{j+1}, r_k), \\ F22_{l,k} &= f(s_i, t_{j+1}, r_{k+1}), \\ F31_{l,k} &= f(s_{i+1}, t_j, r_k), \\ F32_{l,k} &= f(s_{i+1}, t_j, r_{k+1}), \\ F31_{l,k} &= f(s_{i+1}, t_j, r_k), \\ F32_{l,k} &= f(s_{i+1}, t_j, r_{k+1}), \\ F41_{l,k} &= f(s_{i+1}, t_{j+1}, r_k), \\ F42_{l,k} &= f(s_{i+1}, t_{j+1}, r_{k+1}). \end{aligned} \quad (29)$$

- (4) The expansion of the function $k(s, t, x, y)$ of four variables on $(\Omega \times \Omega)$ with respect to 2D-TFs is as follows:

$$k(s, t, x, y) = T^T(s, t) \cdot K \cdot T(x, y), \quad (30)$$

where $T(S, T)$ and $T(x, y)$ are 2D-TF vectors of dimension $4m_1m_2$ and $4m_3m_4$, respectively, and K is a $(4m_1m_2 \times 4m_3m_4)$ 2D-TF coefficient matrix. This matrix can be represented as

$$K = \begin{bmatrix} K11 & K12 & K13 & K14 \\ K21 & K22 & K23 & K24 \\ K31 & K32 & K33 & K34 \\ K41 & K42 & K43 & K44 \end{bmatrix}, \quad (31)$$

where each block of K is an $(m_1 m_2 \times m_3 m_4)$ matrix that can be computed by sampling the function $k(s, t, x, y)$ at grid points $(s_{i_1}, t_{j_1}, x_{i_2}, y_{j_2})$ such that

$$\begin{aligned} s_{i_1} &= i_1 h_1, \quad i_1 = 0, 1, \dots, m_1 - 1, \quad h_1 = \frac{1}{m_1}, \\ t_{j_1} &= j_1 h_2, \quad j_1 = 0, 1, \dots, m_2 - 1, \quad h_2 = \frac{1}{m_2}, \\ x_{i_2} &= i_2 h_3, \quad i_2 = 0, 1, \dots, m_3 - 1, \quad h_3 = \frac{1}{m_3}, \\ y_{j_2} &= j_2 h_4, \quad j_2 = 0, 1, \dots, m_4 - 1, \quad h_4 = \frac{1}{m_4}. \end{aligned} \quad (32)$$

Let $p = i_1 m_2 + j_1$ and $q = i_2 m_4 + j_2$; then,

$$\begin{aligned} K11_{p,q} &= k(s_{i_1}, t_{j_1}, x_{i_2}, y_{j_2}), \\ K12_{p,q} &= k(s_{i_1}, t_{j_1}, x_{i_2}, y_{j_2+1}), \\ K13_{p,q} &= k(s_{i_1}, t_{j_1}, x_{i_2+1}, y_{j_2}), \\ K14_{p,q} &= k(s_{i_1}, t_{j_1}, x_{i_2+1}, y_{j_2+1}), \\ K21_{p,q} &= k(s_{i_1}, t_{j_1+1}, x_{i_2}, y_{j_2}), \\ K22_{p,q} &= k(s_{i_1}, t_{j_1+1}, x_{i_2}, y_{j_2+1}), \\ K23_{p,q} &= k(s_{i_1}, t_{j_1+1}, x_{i_2+1}, y_{j_2}), \\ K24_{p,q} &= k(s_{i_1}, t_{j_1+1}, x_{i_2+1}, y_{j_2+1}), \\ K31_{p,q} &= k(s_{i_1+1}, t_{j_1}, x_{i_2}, y_{j_2}), \\ K32_{p,q} &= k(s_{i_1+1}, t_{j_1}, x_{i_2}, y_{j_2+1}), \\ K33_{p,q} &= k(s_{i_1+1}, t_{j_1}, x_{i_2+1}, y_{j_2}), \\ K34_{p,q} &= k(s_{i_1+1}, t_{j_1}, x_{i_2+1}, y_{j_2+1}), \\ K41_{p,q} &= k(s_{i_1+1}, t_{j_1+1}, x_{i_2}, y_{j_2}), \\ K42_{p,q} &= k(s_{i_1+1}, t_{j_1+1}, x_{i_2}, y_{j_2+1}), \\ K43_{p,q} &= k(s_{i_1+1}, t_{j_1+1}, x_{i_2+1}, y_{j_2}), \\ K44_{p,q} &= k(s_{i_1+1}, t_{j_1+1}, x_{i_2+1}, y_{j_2+1}). \end{aligned} \quad (33)$$

In this paper, we suppose that $m_1 = m_2 = m_3 = m_4 = M$ for convergence.

3. Fuzzy Functions

We now remember through the paper some definitions that are required.

Definition 2. A fuzzy number is a fuzzy set $u: R^1 \rightarrow [0, 1]$ that conforms to the following condition [23]:

- (a) u is upper semicontinuous
- (b) $u(x) = 0$ outside some interval $[c, d]$
- (c) There are real numbers a and b , $c \leq a \leq b \leq d$, for which
 - (i) $u(x)$ is increasing in monotonic manner on $[c, a]$
 - (ii) $u(x)$ is decreasing in monotonic manner on $[b, d]$
 - (iii) $u(x) = 1$ for $a \leq x \leq b$

Definition 3. A fuzzy number u is a pair $(\underline{u}(r), \bar{u}(r))$ of functions $\underline{u}(r)$ and $\bar{u}(r)$, $0 \leq r \leq 1$, satisfying the following requirement [5]:

- (a) $\underline{u}(r)$ is bounded monotonic increasing left continuous function
- (b) $\bar{u}(r)$ is bounded monotonic decreasing left continuous function
- (c) $\underline{u}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$

For arbitrary $u = (\underline{u}(r), \bar{u}(r))$, $v = (\underline{v}(r), \bar{v}(r))$, and $k > 0$, we define addition $(u + v)$ and multiplication by k as

$$\begin{aligned} (\underline{u} + \underline{v})(r) &= \underline{u}(r) + \underline{v}(r) \\ (\overline{u} + \overline{v})(r) &= \bar{u}(r) + \bar{v}(r), \\ (\underline{ku})(r) &= k \underline{u}(r), \\ (\overline{ku})(r) &= k \bar{u}(r). \end{aligned} \quad (34)$$

4. Solving Coupled System of Matrix Equations Using Finite Iterative Algorithm [5]

Matrix equations can be solved using various forms of the finite iterative algorithms example [1–3, 5]. We consider iterative solutions to coupled system similar to the forms of Sylvester matrix equations [5].

$$AV + BW = C, \quad (35)$$

and second algorithm to solve coupled system of Sylvester matrix equations:

$$\begin{aligned} A_1 V + B_1 W &= C_1, \\ A_2 V + B_2 W &= C_2. \end{aligned} \quad (36)$$

Algorithm 1 (see [5]). A finite iterative algorithm is developed to solve equation (35) as follows:

- (1) Input A, B, C .
- (2) Pick arbitrary matrices $V \in \mathbb{R}^{n \times p}$ and $W_1 \in \mathbb{R}^{r \times p}$.
- (3) Set

$$\begin{aligned}
R_1 &= C - AV_1 - BW_1, \\
P_1 &= A^T R_1, \\
Q_1 &= B^T R_1, \\
K &= 1.
\end{aligned} \tag{37}$$

- (4) If $R_K = 0$, then stop and V_K and W_K are the final solutions; else, let $K = K + 1$ and go to step 5.
(5) Calculate

$$\begin{aligned}
V_{K+1} &= V_K + \frac{\|R_K\|^2}{\|P_K\|^2 + \|Q_K\|^2} P_K, \\
W_{K+1} &= W_K + \frac{\|R_K\|^2}{\|P_K\|^2 + \|Q_K\|^2} Q_K, \\
R_{K+1} &= C - AV_{K+1} - BW_{K+1} \\
&= R_K - \frac{\|R_K\|^2}{\|P_K\|^2 + \|Q_K\|^2} [AP_K + BQ_K], \\
P_{K+1} &= A^T R_{K+1} + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} P_K, \\
Q_{K+1} &= B^T R_{K+1} + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} Q_K.
\end{aligned} \tag{38}$$

Algorithm 2 (see [5]). The following finite iterative algorithm is proposed to solve coupled system of Sylvester matrix equation (36):

- (1) Input matrices: $A_1; B_1; A_2; B_2; C_1; C_2$.
- (2) Pick arbitrary matrices $Y_{1_1} \in C^{n \times p}$ and $Y_{2_1} \in C^{r \times p}$.
- (3) Set

$$\begin{aligned}
R_1 &= \text{diag}(C_1 - f(Y_{1_1}, Y_{2_1}), C_2 - g(Y_{1_1}, Y_{2_1})), \\
S_1 &= A_1^T (C_1 - f(Y_{1_1}, Y_{2_1})) + A_2^T (C_2 - g(Y_{1_1}, Y_{2_1})), \\
T_1 &= B_1^T (C_1 - f(Y_{1_1}, Y_{2_1})) + B_2^T (C_2 - g(Y_{1_1}, Y_{2_1})).
\end{aligned} \tag{39}$$

- (4) If $R_K = 0$, then stop and Y_{1_K} and Y_{2_K} are the solutions; else set $K = K + 1$ and then go to step 5.
- (5) Calculate

$$\begin{aligned}
Y_{1_{K+1}} &= Y_{1_K} + \frac{\|R_K\|^2}{\|S_K\|^2 + \|T_K\|^2} S_K, \\
Y_{2_{K+1}} &= Y_{2_K} + \frac{\|R_K\|^2}{\|S_K\|^2 + \|T_K\|^2} T_K, \\
R_{K+1} &= \text{diag}(C_1 - f(Y_{1_{K+1}}, Y_{2_{K+1}}), C_2 - g(Y_{1_{K+1}}, Y_{2_{K+1}})) \\
&= R_K - \frac{\|R_K\|^2}{\|S_K\|^2 + \|T_K\|^2} \text{diag}(f(S_K, T_K), g(S_K, T_K)) \\
S_{K+1} &= A_1^T (C_1 - f(Y_{1_{K+1}}, Y_{2_{K+1}})) + A_2^T (C_2 - g(Y_{1_{K+1}}, Y_{2_{K+1}})) + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} S_K, \\
T_{K+1} &= B_1^T (C_1 - f(Y_{1_{K+1}}, Y_{2_{K+1}})) + B_2^T (C_2 - g(Y_{1_{K+1}}, Y_{2_{K+1}})) + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} T_K.
\end{aligned} \tag{40}$$

5. Two-Dimensional Fuzzy Fredholm Integral Equation

Two-dimensional FIE of the second kind is defined as follows [24]:

$$u(x, y) = f(x, y) + \lambda \int_a^b \int_c^d k(x, y, s, t) u(s, t) ds dt. \tag{41}$$

The linear (2D-FFIE-2) is defined as

$$\tilde{u}(x, y) = \tilde{f}(x, y) + \lambda \int_a^b \int_c^d (x, y, s, t) \tilde{u}(s, t) ds dt, \tag{42}$$

where $\tilde{u}(x, y)$ and $\tilde{f}(x, y)$ are fuzzy real functions on $S = [a, b] \times [c, d]$, $k(x, y, s, t)$ is an arbitrary kernel function over $V = [a, b] \times [c, d] \times [a, b] \times [c, d]$, and $\tilde{u}(x, y)$ is unknown on S .

Throughout this paper, we consider 2D-FFIE-2 with $a = c = 0$, $b = d = 1$, and $\lambda = 1$.

Now, introduce parametric form of a 2D-FFIE-2 with respect to Definition 3. Let $(\underline{f}(x, y, r), \bar{f}(x, y, r))$ and

$(\underline{u}(x, y, r), \bar{u}(x, y, r))$, $0 \leq r \leq 1$, $(x, y) \in S$, be parametric form of $\tilde{f}(x, y)$ and $\tilde{u}(x, y)$, respectively. Then parametric form of 2D-FFIE-2 is as follows:

$$\underline{u}(x, y, r) = \underline{f}(x, y, r) + \int_0^1 \int_0^1 v_1(x, y, s, t, \underline{u}(x, y, r), \bar{u}(x, y, r)) ds dt, \quad (43)$$

$$\bar{u}(x, y, r) = \bar{f}(x, y, r) + \int_0^1 \int_0^1 v_2(x, y, s, t, \underline{u}(x, y, r), \bar{u}(x, y, r)) ds dt, \quad (44)$$

$$v_1(x, y, s, t, \underline{u}(x, y, r), \bar{u}(x, y, r)) = \begin{cases} k(x, y, s, t) \underline{u}(s, t, r), & k(x, y, s, t) \geq 0, \\ k(x, y, s, t) \bar{u}(s, t, r), & k(x, y, s, t) < 0, \end{cases} \quad (45)$$

$$v_2(x, y, s, t, \underline{u}(x, y, r), \bar{u}(x, y, r)) = \begin{cases} k(x, y, s, t) \bar{u}(s, t, r), & k(x, y, s, t) \geq 0, \\ k(x, y, s, t) \underline{u}(s, t, r), & k(x, y, s, t) < 0, \end{cases}$$

for each $0 \leq x, y \leq 1$ and $0 \leq r \leq 1$. We can see that equations (43) and (44) are system of Fredholm integral equation of the second kind with three variables in crisp case.

6. Proposed Hybrid Iterative Technique

6.1. Converting Linear Two-Dimensional FIEs of Second Kind to Two Crisp Coupled Systems. This section presents an efficient method for solving a 2D-FFIE-2 by using 2D-TFs.

First, consider the following equation:

$$\tilde{u}(x, y) = \tilde{f}(x, y) + \lambda \int_a^b \int_c^d k(x, y, s, t) \tilde{u}(s, t) ds dt. \quad (46)$$

Now, the problem is to find the TF coefficients of $\tilde{u}(x, y)$ from the known functions $\tilde{f}(x, y)$ and kernel $k(x, y, s, t)$. 2D-TFs are applied for equations

$$\underline{u}(x, y, r) = \underline{f}(x, y, r) + \int_0^1 \int_0^1 k(x, y, s, t) \underline{u}(s, t, r) ds dt, \quad (47)$$

$$\bar{u}(x, y, r) = \bar{f}(x, y, r) + \int_0^1 \int_0^1 k(x, y, s, t) \bar{u}(s, t, r) ds dt. \quad (48)$$

To describe the approach of equation (47), first expand $\underline{u}(x, y, r)$, $\underline{f}(x, y, r)$, and $k(x, y, s, t)$ by 2D-TFs as follows:

$$\underline{u}(x, y, r) \approx T^T(x, y) U T(r), \quad (49)$$

$$\begin{aligned} \underline{u}(x, y, r) \approx & T11^T(x, y) U11T1(r) + T12^T(x, y) U21T1(r) + T21^T(x, y) U31T1(r) \\ & + T22^T(x, y) U41T1(r) + T11^T(x, y) U12T2(r) + T12^T(x, y) U22T2(r) \\ & + T21^T(x, y) U32T2(r) + T22^T(x, y) U42T2(r), \end{aligned} \quad (50)$$

$$\underline{f}(x, y, r) \approx T^T(x, y) F T(r), \quad (51)$$

$$\begin{aligned} \underline{f}(x, y, r) \approx & T11^T(x, y) F11T1(r) + T12^T(x, y) F21T1(r) \\ & + T21^T(x, y) F31T1(r) + T22^T(x, y) F41T1(r) + T11^T(x, y) F12T2(r) \\ & + T12^T(x, y) F22T2(r) + T21^T(x, y) F32T2(r) + T22^T(x, y) F42T2(r), \end{aligned} \quad (52)$$

$$k(x, y, s, t) \approx T^T(x, y) K T(s, t), \quad (53)$$

where $T(x, y)$ and $T(r)$ are defined in equations (3) and (21), respectively, U and F are $(4M^2 \times 2M)$ matrix of 2D-TF

coefficients of $\underline{u}(x, y, r)$ and $\underline{f}(x, y, r)$, respectively, and K is $(4M^2 \times 4M^2)$ -matrix 2D-TF coefficients of (x, y, s, t) .

To obtain the solution of equation (47) from equations (49), (51), and (53), we have

$$T^T(x, y)UT(r) = T^T(x, y)FT(r) + \int \int_0^1 T^T(x, y)KT(s, t)T^T(x, y)UT(r)ds dt, \quad (54)$$

$$T^T(x, y)UT(r) = T^T(x, y)FT(r) + T^T(x, y)K\left(\int \int_0^1 T^T(x, y)KT(s, t)T^T(x, y)ds dt\right)UT(r).$$

Using equation (22), we have

$$T^T(x, y)UT(r) = T^T(x, y)FT(r) + T^T(x, y)KDU T(r), \quad (55)$$

where U and F are $4M^2 \times 2M$ -matrix and KD is $4M^2 \times 4M^2$ -matrix, so KDU is $4M^2 \times 2M$ -matrix, where U is unknown.

Then, we have

and then

$$U = F + KDU, \quad (56)$$

$$(I - KD)U = F, \quad (57)$$

with

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{31} & U_{32} \\ U_{41} & U_{42} \end{bmatrix},$$

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \\ F_{31} & F_{32} \\ F_{41} & F_{42} \end{bmatrix},$$

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix},$$

$$D = \begin{bmatrix} \frac{h_1 I_1}{3} \otimes \frac{h_2 I_2}{3} & \frac{h_1 I_1}{3} \otimes \frac{h_2 I_2}{6} & \frac{h_1 I_1}{6} \otimes \frac{h_2 I_2}{3} & \frac{h_1 I_1}{6} \otimes \frac{h_2 I_2}{6} \\ \frac{h_1 I_1}{3} \otimes \frac{h_2 I_2}{6} & \frac{h_1 I_1}{3} \otimes \frac{h_2 I_2}{3} & \frac{h_1 I_1}{6} \otimes \frac{h_2 I_2}{6} & \frac{h_1 I_1}{6} \otimes \frac{h_2 I_2}{3} \\ \frac{h_1 I_1}{6} \otimes \frac{h_2 I_2}{3} & \frac{h_1 I_1}{6} \otimes \frac{h_2 I_2}{6} & \frac{h_1 I_1}{3} \otimes \frac{h_2 I_2}{3} & \frac{h_1 I_1}{3} \otimes \frac{h_2 I_2}{6} \\ \frac{h_1 I_1}{6} \otimes \frac{h_2 I_2}{6} & \frac{h_1 I_1}{6} \otimes \frac{h_2 I_2}{3} & \frac{h_1 I_1}{3} \otimes \frac{h_2 I_2}{6} & \frac{h_1 I_1}{3} \otimes \frac{h_2 I_2}{3} \end{bmatrix},$$

$$I_{4M^2 \times 4M^2} = \begin{bmatrix} I_{M^2} & 0 & 0 & 0 \\ 0 & I_{M^2} & 0 & 0 \\ 0 & 0 & I_{M^2} & 0 \\ 0 & 0 & 0 & I_{M^2} \end{bmatrix},$$

$$KD = \begin{bmatrix} K_{11}D_{11} + K_{12}D_{21} + K_{13}D_{31} + K_{14}D_{41} & K_{11}D_{12} + K_{12}D_{22} + K_{13}D_{32} + K_{14}D_{42} & K_{11}D_{13} + K_{12}D_{23} + K_{13}D_{33} + K_{14}D_{43} & K_{11}D_{14} + K_{12}D_{24} + K_{13}D_{34} + K_{14}D_{44} \\ K_{21}D_{11} + K_{22}D_{21} + K_{23}D_{31} + K_{24}D_{41} & K_{21}D_{12} + K_{22}D_{22} + K_{23}D_{32} + K_{24}D_{42} & K_{21}D_{13} + K_{22}D_{23} + K_{23}D_{33} + K_{24}D_{43} & K_{21}D_{14} + K_{22}D_{24} + K_{23}D_{34} + K_{24}D_{44} \\ K_{31}D_{11} + K_{32}D_{21} + K_{33}D_{31} + K_{34}D_{41} & K_{31}D_{12} + K_{32}D_{22} + K_{33}D_{32} + K_{34}D_{42} & K_{31}D_{13} + K_{32}D_{23} + K_{33}D_{33} + K_{34}D_{43} & K_{31}D_{14} + K_{32}D_{24} + K_{33}D_{34} + K_{34}D_{44} \\ K_{41}D_{11} + K_{42}D_{21} + K_{43}D_{31} + K_{44}D_{41} & K_{41}D_{12} + K_{42}D_{22} + K_{43}D_{32} + K_{44}D_{42} & K_{41}D_{13} + K_{42}D_{23} + K_{43}D_{33} + K_{44}D_{43} & K_{41}D_{14} + K_{42}D_{24} + K_{43}D_{34} + K_{44}D_{44} \end{bmatrix},$$

$$(I - KD) = \begin{bmatrix} I_{M^2} - (K_{11}D_{11} + K_{12}D_{21} + K_{13}D_{31} + K_{14}D_{41}) & -(K_{11}D_{12} + K_{12}D_{22} + K_{13}D_{32} + K_{14}D_{42}) & -(K_{11}D_{13} + K_{12}D_{23} + K_{13}D_{33} + K_{14}D_{43}) & -(K_{11}D_{14} + K_{12}D_{24} + K_{13}D_{34} + K_{14}D_{44}) \\ -(K_{21}D_{11} + K_{22}D_{21} + K_{23}D_{31} + K_{24}D_{41}) & I_{M^2} - (K_{21}D_{12} + K_{22}D_{22} + K_{23}D_{32} + K_{24}D_{42}) & -(K_{21}D_{13} + K_{22}D_{23} + K_{23}D_{33} + K_{24}D_{43}) & -(K_{21}D_{14} + K_{22}D_{24} + K_{23}D_{34} + K_{24}D_{44}) \\ -(K_{31}D_{11} + K_{32}D_{21} + K_{33}D_{31} + K_{34}D_{41}) & -(K_{31}D_{12} + K_{32}D_{22} + K_{33}D_{32} + K_{34}D_{42}) & I_{M^2} - (K_{31}D_{13} + K_{32}D_{23} + K_{33}D_{33} + K_{34}D_{43}) & -(K_{31}D_{14} + K_{32}D_{24} + K_{33}D_{34} + K_{34}D_{44}) \\ -(K_{41}D_{11} + K_{42}D_{21} + K_{43}D_{31} + K_{44}D_{41}) & -(K_{41}D_{12} + K_{42}D_{22} + K_{43}D_{32} + K_{44}D_{42}) & -(K_{41}D_{13} + K_{42}D_{23} + K_{43}D_{33} + K_{44}D_{43}) & I_{M^2} - (K_{41}D_{14} + K_{42}D_{24} + K_{43}D_{34} + K_{44}D_{44}) \end{bmatrix},$$

$$(I - KD)U = F$$

$$\begin{aligned} (I_{M^2} - (K_{11}D_{11} + K_{12}D_{21} + K_{13}D_{31} + K_{14}D_{41}))U_{11} - (K_{11}D_{12} + K_{12}D_{22} + K_{13}D_{32} + K_{14}D_{42})U_{21} - (K_{11}D_{13} + K_{12}D_{23} + K_{13}D_{33} + K_{14}D_{43})U_{31} - (K_{11}D_{14} + K_{12}D_{24} + K_{13}D_{34} + K_{14}D_{44})U_{41} &= F_{11}, \\ -(K_{21}D_{11} + K_{22}D_{21} + K_{23}D_{31} + K_{24}D_{41})U_{11} + I_{M^2} - (K_{21}D_{12} + K_{22}D_{22} + K_{23}D_{32} + K_{24}D_{42})U_{21} - (K_{21}D_{13} + K_{22}D_{23} + K_{23}D_{33} + K_{24}D_{43})U_{31} - (K_{21}D_{14} + K_{22}D_{24} + K_{23}D_{34} + K_{24}D_{44})U_{41} &= F_{21}, \\ -(K_{31}D_{11} + K_{32}D_{21} + K_{33}D_{31} + K_{34}D_{41})U_{11} - (K_{31}D_{12} + K_{32}D_{22} + K_{33}D_{32} + K_{34}D_{42})U_{21} + (I_{M^2} - (K_{31}D_{13} + K_{32}D_{23} + K_{33}D_{33} + K_{34}D_{43}))U_{31} - (K_{31}D_{14} + K_{32}D_{24} + K_{33}D_{34} + K_{34}D_{44})U_{41} &= F_{31}, \\ -(K_{41}D_{11} + K_{42}D_{21} + K_{43}D_{31} + K_{44}D_{41})U_{11} - (K_{41}D_{12} + K_{42}D_{22} + K_{43}D_{32} + K_{44}D_{42})U_{21} - (K_{41}D_{13} + K_{42}D_{23} + K_{43}D_{33} + K_{44}D_{43})U_{31} + (I_{M^2} - (K_{41}D_{14} + K_{42}D_{24} + K_{43}D_{34} + K_{44}D_{44}))U_{41} &= F_{41}. \end{aligned}$$

Set

$$\begin{aligned}
 A_1 &= I_{M^2} - (K_{11}D_{11} + K_{12}D_{21} + K_{13}D_{31} + K_{14}D_{41}), \\
 A_2 &= -(K_{11}D_{12} + K_{12}D_{22} + K_{13}D_{32} + K_{14}D_{42}), \\
 A_3 &= -(K_{11}D_{13} + K_{12}D_{23} + K_{13}D_{33} + K_{14}D_{43}), \\
 A_4 &= -(K_{11}D_{14} + K_{12}D_{24} + K_{13}D_{34} + K_{14}D_{44}), \\
 B_1 &= -(K_{21}D_{11} + K_{22}D_{21} + K_{23}D_{31} + K_{24}D_{41}), \\
 B_2 &= I_{M^2} - (K_{21}D_{12} + K_{22}D_{22} + K_{23}D_{32} + K_{24}D_{42}), \\
 B_3 &= -(K_{21}D_{13} + K_{22}D_{23} + K_{23}D_{33} + K_{24}D_{43}), \\
 B_4 &= -(K_{21}D_{14} + K_{22}D_{24} + K_{23}D_{34} + K_{24}D_{44}), \\
 C_1 &= -(K_{31}D_{11} + K_{32}D_{21} + K_{33}D_{31} + K_{34}D_{41}), \\
 C_2 &= -(K_{31}D_{12} + K_{32}D_{22} + K_{33}D_{32} + K_{34}D_{42}), \\
 D_1 &= -(K_{41}D_{11} + K_{42}D_{21} + K_{43}D_{31} + K_{44}D_{41}), \\
 D_2 &= -(K_{41}D_{12} + K_{42}D_{22} + K_{43}D_{32} + K_{44}D_{42}), \\
 D_3 &= -(K_{41}D_{13} + K_{42}D_{23} + K_{43}D_{33} + K_{44}D_{43}), \\
 D_4 &= I_{M^2} - (K_{41}D_{14} + K_{42}D_{24} + K_{43}D_{34} + K_{44}D_{44}),
 \end{aligned} \tag{59}$$

which lead to the following two crisp linear systems:

$$A_1U_{11} + A_2U_{21} + A_3U_{31} + A_4U_{41} = F_{11}, \tag{60}$$

$$B_1U_{11} + B_2U_{21} + B_3U_{31} + B_4U_{41} = F_{21}, \tag{61}$$

$$C_1U_{11} + C_2U_{21} + C_3U_{31} + C_4U_{41} = F_{31}, \tag{62}$$

$$D_1U_{11} + D_2U_{21} + D_3U_{31} + D_4U_{41} = F_{41}, \tag{63}$$

and

$$A_1U_{12} + A_2U_{22} + A_3U_{32} + A_4U_{42} = F_{12}, \tag{64}$$

$$B_1U_{12} + B_2U_{22} + B_3U_{32} + B_4U_{42} = F_{22}, \tag{65}$$

$$C_1U_{12} + C_2U_{22} + C_3U_{32} + C_4U_{42} = F_{32}, \tag{66}$$

$$D_1U_{12} + D_2U_{22} + D_3U_{32} + D_4U_{42} = F_{42}. \tag{67}$$

Similarly, we expand $\bar{u}(x, y, r)$ and $\bar{f}(x, y, r)$ by 2D-TFs, and by substituting them into equation (42), two coupled crisp linear systems, similar to (61) and (65), are obtained. It is clear that all matrices in the two coupled crisp linear systems (61) and (65) are square matrices of dimensions $M \times M$. Thus, we need to obtain the coefficient matrices $U_{11}, U_{12}, U_{21}, U_{22}, U_{31}, U_{32}, U_{41}$, and U_{42} in order to get the approximate numerical solution of the form:

$$\underline{u}(x, y, r) = T^T(x, y)UT(r). \tag{68}$$

6.2. Proposed Iterative Algorithm for Solving Coupled Systems (61) and (65). An iterative algorithm is proposed here to solve the two coupled systems (61) and (65) as a generalization of Algorithm 2.

Algorithm 3. Algorithm 2 is modified and generalized to work out for systems (61) and (65) as follows.

First, for coupled system (61):

- (1) Input $A_1; A_2; A_3; A_4; B_1; B_2; B_3; B_4; C_1; C_2; C_3; C_4; D_1; D_2; D_3; D_4; F_{11}; F_{21}; F_{31}; F_{41}$.
- (2) Choose arbitrary matrices $U_{11}, U_{21}, U_{31}, U_{41}$.
- (3) For $k = 1$, set

$$\begin{aligned}
 R_k &= \text{diag}(F_{11} - f_1(U_{11}, U_{21}, U_{31}, U_{41}), F_{21} - f_2(U_{11}, U_{21}, U_{31}, U_{41}), F_{31} - f_3(U_{11}, U_{21}, U_{31}, U_{41}), F_{41} \\
 &\quad - f_4(U_{11}, U_{21}, U_{31}, U_{41}))
 \end{aligned}$$

$$\begin{aligned}
 S1_k &= A_1^T (F_{11} - f_1(U_{11}, U_{21}, U_{31}, U_{41})) + B_1^T (F_{21} - f_2(U_{11}, U_{21}, U_{31}, U_{41})) + C_1^T (F_{31} - f_3(U_{11}, U_{21}, U_{31}, U_{41})) \\
 &\quad + D_1^T (F_{41} - f_4(U_{11}, U_{21}, U_{31}, U_{41})),
 \end{aligned}$$

$$\begin{aligned}
 S2_k &= A_2^T (F_{11} - f_1(U_{11}, U_{21}, U_{31}, U_{41})) + B_2^T (F_{21} - f_2(U_{11}, U_{21}, U_{31}, U_{41})) + C_2^T (F_{31} - f_3(U_{11}, U_{21}, U_{31}, U_{41})) \\
 &\quad + D_2^T (F_{41} - f_4(U_{11}, U_{21}, U_{31}, U_{41})),
 \end{aligned}$$

$$\begin{aligned}
 S3_k &= A_3^T (F_{11} - f_1(U_{11}, U_{21}, U_{31}, U_{41})) + B_3^T (F_{21} - f_2(U_{11}, U_{21}, U_{31}, U_{41})) + C_3^T (F_{31} - f_3(U_{11}, U_{21}, U_{31}, U_{41})) \\
 &\quad + D_3^T (F_{41} - f_4(U_{11}, U_{21}, U_{31}, U_{41})),
 \end{aligned}$$

$$\begin{aligned}
 S4_k &= A_4^T (F_{11} - f_1(U_{11}, U_{21}, U_{31}, U_{41})) + B_4^T (F_{21} - f_2(U_{11}, U_{21}, U_{31}, U_{41})) + C_4^T (F_{31} - f_3(U_{11}, U_{21}, U_{31}, U_{41})) \\
 &\quad + D_4^T (F_{41} - f_4(U_{11}, U_{21}, U_{31}, U_{41})).
 \end{aligned}$$

(69)

(4) If $R_K = 0$, then stop and $U11, U21, U31, U41$ are the solutions; else, let $K = K + 1$ and go to step 5.

(5) Compute

$$U11 = U11 + \frac{\|R_K\|^2}{\|S1_k\|^2 + \|S2_k\|^2 + \|S3_k\|^2 + \|S4_k\|^2} S1_k,$$

$$U21 = U21 + \frac{\|R_K\|^2}{\|S1_k\|^2 + \|S2_k\|^2 + \|S3_k\|^2 + \|S4_k\|^2} S2_k,$$

$$U31 = U31 + \frac{\|R_K\|^2}{\|S1_k\|^2 + \|S2_k\|^2 + \|S3_k\|^2 + \|S4_k\|^2} S3_k,$$

$$U41 = U41 + \frac{\|R_K\|^2}{\|S1_k\|^2 + \|S2_k\|^2 + \|S3_k\|^2 + \|S4_k\|^2} S4_k,$$

$$R_{k+1} = \text{diag}(F11 - f1(U11, U21, U31, U41), F21 - f2(U11, U21, U31, U41), F31 - f3(U11, U21, U31, U41), \\ \cdot F41 - f4(U11, U21, U31, U41)),$$

$$S1_{k+1} = A_1^T (F11 - f1(U11, U21, U31, U41)) + B_1^T (F21 - f2(U11, U21, U31, U41)) + C_1^T (F31 - f3(U11, U21, U31, U41)) \\ + D_1^T (F41 - f4(U11, U21, U31, U41)) + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} S1_K,$$

$$S2_{k+1} = A_2^T (F11 - f1(U11, U21, U31, U41)) + B_2^T (F21 - f2(U11, U21, U31, U41)) + C_2^T (F31 - f3(U11, U21, U31, U41)) \\ + D_2^T (F41 - f4(U11, U21, U31, U41)) + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} S2_K,$$

$$S3_{k+1} = A_3^T (F11 - f1(U11, U21, U31, U41)) + B_3^T (F21 - f2(U11, U21, U31, U41)) + C_3^T (F31 - f3(U11, U21, U31, U41)) \\ + D_3^T (F41 - f4(U11, U21, U31, U41)) + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} S3_K,$$

$$S4_{k+1} = A_4^T (F11 - f1(U11, U21, U31, U41)) + B_4^T (F21 - f2(U11, U21, U31, U41)) + C_4^T (F31 - f3(U11, U21, U31, U41)) \\ + D_4^T (F41 - f4(U11, U21, U31, U41)) + \frac{\|R_{K+1}\|^2}{\|R_K\|^2} S4_K.$$

(70)

For coupled system (65), the algorithm is repeated with replacing $U11, U21, U31, U41$ by $U12, U22, U32, U42$ and $F11, F21, F31, F41$ by $F12, F22, F32, F42$ where the $2M \times 2M$ block U matrix is computed.

The approximate crisp numerical solution for equation (51) of the form $\underline{u}(x, r)_{\text{approx.}} = T^T(x)UT(r)$ is then obtained.

In a similar manner, the crisp numerical solution for equation (53) of the form $\bar{u}(x, r)_{\text{approx.}} = T^T(x)UT(r)$ can

be obtained by carrying out the above-proposed algorithm for the other coupled crisp systems similar to (61) and (65).

Finally, the solution for linear 2D-FFIE-2 by 2D-TFs is then given as

$$u(x, y) = (\underline{u}_{\text{approx}}(x, y, r), \bar{u}_{\text{approx}}(x, y, r)), \quad 0 \leq r \leq 1, x, y \in [0, 1]. \quad (71)$$

7. Convergence Analysis of the Proposed Method

In this section, we obtain error estimate for the numerical method proposed in previous section.

Theorem 1. *The solution of the two-dimensional Fredholm fuzzy integral equations given by equation (1) by using 2D-TFs converges to exact solution if*

$$S = \max_{0 \leq x, y, s, t \leq 1} |k(x, y, s, t)| < 1. \quad (72)$$

Proof. Assume $\tilde{u}_{\text{exact}}(x, y)$ and $\tilde{u}_{\text{approx}}(x, y)$ represent the exact and approximate solutions of equation (1), respectively. Therefore,

$$\begin{aligned} \tilde{u}_{\text{approx}}(x, y) = & \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} c_{i,j} T_{i,j}^{1,1}(s, t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} d_{i,j} T_{i,j}^{1,2}(s, t) \\ & + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} e_{i,j} T_{i,j}^{2,1}(s, t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} l_{i,j} T_{i,j}^{2,2}(s, t). \end{aligned} \quad (73)$$

By using equation (1), we can write

$$\begin{aligned} e(x, y) = & \|\tilde{u}_{\text{exact}}(x, y) - \tilde{u}_{\text{approx}}(x, y)\| = \max_{(x,y) \in [0,1]} |\tilde{u}_{\text{exact}}(x, y) - \tilde{u}_{\text{approx}}(x, y)| \\ = & \max_{(x,y) \in [0,1]} \left| \left(\int_0^1 \int_0^1 k(x, y, s, t) \tilde{u}_{\text{exact}}(s, t) ds dt - \int_0^1 \int_0^1 k(x, y, s, t) \left(\sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} c_{i,j} T_{i,j}^{1,1}(s, t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} d_{i,j} T_{i,j}^{1,2}(s, t) \right. \right. \right. \\ & \left. \left. + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} e_{i,j} T_{i,j}^{2,1}(s, t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} l_{i,j} T_{i,j}^{2,2}(s, t) \right) ds dt \right| \\ \leq & S \left(\int_0^1 \int_0^1 \max_{(x,y) \in [0,1]} |\tilde{u}_{\text{exact}}(s, t) - \tilde{u}_{\text{approx}}(s, t)| ds dt \right) = S \left(\int_0^1 \int_0^1 \|\tilde{u}_{\text{exact}}(s, t) - \tilde{u}_{\text{approx}}(s, t)\| ds dt \right), \end{aligned} \quad (74)$$

where

$$S = \max_{0 \leq x, y, s, t \leq 1} |k(x, y, s, t)| < \infty. \quad (75)$$

Also, we have $\lim_{M \rightarrow \infty} \tilde{u}_{\text{approx}}(x, y) = \tilde{u}_{\text{exact}}(x, y)$, so $\|\tilde{u}_{\text{exact}}(x, y) - \tilde{u}_{\text{approx}}(x, y)\| \rightarrow 0$ as $M \rightarrow \infty$ and since S is bounded.

Thus,

$$\lim_{M \rightarrow \infty} \|\tilde{u}_{\text{exact}}(x, y) - \tilde{u}_{\text{approx}}(x, y)\| \rightarrow 0, \quad (76)$$

so the proof of the theorem is completed.

Remark 1. In our theoretical investigation for the proposed method, we take $m_1 = m_2 = M$.

8. Numerical Results and Discussion

This section demonstrates the effectiveness and the accuracy of our proposed hybrid method, 2D-TFs and an iterative algorithm, on some examples. The solution of each example is obtained for different values of x, y, r , and M and is compared with the exact solution, the direct method, and the

presented method when the tolerance criteria residual is $>e^{-4}$ and $>e^{-8}$.

Example 1. The 2D-FFIE-2 given in Mirzaee et al. [15] is considered as

$$\underline{f}(x, y, r) = r \left(xy + \frac{1}{676} (x^2 + y^2 - 2) \right),$$

$$\bar{f}(x, y, r) = (2 - r) \left(xy + \frac{1}{676} (x^2 + y^2 - 1) \right),$$

$$k(x, y, s, t) = \frac{1}{169} (x^2 + y^2 - 2)(s^2 + t^2 - 2), \quad 0 \leq x, t \leq 1 \text{ and } \lambda = 1. \quad (77)$$

In this case, the exact solution is given by

$$\tilde{u}(x, y, r) = (\underline{u}(x, y, r), \bar{u}(x, y, r)) = (rxy, (2 - r)xy). \quad (78)$$

The number of iterations for solving the two coupled matrix equations to obtain the coefficient matrices taken by our proposed iterative algorithm is $k = 3$ when the tolerance criteria residual is $>e^{-4}$ which indicates that the hybrid

TABLE 1: Numerical results with TF system for Example 1 for $x = 0.1$, $y = 0.4$, and $M = 4$.

r	Exact solution $\underline{u}(x, y, r)$	Direct method [15]	Absolute error	Presented method	Absolute error
0	0.00000000	0.00000000	$8.4015e-006$	0.00000002	$1.89187044e-008$
0.1	0.00400000	0.00398678	$1.3220e-005$	0.00399416	$5.83560842e-006$
0.2	0.00800000	0.00797356	$2.6440e-005$	0.00798831	$1.16901355e-005$
0.3	0.01200000	0.01196034	$3.9660e-005$	0.01198246	$1.75446627e-005$
0.4	0.01600000	0.01594712	$5.2880e-005$	0.01597660	$2.33991898e-005$
0.5	0.02000000	0.01993390	$6.6100e-005$	0.01997075	$2.92537169e-005$
0.6	0.02400000	0.02392068	$7.9320e-005$	0.02396489	$3.51082441e-005$
0.7	0.02800000	0.02700746	$9.9254e-004$	0.02796099	$4.09627712e-005$
0.8	0.03200000	0.03189424	$1.0576e-004$	0.03195318	$4.68172983e-005$
0.9	0.03600000	0.03588102	$1.1898e-004$	0.03594733	$5.26718254e-005$

TABLE 2: Numerical results with TF system for Example 1 for $x = 0.1$, $y = 0.4$, and $M = 4$.

r	Exact solution $\bar{u}(x, y, r)$	Direct method [15]	Absolute error	Presented method	Absolute error
0	0.08000000	0.07973560	$2.6440e-004$	0.07988291	$1.17085832e-004$
0.1	0.07600000	0.07574882	$2.5118e-004$	0.07588878	$1.11222414e-004$
0.2	0.07200000	0.07176204	$2.3796e-004$	0.07189464	$1.05358995e-004$
0.3	0.06800000	0.06777526	$2.2474e-004$	0.06790050	$9.94955768e-005$
0.4	0.06400000	0.06378849	$2.1151e-004$	0.06390637	$9.36321584e-005$
0.5	0.06000000	0.05980170	$1.9830e-004$	0.05991223	$8.77687400e-005$
0.6	0.05600000	0.05581492	$1.8508e-004$	0.05592806	$8.19053216e-005$
0.7	0.05200000	0.05182814	$1.7186e-004$	0.05192396	$7.60419032e-005$
0.8	0.04800000	0.04784136	$1.5864e-004$	0.04792982	$7.01784848e-005$
0.9	0.04400000	0.04385458	$1.4542e-004$	0.04393568	$6.43150664e-005$

proposed method is quite efficient and has good accuracy as seen from Tables 1 and 2.

Remark 2. The numerical results for the approximate solution using the direct method in Tables 1 and 2 are taken from Table 3 in [15], while the numerical results using the direct and the proposed iterative methods in Tables 3 and 4 are obtained using our own program written using MATLAB R2018b. Also, the number of iterations for solving the two coupled matrix equations to obtain the coefficient matrices taken by our proposed iterative algorithm is $k = 3$ when the tolerance criteria residual is $>e^{-4}$ which indicates that the hybrid proposed method is quite efficient and has good accuracy as seen from Tables 3 and 4.

Example 2. Consider the following made up linear two-dimensional fuzzy Fredholm integral equations (2D-FFIE-2):

$$\begin{aligned}\underline{f}(x, y, r) &= \frac{5}{12}r(x+y), \\ \bar{f}(x, y, r) &= \frac{5}{12}(2-r)(x+y),\end{aligned}\quad (79)$$

$$k(x, y, s, t) = xs + yt, \quad 0 \leq x, t \leq 1 \text{ and } \lambda = 1.$$

In this case, the exact solution is given by

$$\bar{u}(x, y, r) = (\underline{u}(x, y, r), \bar{u}(x, y, r)) = (r(x+y), (2-r)(x+y)). \quad (80)$$

Remark 3. The number of iterations for solving the two coupled matrix equations to obtain the coefficient matrices taken by our proposed iterative algorithm is $k = 3$ when the tolerance criteria residual is $>e^{-4}$ which indicates that the hybrid proposed method is quite efficient and has good accuracy as seen from Tables 5 and 6.

Example 3. Consider the following 2D-FFIE-2:

$$\underline{f}(x, y, r) = \frac{11}{36}rx^2y^2,$$

$$\bar{f}(x, y, r) = \frac{11}{36}(2-r)x^2y^2,$$

$$k(x, y, s, t) = x^2y^2(1+2t)(1+2s), \quad 0 \leq x, t \leq 1 \text{ and } \lambda = 1. \quad (81)$$

In this case, the exact solution is given by

$$\begin{aligned}\underline{u}(x, y, r) &= rx^2y^2, \\ \bar{u}(x, y, r) &= (2-r)x^2y^2.\end{aligned}\quad (82)$$

TABLE 3: Numerical results with TF system for Example 1 for $x = 0.1$, $y = 0.4$, and $M = 6$.

r	Exact solution $\underline{u}(x, y, r)$	Direct method	Absolute error	Presented method	Absolute error
0	0.00000000	0.00000239	$2.39402667e-006$	0.00000010	$9.77129847e-008$
0.1	0.00400000	0.00399511	$4.89026667e-006$	0.00399758	$2.42072449e-006$
0.2	0.00800000	0.00799261	$7.38650667e-006$	0.00799506	$4.93916197e-006$
0.3	0.01200000	0.01199012	$9.88274667e-006$	0.01199254	$7.45759945e-006$
0.4	0.01600000	0.01598762	$1.23789867e-005$	0.01599002	$9.97603693e-006$
0.5	0.02000000	0.01998512	$1.48752267e-005$	0.01998751	$1.24944744e-005$
0.6	0.02400000	0.02398263	$1.73714667e-005$	0.02398499	$1.50129119e-005$
0.7	0.02800000	0.02798013	$1.98677067e-005$	0.02798247	$1.75313494e-005$
0.8	0.03200000	0.03197764	$2.23639467e-005$	0.03197995	$2.00497868e-005$
0.9	0.03600000	0.03597514	$2.48601867e-005$	0.03597743	$2.25682243e-005$

TABLE 4: Numerical results with TF system for Example 1 for $x = 0.1$, $y = 0.4$, and $M = 6$.

r	Exact solution $\bar{u}(x, y, r)$	Direct method	Absolute error	Presented method	Absolute error
0	0.08000000	0.07994709	$5.29077333e-005$	0.07994927	$5.07341947e-005$
0.1	0.07600000	0.07594863	$5.13736533e-005$	0.07595198	$4.80210603e-005$
0.2	0.07200000	0.07195016	$4.98395733e-005$	0.07195469	$4.53079259e-005$
0.3	0.06800000	0.06795169	$4.83054933e-005$	0.06795741	$4.25947915e-005$
0.4	0.06400000	0.06395323	$4.67714133e-005$	0.06396012	$3.98816571e-005$
0.5	0.06000000	0.05995476	$4.52373333e-005$	0.05996283	$3.71685227e-005$
0.6	0.05600000	0.05595630	$4.37032533e-005$	0.05596554	$3.44553883e-005$
0.7	0.05200000	0.05195783	$4.21691733e-005$	0.05196826	$3.17422539e-005$
0.8	0.04800000	0.04795936	$4.06350933e-005$	0.04797097	$2.90291195e-005$
0.9	0.04400000	0.04396090	$3.91010133e-005$	0.04397368	$2.63159851e-005$

TABLE 5: Numerical results with TF system for Example 2 for $x = 0.1$, $y = 0.1$, and $M = 4$.

r	Exact solution $\underline{u}(x, y, r)$	Presented method	Absolute error	Exact solution $\bar{u}(x, y, r)$	Presented method	Absolute error
0	0.00000000	0.00000000	$1.734723476e-17$	0.40000000	0.40000000	$0.000000000e+00$
0.1	0.02000000	0.02000000	$1.387778781e-17$	0.38000000	0.38000000	$0.000000000e+00$
0.2	0.04000000	0.04000000	$1.387778781e-17$	0.36000000	0.36000000	$5.551115123e-17$
0.3	0.06000000	0.06000000	$0.000000000e+00$	0.34000000	0.34000000	$0.000000000e+00$
0.4	0.08000000	0.08000000	$0.000000000e+00$	0.32000000	0.32000000	$0.000000000e+00$
0.5	0.10000000	0.10000000	$0.000000000e+00$	0.30000000	0.30000000	$0.000000000e+00$
0.6	0.12000000	0.12000000	$2.775557562e-17$	0.28000000	0.28000000	$5.551115123e-17$
0.7	0.14000000	0.14000000	$2.775557562e-17$	0.26000000	0.26000000	$5.551115123e-17$
0.8	0.16000000	0.16000000	$0.000000000e+00$	0.24000000	0.24000000	$2.775557562e-17$
0.9	0.18000000	0.18000000	$1.734723476e-17$	0.22000000	0.22000000	$5.551115123e-17$

TABLE 6: Numerical results with TF system for Example 2 for $x = 0.1$, $y = 0.1$, and $M = 6$.

r	Exact solution $\underline{u}(x, y, r)$	Presented method	Absolute error	Exact solution $\bar{u}(x, y, r)$	Presented method	Absolute error
0	0.00000000	0.00000000	$0.000000000e+00$	0.40000000	0.40000000	$0.000000000e+00$
0.1	0.02000000	0.02000000	$0.000000000e+00$	0.38000000	0.38000000	$0.000000000e+00$
0.2	0.04000000	0.04000000	$0.000000000e+00$	0.36000000	0.36000000	$0.000000000e+00$
0.3	0.06000000	0.06000000	$0.000000000e+00$	0.34000000	0.34000000	$0.000000000e+00$
0.4	0.08000000	0.08000000	$0.000000000e+00$	0.32000000	0.32000000	$0.000000000e+00$
0.5	0.10000000	0.10000000	$0.000000000e+00$	0.30000000	0.30000000	$0.000000000e+00$
0.6	0.12000000	0.12000000	$0.000000000e+00$	0.28000000	0.28000000	$0.000000000e+00$
0.7	0.14000000	0.14000000	$0.000000000e+00$	0.26000000	0.26000000	$0.000000000e+00$
0.8	0.16000000	0.16000000	$0.000000000e+00$	0.24000000	0.24000000	$0.000000000e+00$
0.9	0.18000000	0.18000000	$0.000000000e+00$	0.22000000	0.22000000	$0.000000000e+00$

TABLE 7: Numerical results with TF system for Example 3 for $x = 0.25$, $y = 0.25$, and $M = 6$.

r	Exact solution $\underline{u}(x, y, r)$	Presented method	Absolute error	Exact solution $\overline{u}(x, y, r)$	Presented method	Absolute error
0	0.00000000	0.00000232	$2.32197716e-006$	0.00781250	0.00609664	$1.71585866e-003$
0.1	0.00039063	0.00030680	$8.38225958e-005$	0.00742187	0.00579187	$1.63000792e-003$
0.2	0.00078125	0.00061128	$1.69967169e-004$	0.00703125	0.00548709	$1.54415718e-003$
0.3	0.00117188	0.00091576	$2.56111742e-004$	0.00664062	0.00518232	$1.45830644e-003$
0.4	0.00156250	0.00122024	$3.42256315e-004$	0.00625000	0.00487754	$1.37245569e-003$
0.5	0.00195313	0.00152472	$4.28400888e-004$	0.00585938	0.00457277	$1.28660495e-003$
0.6	0.00234375	0.00182920	$5.14545461e-004$	0.00546875	0.00426800	$1.20075421e-003$
0.7	0.00273437	0.00213368	$6.00690034e-004$	0.00507813	0.00396322	$1.11490347e-003$
0.8	0.00312500	0.00243817	$6.86834607e-004$	0.00468750	0.00365845	$1.02905273e-003$
0.9	0.00351562	0.00274265	$7.72979179e-004$	0.00429688	0.00335367	$9.43201983e-004$

Remark 4. In Table 7, the number of iterations for this example by our proposed iterative algorithm is $k = 3$ when the tolerance criteria residual is $>e^{-4}$ which indicates that the hybrid proposed method is quite efficient. Moreover, we can see that our method has good accuracy which can be further improved by increasing the residual.

9. Conclusion

Fuzzy control applications and a large proportion of applied mathematical topics require the solution of the fuzzy integral equations. The paper introduced the 2D-TFs method for approximating the solution of linear 2D-FFIE-2, which is a hybrid of triangular functions and an iterative algorithm. The method is simple, efficient, and accurate and is based on converting the original equation into two crisp systems (2D-FFIE-2). The efficiency and simplicity of the proposed method are demonstrated via numerical examples with known exact solutions, and the results are given. Furthermore, the exceptional value of the proposed method is low cost of the equation setting with no need for any projection method or integration.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the manuscript.

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