

Research Article

Optimal Order Error Estimates of a Modified Nonconforming Rotated Q_1 IFEM for Interface Problems

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This paper presents a new numerical method and analysis for solving second-order elliptic interface problems. The method uses a modified nonconforming rotated Q_1 immersed finite element (IFE) space to discretize the state equation required in the variational discretization approach. Optimal order error estimates are derived in L^2 -norm and broken energy norm. Numerical examples are provided to confirm the theoretical results.

1. Introduction

Interface problems arise in many applications, such as mechanical analysis in material sciences or fluid dynamics, where two distinct materials or fluids with different conductivities or densities encounter on an interface [1, 2]. Because of the discontinuity of the properties along the interface and different control equations corresponding to different materials, the solutions of such problems have low regularity on the whole physical domain. Hence, it is a challenge to develop efficient numerical methods for such elliptic interface problems.

In recent years, the convergence analysis of the finite element method (FEM) for the interface problems has been discussed in many publications. For instance, Babuška [3] investigated the elliptic interface problem with a smooth interface. By the use of the boundary and jump conditions incorporated in the cost functions, an equivalent minimization problem was built, and error estimates in the energy norm were derived. Han [4] obtained the error estimates of the infinite element method for the elliptic interface problems. However, the method proposed in [4] can only deal with the case that interfaces consist of straight lines. Chen and Zou [5] proved the error estimates in the energy norm and L^2 -norm of interface problems with the interface being

C^2 -smooth. Moreover, it was shown that the error estimate in the energy norm could be optimal when the exact solution is much smoother (in $W^{1,\infty}$) near the interface (Remark 2.4 in [5]). Based on this assumption, the authors of [6] achieved the optimal energy norm and suboptimal L^2 -norm error estimates. Later on, the above result was applied to semi-linear elliptic and parabolic interface problems in [7], and the optimal order error estimates of the energy norm were derived. And Guan and Shi [8] obtained the same convergence order by using the P_1 -nonconforming triangular element.

Immersed methods are effective for solving interface problems, which contain immersed finite difference method (IFDM) and immersed finite element method (IFEM). LeVeque and Li [9] modified an immersed centered FDM for elliptic interface problems defined on a simple domain and obtained the second-order accuracy on the uniform grid. Moreover, this IFDM was applied to Stokes flow interface problems and moving interface problems, respectively (cf. [10, 11]). Because the FEM has many advantages in engineering calculations, such as the low requirements for the smoothness of the exact solutions, the flexibility of the divisions, and the generality of application programs, a lot of research studies have been devoted to the IFEM based on the

basic idea of adopting a proper FE space to conquer the trouble caused by the interface. For example, Camp et al. [12] constructed a class of quadratic IFE spaces and discussed the approximation capabilities for solving the second-order elliptic interface problems. The IFEM was applied in [13] to solve elliptic interface problems with nonhomogeneous jump conditions for the Galerkin formulation, which can be considered as an extension of those IFEMs in the literature developed for homogeneous jump conditions. A bilinear IFE space was proposed in [14] for solving second-order elliptic boundary value problems, and the error estimates were given for the interpolation, which indicated that this IFE space has the usual approximation capability. Although the IFE solution was proved in [15] to be convergent to the exact solution, unfortunately, the error estimate in the energy norm was of order $O(h^{1/2})$, which was not optimal, and the L^2 -norm error estimate was not considered. The above linear and bilinear IFEM was also applied to elliptic interface optimal control problems, Stokes interface problems, and eigenvalue interface problems, see [16–18] for details. In addition, the above IFEMs are all denoted to conforming linear or bilinear FE approximations.

However, the works mentioned above are mainly contributions to the conforming FEM. In fact, nonconforming finite element method (NFEM) has some advantages compared with the conforming ones. For example, for the Crouzeix–Raviart-type nonconforming finite element, since the unknowns of the elements are associated with the element edges or faces, each degree of freedom belongs to at most two elements, so using the nonconforming elements facilitates the exchange of information across each subdomain and provides spectral radius estimates for the iterative domain decomposition operator, see [19]. Recently, some research studies showed the advantages of the NFEM for PDEs, such as [20–22]. In [23], the authors introduced a nonconforming Crouzeix–Raviart IFEM to solve second-order elliptic problems, but the interface elements were still constructed by conforming the linear triangular element. In this paper, we will adapt the nonconforming finite element different from [23] to deal with the interface problems over all the domains considered.

The remainder of the paper is organized as follows. Firstly, a nonconforming modified rotated Q_1 (cf. [24]) IFE space will be constructed for the elliptic interface problems, which just satisfies the jump conditions. And a remark is given to explain that the original rotated Q_1 -element proposed in [25] cannot be used for this IFEM, although this element has some advantages over other elements for anisotropic noninterior meshes (cf. [26]). We should point out that the convergence order in the energy norm is half order higher than that in [15] and in which the L^2 -norm was not mentioned. Secondly, optimal order error estimates will be carried out in L^2 -norm and broken energy norm by employing some novel analysis techniques. Finally, some numerical results are provided to verify our theoretical analysis.

2. Elliptic Interface Problem

Let Ω be a convex polygonal domain in R^2 , $\Omega^- \subset \Omega$ be an open domain with smooth curve bound $\Gamma \subset \Omega$, and $\Omega^+ = \Omega - \Omega^-$ (see Figure 1). We consider the following elliptic interface problem:

$$\begin{cases} -\nabla \cdot (\beta \nabla u) = f, & \forall x = (x_1, x_2) \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (1)$$

with the jump conditions at the interface Γ :

$$\begin{aligned} [u]_{\Gamma} &= 0, \\ \left[\beta \frac{\partial u}{\partial n} \right]_{\Gamma} &= 0. \end{aligned} \quad (2)$$

In (2), $[u]_{\Gamma}$ denotes the jump of u across the interface Γ and n the outward unit normal vector to Γ . The coefficient β is a positive piecewise constant function defined by

$$\beta(x) = \beta^s, \quad x \in \Omega^s, \quad (3)$$

where $s = -$ or $+$ throughout this paper.

The corresponding variational form of (1) is as follows. Find $u \in H^1(\Omega)$ such that

$$\begin{cases} a(u, v) = (f, v), & \forall v \in H_0^1(\Omega), \\ u|_{\partial\Omega} = 0, \end{cases} \quad (4)$$

which also satisfies the jump condition (2). In (4), $a(u, v) = \int_{\Omega} \beta \nabla u \nabla v dx$, $(f, v) = \int_{\Omega} f v dx$.

Because of the low global regularity of the exact solution in (1), the following spaces and norms are defined as

$$PW^{m,p}(\Omega) = \{u|_{\Omega^s} \in W^{m,p}(\Omega^s), \quad p \geq 1, m = 0, 1, 2\}, \quad (5)$$

equipped with the norm $\|\cdot\|_{m,p,\Omega} = \sqrt{\sum_s \|u\|_{m,p,\Omega^s}^2}$ and seminorm $|\cdot|_{m,p,\Omega} = \sqrt{\sum_s |u|_{m,p,\Omega^s}^2}$.

As the usual Sobolev spaces, when $p = 2$, let

$$PH_{\text{int}}^2(\Omega) = \left\{ u|_{\Omega^s} \in H^2(\Omega^s), \left[\beta \frac{\partial u}{\partial n} \right]_{\Gamma} = 0 \right\}, \quad (6)$$

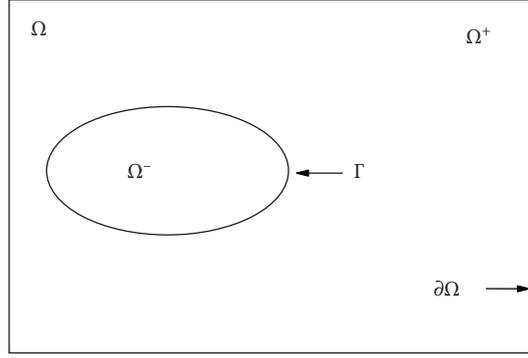
equipped with the norm $\|\cdot\|_{2,\Omega} = \sqrt{\sum_s \|u\|_{2,\Omega^s}^2}$ and seminorm $|\cdot|_{2,\Omega} = \sqrt{\sum_s |u|_{2,\Omega^s}^2}$.

3. The Nonconforming IFE Space

In this section, local nonconforming Q_1^{rot} IFE basis functions will be introduced, and the well-posedness of the nonconforming IFE interpolation will be proven.

Assume that \hat{K} is the square reference element with four vertices $\hat{A}_1 = (-1, -1)$, $\hat{A}_2 = (1, -1)$, $\hat{A}_3 = (1, 1)$, and $\hat{A}_4 = (-1, 1)$; four edges are $\hat{F}_1 = \hat{A}_1\hat{A}_2$, $\hat{F}_2 = \hat{A}_2\hat{A}_3$, $\hat{F}_3 = \hat{A}_3\hat{A}_4$, and $\hat{F}_4 = \hat{A}_4\hat{A}_1$. Define the FE $(\hat{K}, \hat{P}, \hat{\Sigma})$ on \hat{K} as follows:

$$\begin{aligned} \hat{P} &= \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1^2\}, \\ \hat{\Sigma} &= \{\hat{v}_i, i = 1, 2, 3, 4\}, \end{aligned} \quad (7)$$


 FIGURE 1: Sketch of the domain $\Omega = \Omega^- \cup \Omega^+$.

in which $\hat{v}_i = (1/|\hat{F}_i|) \int_{\hat{F}_i} \hat{v} d\hat{s}$.

Remark 1. It is well known that the standard Q_1 reference element $(\hat{K}, \hat{P}, \hat{\Sigma})$ can be defined as

$$\begin{aligned} \hat{P} &= \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1 \hat{x}_2\}, \\ \hat{\Sigma} &= \{\hat{v}_i, i = 1, 2, 3, 4\}, \end{aligned} \quad (8)$$

where $\hat{v}_i = v(A_i)$ are the function values of $\hat{v}(\hat{x})$ at the four vertices A_i of \hat{K} . The IFEM and the convergence analysis of this conforming bilinear element could be found in [13–15]. This paper focuses on the IFEM of the nonconforming Q_1^{rot} element (7).

So, by direct calculation, the corresponding finite element interpolation function can be expressed as

$$\begin{aligned} \hat{\Pi} \hat{v} &= \frac{3\hat{v}_1 - \hat{v}_2 + 3\hat{v}_3 - \hat{v}_4}{4} + \frac{\hat{v}_2 - \hat{v}_4}{2} \hat{x}_1 + \frac{\hat{v}_3 - \hat{v}_1}{2} \hat{x}_2 \\ &\quad + \frac{3(-\hat{v}_1 + \hat{v}_2 - \hat{v}_3 + \hat{v}_4)}{4} \hat{x}_1^2. \end{aligned} \quad (9)$$

Let $\varphi_K: \hat{K} \rightarrow K$ be an invertible mapping from the reference element \hat{K} to the general quadrilateral element K , and the FE space be defined as

$$V_h = \left\{ v_h | \hat{v}_h = v_h|_K \circ \varphi_K \in \hat{P}, \forall K \in T_h, \int_F [v_h]_F ds = 0, F \subset \partial K \right\}, \quad (10)$$

where $[v_h]_F = v_h$, while $F \subset \partial \Omega$ is the boundary edge. Let $\Pi_h: H^1(\Omega) \rightarrow V_h$ be the associated interpolation operator on V_h ; $\Pi_K = \Pi_h|_K$ satisfies

$$\int_{F_i} (v - \Pi_K v) ds = 0, \quad i = 1, 2, 3, 4. \quad (11)$$

There holds the following interpolation error estimate for any given $u \in H^2(\Omega)$:

$$\|u - \Pi_h u\|_0 + h \|u - \Pi_h u\|_h \leq ch^2 |u|_2, \quad (12)$$

where $\|v_h\|_{1,h} = \sqrt{\sum_{K \in T_h} |v_h|_{1,K}^2}$ is a broken energy norm on V_h .

Consider the discrete variational form of (7) as follows. Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (13)$$

where $a_h(u_h, v_h) = \sum_{K \in T_h} \int_K \beta \nabla u_h \nabla v_h dx$.

Now, some reasonable restrictions on the quadrilateral subdivision T_h ($0 < h < 1$ be the mesh size) are given as follows. The interface is allowed to cut through the elements, which are named interface elements K_{int} . Otherwise, the elements are called noninterface elements K_{non} . For the interface elements K_{int} , assume that

- (1) The edges meet the interface at no more than two points
- (2) Each edge is passed through at most once except passed at two vertices

It is easy to check that $V_h = V_h^{\text{int}} \cup V_h^{\text{non}}$, where V_h^{int} and V_h^{non} denote the FE spaces defined on interface elements K_{int} and noninterface elements K_{non} , respectively. In fact, the main concern is the interface elements K_{int} separated by the interface Γ into two subsets K^- and K^+ . The corresponding piecewise interpolation function should be constructed on K^- and K^+ , respectively. The key is how to make them together so that the jump conditions across the interface are maintained.

To describe the local IFE space on an interface element K_{int} , we assume that the vertices are $A_i, i = 1, 2, 3, 4$. Without loss of generality, we assume that ∂K_{int} intersects with Γ at two points D and E . There are two types of rectangle interface elements. Type I interface elements are those for which the interface intersects with two of their adjacent edges; Type II interface elements are those for which the interface intersects with two of their opposite edges.

Note that each piecewise polynomial in V_h has four freedoms (coefficients). The values $v_i (i = 1, 2, 3, 4)$ on K provide four restrictions. The normal derivative jump condition on DE provides another restriction. Then, three more restrictions can be provided by requiring the continuity of the finite element function at interface points D, E , and $(D + E)/2$. Intuitively, these eight conditions can yield the desired piecewise bilinear polynomial in an interface rectangle. In fact, since \overline{DE} can be considered as an approximation of the C^2 -curve \overline{DE} , the interface is perturbed by a $O(h^2)$ term. From [15], one can see for the interpolation polynomial defined below, such a perturbation will only affect the interpolation error to the order of $O(h^2)$. This idea leads us to consider functions defined as follows:

$$\Psi(x) = \begin{cases} \Psi^-(x) = a^-x_1 + b^-x_2 + c^- + d^-x_1^2, & \forall x \in K^-, \\ \Psi^+(x) = a^+x_1 + b^+x_2 + c^+ + d^+x_1^2, & \forall x \in K^+, \\ \Psi^-(D) = \Psi^+(D), \\ \Psi^-(E) = \Psi^+(E), \\ \Psi^-\left(\frac{D+E}{2}\right) = \Psi^+\left(\frac{D+E}{2}\right), \\ \int_{\overline{DE}} \left(\beta^- \frac{\partial \Psi^-}{\partial n} - \beta^+ \frac{\partial \Psi^+}{\partial n} \right) ds = 0. \end{cases} \quad (14)$$

Lemma 1. *Given a reference interface element, the piecewise function defined by (14) is uniquely determined.*

Proof. For any function Ψ defined on a rectangular element K , we let $\hat{\Psi}$ be the corresponding function on \hat{K} induced by Ψ with $\hat{\Psi}(\hat{x}) = \Psi(\varphi_K(\hat{x}))$.

Under this affine mapping φ_K , points D and E are mapped to

$$\begin{aligned} \hat{D} &= (-1, \hat{y}_0), \\ \hat{E} &= (\hat{x}_0, -1), \end{aligned} \quad (15)$$

for Type I interface elements and

$$\begin{aligned} \hat{D} &= (-1, \hat{y}_0), \\ \hat{E} &= (\hat{1}, \hat{y}_1), \end{aligned} \quad (16)$$

for Type II interface elements.

So, we only need to prove that the desired result holds on the reference element.

The values $\hat{\Psi}_i$ ($i = 1, 2, 3, 4$) on \hat{K} provide four restrictions as follows:

$$\begin{aligned} \hat{\Psi}_1 &= \int_{\hat{A}_1\hat{E}} (a^-x_1 + b^-x_2 + c^- + d^-x_1^2) d\hat{x}_1 \\ &+ \int_{\hat{E}\hat{A}_2} (a^+x_1 + b^+x_2 + c^+ + d^+x_1^2) d\hat{x}_2. \end{aligned} \quad (17)$$

□

Remark 2. If we choose $\hat{P} = \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_2^2\}$, the IFE space is also well-posed. However, the original rotated Q_1 element proposed in [25] was adapted, $\hat{P} = \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1^2 - \hat{x}_2^2\}$ instead of $\text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1^2\}$, and it can be checked that $\det(B_l) = \det(B_r) = 0$, which will bring about the non-uniqueness of \hat{a} . Thus, this IFE space is not well-posed.

4. Convergence Analysis for the Elliptic Interface Problem

In this section, the convergence analysis and error estimates of the IFEM will be carried out for the elliptic interface

problem. In order to do this, the following two important lemmas are proven as follows.

Lemma 2. *Let K be a general element with four edges F_i ($i = 1, 2, 3, 4$); then, for all $u \in PH_{\text{int}}^2(K)$, $v_h \in V_h$, we have*

$$\begin{aligned} \|v_h - P_{0i}v_h\|_{0,F_i} &\leq ch^{1/2}|v_h|_{1,K}, \\ \left\| \beta \frac{\partial u}{\partial n} - P_{0i}\beta \frac{\partial u}{\partial n} \right\|_{0,F_i} &\leq ch^{1/2}|u|_{2,K}, \end{aligned} \quad (18)$$

where $P_{0i}v_h = (1/|F_i|)\int_{F_i} v_h ds$, here and later, and c is a generic positive constant independent of h .

Proof. If K is a noninterface element, the results can be found in [27]. Now, only the case of K is an interface element needs to be proven. Without loss of generality, we prove (18) for $i = 1$.

Firstly, by the trace theorem, it can be derived that

$$\|v_h - P_{01}v_h\|_{0,F_1}^2 \leq ch|v_h|_{1,K}^2. \quad (19)$$

Secondly, let $\beta^-(\partial u^-/\partial n)$ be the restriction of $\beta(\partial u/\partial n)$ on K^- . Because of $[\beta(\partial u/\partial n)]_\Gamma = 0$, the function $\beta^-(\partial u^-/\partial n)$ can be extended onto the whole element K , and a function $\beta(\partial u/\partial n)$ can be obtained such that $\beta(\partial u/\partial n) = \beta^-(\partial u^-/\partial n)$ in K^- (see [14] for the details). There holds

$$\left\| \widetilde{\beta \frac{\partial u}{\partial n}} \right\|_{1,K} \leq c \left\| \beta^- \frac{\partial u^-}{\partial n} \right\|_{1,K^-}. \quad (20)$$

Thus, it can be derived that

$$\begin{aligned} \left\| \beta \frac{\partial u}{\partial n} - P_{01}\beta \frac{\partial u}{\partial n} \right\|_{0,F_1}^2 &= \left\| \beta^- \frac{\partial u^-}{\partial n} - P_{01}\beta^- \frac{\partial u^-}{\partial n} \right\|_{0,F_1}^2, \\ &= \int_{F_1} \left(\beta^- \frac{\partial u^-}{\partial n} - P_{01}\beta^- \frac{\partial u^-}{\partial n} \right)^2 dx_1 \\ &= h \int_{F_1} \left(\beta^- \frac{\partial u^-}{\partial \hat{n}} - P_{01}\beta^- \frac{\partial u^-}{\partial \hat{n}} \right)^2 d\hat{x}_1 \\ &= h \left\| \beta^- \frac{\partial u^-}{\partial \hat{n}} - \hat{P}_{01}\beta^- \frac{\partial u^-}{\partial \hat{n}} \right\|_{0,\hat{F}_1}^2 \\ &\leq ch \left\| \beta^- \frac{\partial u^-}{\partial \hat{n}} - \hat{P}_{01}\beta^- \frac{\partial u^-}{\partial \hat{n}} \right\|_{0,\hat{K}^-}^2 \\ &\leq ch \left\| \beta^- \frac{\partial u^-}{\partial \hat{n}} - \hat{P}_{01}\beta^- \frac{\partial u^-}{\partial \hat{n}} \right\|_{0,\hat{K}}^2 \leq ch \left\| \beta^- \frac{\partial u^-}{\partial \hat{n}} \right\|_{1,\hat{K}}^2 \\ &\leq ch \left\| \widetilde{\beta \frac{\partial u}{\partial n}} \right\|_{1,K}^2 \leq ch \left\| \beta^- \frac{\partial u^-}{\partial n} \right\|_{1,K^-}^2 \leq ch|u|_{2,K}^2, \end{aligned} \quad (21)$$

where in the last fourth inequality, the norm equivalent property was used on a reference element, see Section 4 in [28] for details. The proof is completed. □

Lemma 3. Let $u \in PH_{\text{int}}^2(\Omega) \cap H_0^1(\Omega)$; then, for $v_h \in V_h$, there holds

$$\left| \sum_{K \in T_h} \int_{\partial K} \beta \frac{\partial u}{\partial n} v_h ds \right| \leq ch |u|_{2,\Omega} \|v_h\|_{1,h}. \quad (22)$$

Proof. Noticing that $\int_{\partial K} (v_h - P_{0i} v_h) ds = 0$, there yields

$$\left| \sum_{K \in T_h} \int_{\partial K} \beta \frac{\partial u}{\partial n} v_h ds \right| = \left| \sum_{K \in T_h} \sum_{i=1}^4 \int_{F_i} \beta \frac{\partial u}{\partial n} - P_{0i} \beta \frac{\partial u}{\partial n} (v_h - P_{0i} v_h) ds \right|. \quad (23)$$

Applying Lemma 1 and Cauchy inequality to the right-hand side of (23) leads to the desired result. \square

Theorem 1. Let $u \in PH_{\text{int}}^2(\Omega) \cap H_0^1(\Omega)$ and $u_h \in V_h$ be the solutions of (18) and (27), respectively; then, there hold

$$\|u - u_h\|_{1,h} \leq ch |u|_{2,\Omega}, \quad (24)$$

$$\|u - u_h\|_{0,\Omega} \leq ch^2 |u|_{2,\Omega}. \quad (25)$$

Proof. By Strong's second lemma, it can be obtained that

$$\begin{aligned} \|u - u_h\|_{1,h} &\leq c \left(\frac{\inf_{v_h \in V_h} \|u - v_h\|_{1,h} + \sup_{v_h \in V_h} |a_h(u, v_h) - (f, v_h)|}{\|v_h\|_{1,h}} \right) \\ &= c \left(\frac{\inf_{v_h \in V_h} \|u - v_h\|_{1,h} + \sup_{v_h \in V_h} \left| \sum_{K \in T_h} \int_{\partial K} (\partial u / \partial n) v_h ds \right|}{\|v_h\|_{1,h}} \right). \end{aligned} \quad (26)$$

From now on, Π_h denotes the associated interpolation operator on V_h^{int} or V_h^{non} . So, by the interpolation theory and Lemma 3, two error terms on the right-hand side of (26) can be bounded as

$$\inf_{v_h \in V_h} \|u - v_h\|_{1,h} \leq \|u - \Pi_h u\|_{1,h} \leq ch |u|_{2,\Omega}, \quad (27)$$

$$\frac{\sup_{v_h \in V_h} \left| \sum_{K \in T_h} \int_{\partial K} (\partial u / \partial n) v_h ds \right|}{\|v_h\|_{1,h}} \leq ch |u|_{2,\Omega}, \quad (28)$$

respectively, which give the result (24).

In order to prove (25), we introduce the following auxiliary problem:

$$\begin{cases} \text{find } w \in PH_{\text{int}}^2(\Omega) \text{ such that} \\ -\nabla \cdot (\beta \nabla w) = u - u_h, \quad \forall x = (x_1, x_2) \in \Omega, \\ w|_{\partial\Omega} = 0, \end{cases} \quad (29)$$

with the jump conditions

$$[w]_{\Gamma} = 0,$$

$$\left[\beta \frac{\partial w}{\partial n} \right]_{\Gamma} = 0. \quad (30)$$

From [28], it is known that (29) has a unique solution w satisfying $\|w\|_{2,\Omega} \leq c \|u - u_h\|_{0,\Omega}$. Thus,

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 &= (u - u_h, u - u_h) = a_h(u - u_h, w) + \sum_{K \in T_h} \int_{\partial K} \beta \frac{\partial w}{\partial n} (u - u_h) ds \\ &= a_h(u - u_h, w - \Pi_h w) + a_h(u - u_h, \Pi_h w) + \sum_{K \in T_h} \int_{\partial K} \beta \frac{\partial w}{\partial n} (u - u_h) ds \\ &= a_h(u - u_h, w - \Pi_h w) + \sum_{K \in T_h} \int_{\partial K} \beta \frac{\partial u}{\partial n} (w - \Pi_h w) ds + \sum_{K \in T_h} \int_{\partial K} \beta \frac{\partial w}{\partial n} (u - u_h) ds \\ &\leq c \|u - u_h\|_{1,h} \|w - \Pi_h w\|_{1,h} + ch |u|_{2,\Omega} \|w - \Pi_h w\|_{1,h} + ch |w|_{2,\Omega} \|u - u_h\|_{1,h} \\ &\leq ch^2 |u|_{2,\Omega} |w|_{2,\Omega} \leq ch^2 |u|_{2,\Omega} \|u - u_h\|_{0,\Omega}, \end{aligned} \quad (31)$$

which leads to (25). The proof is completed. \square

Remark 3. The error estimate order in Theorem 1 is optimal and is half order higher than [15], which benefits from the proper partition near the interface.

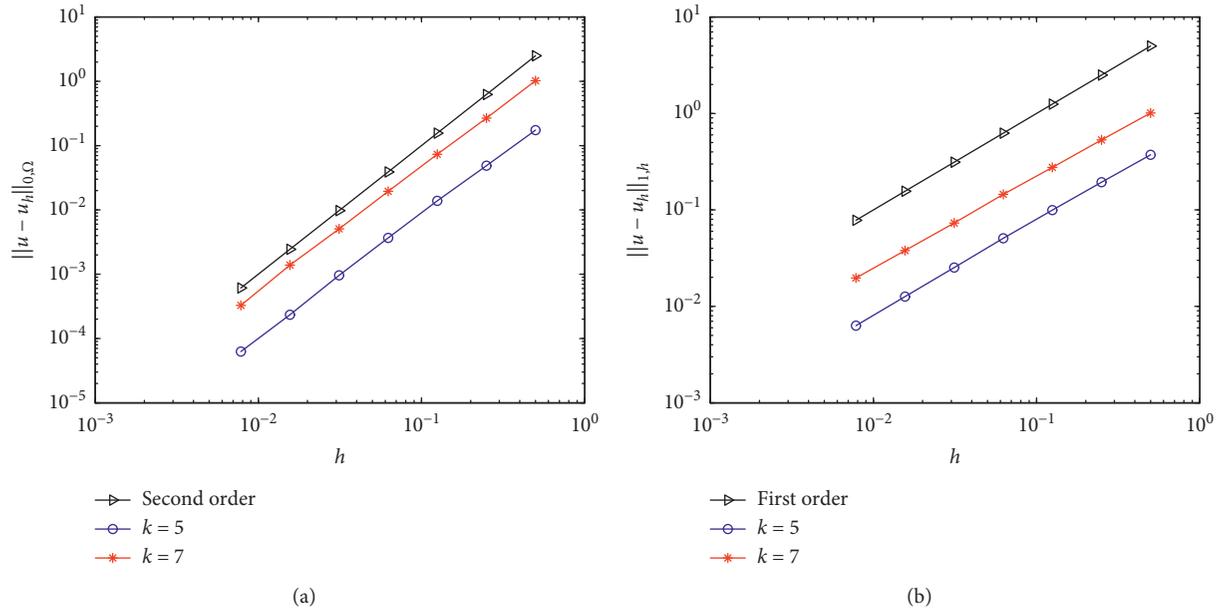
Remark 4. In order to conquer the asymmetry of the basis function space, one can first choose $\hat{P}_1 = \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1^2\}$ and $\hat{P}_2 = \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_2^2\}$ and compute the FE solutions u_h^1 and u_h^2 , respectively. Then, using $u_h = (1/2)(u_h^1 + u_h^2)$ as the approximation solution also satisfies Theorem 1.

TABLE 1: Errors in L^2 -norm and broken energy norm with $k = 5$.

$m \times n$	$\ u - u_h\ _{0,\Omega}$	Order	$\ u - u_h\ _{1,h}$	Order
4×2	0.174001020207	\	0.372938117902	\
8×4	0.048754398102	1.835491491	0.194134998441	0.941876033
16×8	0.013834029168	1.817310966	0.099413440848	0.965547404
32×16	0.003679917736	1.910475983	0.050746033726	0.970145853
64×32	0.000962451092	1.934888380	0.025235556981	1.007837138
128×64	0.000234073307	2.039752808	0.012610758212	1.000802910
256×128	0.000062682198	1.900832744	0.006318603068	0.996977475

TABLE 2: Errors in L^2 -norm and broken energy norm with $k = 7$.

$m \times n$	$\ u - u_h\ _{0,\Omega}$	Order	$\ u - u_h\ _{1,h}$	Order
4×2	1.025151980921	\	1.013510946696	\
8×4	0.268072065256	1.935145013	0.532621122372	0.928180123
16×8	0.073483750865	1.867123716	0.275850326414	0.949223956
32×16	0.019446071569	1.917946532	0.145081776373	0.927019373
64×32	0.005066576895	1.940395475	0.073089894557	0.989122457
128×64	0.001387910935	1.868096366	0.037931441458	0.946277755
256×128	0.000327592545	2.082940559	0.019698456819	0.945311584

FIGURE 2: Convergence rates in L^2 -norm (a) and broken energy norm (b) for Example 1.

5. Numerical Results

In this section, two numerical experiments will be carried out for an elliptic interface problem.

Example 1 (see [6, 29]). In this example, the domain is chosen as $\Omega = [0, 2] \times [0, 1]$, $\Omega^- = [0, 1] \times [0, 1]$, and $\Omega^+ = (1, 2] \times [0, 1]$; the interface Γ occurs at $x_1 = 1$. The exact solution u can be expressed as

$$u = \begin{cases} u^- = \sin(\pi x_1) \sin(\pi x_2), & \text{in } \Omega^-, \\ u^+ = \sin(k\pi x_1) \sin(\pi x_2), & \text{in } \Omega^+, \end{cases} \quad (32)$$

where $\beta^+ = 1$ and $\beta^- = k$ (k is an odd number) just satisfy the jump conditions. Now, the errors are listed in Tables 1 and 2 for $k = 5$ and $k = 7$, respectively. Figure 2 reports the convergence rates of our nonconforming IFEM in L^2 and broken energy norms, respectively.

Example 2. In this example, the domain is chosen as $\Omega = [0, 1] \times [0, 1]$, and the interface Γ occurs at $x_1 + x_2 = 1$. Ω^- is the triangle rounded by x_1 -axis, x_2 -axis, and Γ . $\Omega^+ = \Omega - \Omega^-$. If the corresponding right-hand term is given as

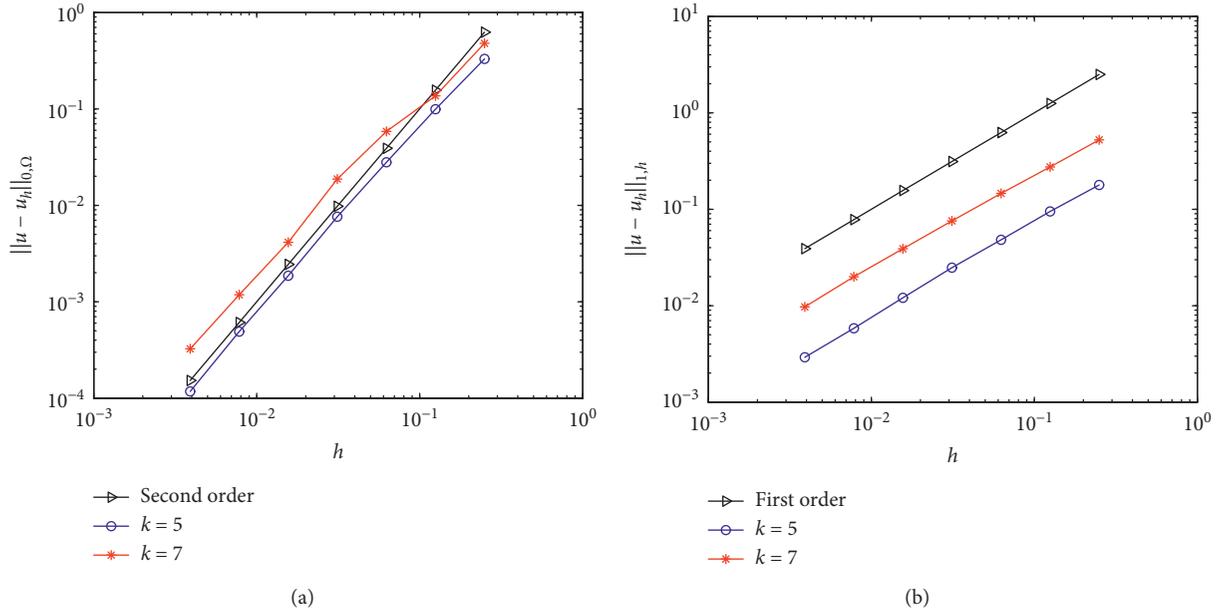


FIGURE 3: Convergence rates in L^2 -norm (a) and broken energy norm (b) for Example 2.

TABLE 3: Errors in L^2 -norm and broken energy norm with $k' = 5$.

$m \times n$	$\ u - u_h\ _{0,\Omega}$	Order	$\ u - u_h\ _{1,h}$	Order
4×4	0.329346507641	\	0.178346555746	\
8×8	0.099368258437	1.728749267	0.094748546340	0.912507640
16×16	0.027942781629	1.830309436	0.048211021354	0.974740816
32×32	0.007621493172	1.874330067	0.024800811014	0.958975695
64×64	0.001861050498	2.033956470	0.012082578254	1.037458961
128×128	0.000491029936	1.922234315	0.005817501298	1.054456806
256×256	0.000117408662	2.064272133	0.002914705261	0.997049624

TABLE 4: Errors in L^2 -norm and broken energy norm with $k' = 7$.

$m \times n$	$\ u - u_h\ _{0,\Omega}$	Order	$\ u - u_h\ _{1,h}$	Order
4×4	0.478733603485	\	0.524504640793	\
8×8	0.136643751524	1.808803586	0.274358019661	0.934895790
16×16	0.058418679138	1.225917846	0.146031238772	0.909782726
32×32	0.018747166324	1.639757193	0.075697236338	0.947964487
64×64	0.004139677913	2.179082117	0.038901806739	0.960403469
128×128	0.001182704768	1.807428537	0.019973384828	0.961758318
256×256	0.000325298041	1.862255947	0.009713801702	1.039970902

$$f = \begin{cases} f^- = -4\pi x_1 \cos 2\pi(x_1 + x_2) - 4\pi x_2 \cos 2\pi(x_1 + x_2) + 8\pi^2 x_1 x_2 \sin 2\pi(x_1 + x_2), & \text{in } \Omega^-, \\ f^+ = -4k\pi x_1 \cos 2k\pi(x_1 + x_2) - 4k\pi x_2 \cos 2k\pi(x_1 + x_2) + 8k^2\pi^2(x_1 - 1)(x_2 - 1)\sin 2k\pi(x_1 + x_2), & \text{in } \Omega^+, \end{cases} \quad (33)$$

where k is a positive constant independent of x_1 and x_2 , the exact solution can be expressed as

$$u = \begin{cases} u^- = x_1 x_2 \sin 2\pi(x_1 + x_2), & \text{in } \Omega^-, \\ u^+ = (x_1 - 1)(x_2 - 1)\sin 2k\pi(x_1 + x_2), & \text{in } \Omega^+. \end{cases} \quad (34)$$

Obviously, $u|_{\partial\Omega} = 0$. If $(\beta^- / \beta^+) = k\iota$, the jump conditions $[u]_\Gamma = 0$ and $[\beta(\partial u / \partial n)]_\Gamma = 0$ are satisfied. In the following, the errors are listed in Tables 1 and 2 for $k' = 5$ and $k' = 7$, respectively.

Figure 3 reports the convergence rates of our non-conforming IFEM in L^2 and broken energy norm, respectively.

From Tables 1–4 and Figures 2 and 3, one can see that the proposed new nonconforming IFEM can solve the linear interface problem with the optimal order error estimates, but how to apply this method to the nonlinear case still remains open.

6. Conclusions

This paper discusses a modified nonconforming rotated Q_1 IFEM for second-order elliptic interface problems. Optimal order error estimates of L^2 -norm and broken energy norm are derived. Numerical examples are provided to confirm the theoretical results.

We should point out that this method is suitable for parabolic-type or hyperbolic-type interface problems by using a suitable full discretization scheme. However, the method cannot be applied to other very popular nonconforming FEs, such as EQ_1^{rot} element [22], Carey element [30], and Wilson element [31].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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