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Research Article

Game Theoretic Analysis of After-Sales Service in Two-Echelon Supply Chain with Warranty Sensitive Demand

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After-sales service contract is widely popular in business. Although both the cases of manufacturer offering warranty and retailer offering warranty are common in market, the differences between them have been few studied. In this paper, we build a two-echelon supply chain in which a manufacturer produces limit quality products and sells them to a retailer. To promote sales, the manufacturer or retailer offers a free-replacement warranty to the customers. Customer's demand is affected by the warranty length. We investigate the game relationships between the supply chain members. We find that the warranty length negatively relates to the product quality in both the manufacturer offering warranty case and the retailer offering warranty case. When retailer's profit margin is not low, the retailer offers a longer warranty than the manufacturer and vice versa. Profit of the supply chain is also analyzed along with a numerical study.

1. Introduction

Warranty contract is a service commitment that the customers can return, repair, or replace the failed products for free or at some cost during the warranty period [1]. For example, a car manufacturer promises to repair his cars in three years after the sales and an online retailer such as tmall.com promises seven days of no reason to return. Generally, a warranty contract usually specifies the warranty period, the repair or replaces policy for the failed products, the responsible party for the warranty, etc.

Warranty can be offered by the manufacturer, retailer, or a third party. The third party warranty contract is a kind of insurance policy to the consumers [2]. Generally, the manufacturer or retailer has more information about the product quality than the customers do, so prudent customers do not always completely believe the advertisement that the manufacturer or retailer boasted on the product quality. Since the warranty provider shares the costs of product fails, longer warranty attracts more customers ceteris paribus. Thus, from the view point of manufacturer or retailer, warranty service is not only a legally compulsory provision but also a sales promotion measure [3–5]. The question is that does longer warranty really mean the product has a better quality and what are the differences between the manufacturer offering warranty and the retailer offering warranty.

Since different supply chain contracts may result in different supply chain performances, proper choosing of supply chain contracts is important to the contract designers [6–9]. In this paper, we build a two echelon supply chain in which a manufacturer produces limit quality products and sells them to a retailer. To promote sales, the manufacturer or retailer offers a free-replacement warranty to the customers. Customer's demand is affected by the warranty length. We investigate the game relationships between the supply chain members. We find that, in both the manufacturer-offered warranty contract and the retailer-offered warranty contract, the optimal warranty period is decreasing in the product quality. This shows that, for both manufacturers and retailers, there is a negative correlation between providing higher quality products and providing higher level of after-sales service. We also prove that when the retailer's profit margin is not low, the case that retailer-offering warranty can generate a longer warranty period than that manufacturer-offering warranty, otherwise the

opposite. Explanations and management insights are also discussed in the proper positions in this paper.

Rest of this paper is as follows. Section 2 reviews the literature. Section 3 sets up the models of manufacturer warranty and retailer warranty. Section 4 analyzes the models and compares the contracts and the warranty lengths with different situations. Section 5 extends the model. Section 6 is the numerical analysis. Section 7 concludes the paper.

2. Literature Review

Former review papers on warranty include [1, 10–12]. They reviewed the literature from the viewpoints of new product warranties, warranty logistics, long-term warranties, and maintenance models. In specific areas, some papers focus on the marketing problem that how consumers value the warranties. For example, Luo and Wu [13] analyze the consumers' attitude to extended warranties. Huang [14] examine consumer affective reactions to product problems when the warranty length varies.

Besides customer warranty perception, some papers study the warranty costs. These papers mainly interest in calculating the cost directly or indirectly related to the warranty policies. Cost estimation models for renewing warranty policies [15–18], nonrepairable products [19], and two-dimensional warranty policies [20, 21] are analyzed by these papers. Another research direction is on the warranty contract design. Li [22] studies determinate demand case. Tong [23] and Zhou [24] study the dynamic market case by using the dynamic programming or optimal control theories. Chen and Chang [25] and Yeo and Yuan [26] further consider special warranties.

When the product has a failure during the warranty period, a repair or replace service occurs by the warranty provider. But the question is what the optimal strategy is for the warranty provider. Some papers focus on this problem. Chien [27] and Jung [28] study the optimal replacement strategies. Generally, repairing a product is cheaper than replacing it but has a greater probability of failing during the remainder warranty period. It is of interest to the warranty providers that to choose a proper repair-replace strategy. Xu [29] makes contributions to this problem.

The green quality and warranty period of products are two major factors which affect consumers' purchasing behavior. Ebrahimi [9] analyzed a supply chain with a monopolistic manufacturer aiming to release a new substitutable green product in addition to the conventional nongreen product and determine the green quality level of the new product. Hosseini-Motlagh [30] studied a competitive closed-loop supply chain and proposed a new function for supply of used products based on the service competition between the dealers. End-of-life (EOL) items play a critical role in the success of remanufacturing systems.

This paper focuses on the differences between the manufacturer-offering warranty contract and the retailer-offering warranty contract (Table 1). In this paper, we develop stochastic models of the two echelon supply chains with the manufacturer or retailer offering a free replacement warranty to consumers. Free replacement warranty is the most common warranty type in practice and is most widely used in reality. In addition, we regard the customer demand as a function of the warranty length. We assume that longer warranty service length generates more demand and vice versa.

3. The Models

Consider a two-echelon supply chain with a manufacturer selling to the retailer. The manufacturer produces a single kind of product with unit cost c and sells them to retailer with wholesale price w. Then, the retailer sells the products to consumers with retail price p. Besides the product itself, the manufacturer or retailer also provides warranty. The warranty provider promises to replace the failed products (with failure rate λ) for free during the warranty period. Consumer demand is affected by the warranty length t. In this paper, we adopt the additive demand $D(t) = k(t) + \varepsilon$, where $k(t)$ is an increasing and concave function of t. In practice, demand is not only affected by warranty service but also by other factors. These factors may be uncertain and unknown. We use the random variable ε to represent the influence of these unknown factors. In addition, we use the form of additive demand function. This shows that these random factors are independent with the warranty. The probability density function of the random variable ε is $f(\varepsilon)$, and the cumulative distribution function is $F(\varepsilon)$.

The order quantity of retailer is q at the first stage (before offering warranty). At the second stage, the retailer may make an additional order for the replacing warranty (in retailer-offering warranty case). The relationship between the manufacturer and the retailer is a stackelberg game where the manufacturer is the leader. We use π to denote the expected profit. To differentiate between different cases, we use superscript $(.)^M$, $(.)^R$, or $(.)^C$ to refer to the case of manufacturer offers warranty, retailer offers warranty, or the centralized supply chain. We use subscript $(.)_m$ or $(.)_r$ to the profit of the manufacturer or retailer (Table 2).

3.1. Manufacturer Offers Warranty. As the manufacturer is a stackelberg leader, we first consider retailer's optimal decisions. In the manufacturer-offering warranty model, the decision of the retailer is to choose the proper order quantity with the purpose of maximizing his own expected profit:

$$\begin{aligned} \operatorname{Max} \pi_r^M(q) = {} & p\int_0^q D(t) f(D(t)) d D(t) \\ & + p\int_q^{+\infty} q f(D(t)) d D(t) - wq. \end{aligned} \tag{1}$$

For the manufacturer, the decision is to choose a proper warranty period to maximize his own profit with considering the retailer's reactions:

Table 1: Comparing current paper with previous studies.

Study	Demand type	Warranty type	Research subject
Zhang [15]	A two-component series system with stochastic demand	Free replacement policy	Warranty cost evaluation
Huang [14]	Pro rata proportions affect demand of the customer	Extended warranty	Propose a new extended warranty policy
Chen [16]	Demand rate is deterministic and constant	Free replacement warranty	Study optimal production run length and warranty period
Dan [31]	Warranty affects demand	Free value-added service	Warranty service strategies in dual-channel supply chain
Current study	Stochastic demand which is affected by the warranty length	Free replacement/repair warranty	Game theoretic analysis on warranty service between the supply chain members

Table 2: Notations and parameters.

w	Wholesale price
p	Retail price
λ	Failure rate of the product
t	Warranty length
D	Demand of the product
$f(\varepsilon)$	Probability density function of the demand
$F(\varepsilon)$	Cumulative distribution function of the demand
$(.)^M$	The case of manufacturer-offering warranty
$(.)^R$	The case of retailer-offering warranty
$(.)^C$	The case of centralized supply chain
$(.)_m$	Parameters of the manufacturer
$(.)_r$	Parameters of the retailer

$$\text{Max}\, \pi_m^M(t) = (w-c)q - \lambda tc\left[\int_0^q D(t)f(D(t))dD(t) + \int_q^{+\infty} qf(D(t))dD(t)\right]. \tag{2}$$

From equation (1), we know that the retailer's decision is a newsvendor problem. But here, the consumers' demand is affected by the manufacturer's warranty decision (i.e., the length of warranty). λt in equation (2) is the expected failure time of the product during the warranty period. $\lambda tc[\int_0^q D(t)f(D(t))dD(t) + \int_q^{+\infty} qf(D(t))dD(t)]$ is the warranty costs of the manufacturer.

3.2. Retailer Offers Warranty. When the retailer offers the warranty, he should simultaneously choose the proper order quantity and the warranty period with the purpose of maximizing his own expected profit:

$$\text{Max}\, \pi_r^R(q,t) = (p-\lambda tw)\left[\int_0^q D(t)f(D(t))dD(t) + \int_q^{+\infty} qf(D(t))dD(t)\right] - wq. \tag{3}$$

The manufacturer gains a constant profit with respect to retailer's total order quantity which can be described as

$$\text{Max}\, \pi_m^R = (w-c)\left[q + \lambda t\left(\int_0^q D(t)f(D(t))dD(t) + \int_q^{+\infty} qf(D(t))dD(t)\right)\right]. \tag{4}$$

When the retailer offers the warranty, the retailer should pay for the warranty cost himself. The warranty cost of unit product for the retailer can be expressed as λtw. Since q is the original order quantity of the retailer, the item $\lambda t(\int_0^q D(t)f(D(t))dD(t) + \int_q^{+\infty} qf(D(t))dD(t))$ is the additional order quantity of the retailer for the reason of replacing the failed products.

4. Model Analysis

4.1. Concavity of Profit Functions. Retailer's decisions in both the cases that manufacturer offers warranty and retailer offers warranty are similar to the newsvendor problem. First, we prove the concavity of both the manufacturer's profit function and the retailer's profit function in the case that manufacturer offers warranty. The optimal solution to the manufacturer offering warranty model can be expressed as the following proposition.

Proposition 1. *If the manufacturer directly offers the warranty to customers, then*

(a) Both the retailer's profit function $\pi_r^M(q)$ and the manufacturer's profit function $\pi_m^M(t)$ are concave.

(b) The unique optimal warranty length t for the manufacturer is found from the following equation:

$$\frac{w-c}{\lambda c} = t + \frac{k(t) + \int_0^{F^{-1}[1-(w/p)]}[1-F(x)]dx}{k'(t)}. \tag{5}$$

The unique optimal order quantity for the retailer is $q = k(t) + F^{-1}[1-(w/p)]$.

Proof. Taking the first and second derivatives of $\pi_r^M(q)$ with respect to q yields, we get

$$\frac{\partial \pi_r^M(q)}{\partial q} = -p[1-F(q-k(t))] - w, \tag{6}$$

$$\frac{\partial^2 \pi_r^M(q)}{\partial q^2} = -pf[q-k(t)]. \tag{7}$$

It can be seen that $(\partial^2 \pi_r^M(q)/\partial q^2) < 0$. That is the profit function of the retailer is concave in q.

By the first-order condition, we can get $q = k(t) + F^{-1}[1-(w/p)]$. Substituting q with $k(t) + F^{-1}(1-(w/p))$into the manufacturer's profit function and then taking the first and second derivatives of $\pi_m^M(t)$ with respect to t yields

$$\frac{\partial \pi_m^M(t)}{\partial t} = k'(t)[w-(1+\lambda)c] - \lambda ct\left[k(t) + \int_0^{F^{-1}[1-(w/p)]}[1-F(x)]\mathrm{d}x\right], \tag{8}$$

$$\frac{\partial^2 \pi_m^M(t)}{\partial t^2} = k''(t)[w-(1+\lambda)c] - \lambda ctk'(t) - \lambda c\left[k(t) + \int_0^{F^{-1}[1-(w/p)]}[1-F(x)]\mathrm{d}x\right]. \tag{9}$$

Since $k(t)$ is an increasing and concave function of t, we have $k'(t) > 0$ and $k''(t) < 0$. It is easy to verify that $(\partial^2 \pi_m^M(t)/\partial t^2) < 0$. That is the profit function of the manufacturer is also concave in t.

When the manufacturer offers warranty directly to the consumers, there is a unique solution to the model of manufacturer offering warranty. Consumers' demand is a nondecreasing function of the warranty period, ceteris paribus (see the definition of customers' demand). Thus, by the first-order condition, the retailer's order quantity under manufacturer-offering warranty $(t > 0)$ is always higher than that with nonwarranty service $(t = 0)$. Similar to Proposition 1, we have the following proposition that describes the uniqueness of the model of retailer-offering warranty. □

Proposition 2. *If the retailer offers warranty to customers, then,*

(a) The retailer's profit function $\pi_r^R(q,t)$ is concave in q and t.

(b) The unique optimal warranty length t is found from the following equation:

$$\frac{p-w}{\lambda w} = t + \frac{k(t) + \int_0^{F^{-1}[1-(w/(p-\lambda wt))]}[1-F(x)]dx}{k'(t)}. \tag{10}$$

The unique optimal-order quantity for the retailer is $q = k(t) + F^{-1}[1-(w/(p-\lambda wt))]$.

Proof. Taking the first derivative of $\pi_r^R(q,t)$ with respect to q and t yields

$$\frac{\partial \pi_r^R(q,t)}{\partial q} = (p-\lambda wt)[1-F(q-k(t))] - w, \tag{11}$$

$$\frac{\partial \pi_r^R(q,t)}{\partial} = -\lambda w\left[\int_0^q D(t)f(D(t))d\,D(t) + \int_q^{+\infty} qf(D(t))d\,D(t)\right] + (p-\lambda wt)\frac{\partial\left[\int_0^q D(t)f(D(t))d\,D(t) + \int_q^{+\infty} qf(D(t))d\,D(t)\right]}{\partial t}. \tag{12}$$

By the first-order condition, we have $q = k(t) + F^{-1}[1-(w/(p-\lambda wt))]$. Taking it into equation (10) yields

$$\frac{\partial^2 \pi_r^R(q,t)}{\partial t} = -\lambda w\left[k(t) + \int_0^{F^{-1}[1-(w/(p-\lambda wt))]}[1-F(x)]\mathrm{d}x\right] - (p-\lambda wt)k'(t). \tag{13}$$

Take the derivative of $\pi_r^R(q,t)$ in equation (11) with respect to q and take the derivative of $\pi_r^R(q,t)$ with respect to t. We have

$$\frac{\partial^2 \pi_r^R(q,t)}{\partial q^2} = -f(q-k(t)) < 0, \tag{14}$$

$$\frac{\partial^2 \pi_r^R(q,t)}{\partial t^2} = -\lambda w\left[k'(t) + \frac{\lambda w^3}{(p-\lambda wt)^3}\right] - (p-\lambda wt)k''(t) + \lambda wk'(t), \tag{15}$$

$$\frac{\partial^2 \pi_r^R(q,t)}{\partial q\,\partial qt} = -\lambda w[1-F(q-k(t))] + (p-\lambda wt)f(q-k(t))k'(t). \tag{16}$$

The Hessian matrix is

$$H\left(\pi_r^R\right) = \begin{vmatrix} \dfrac{\partial^2 \pi_r^R(q,t)}{\partial q^2} & \dfrac{\partial^2 \pi_r^R(q,t)}{\partial q\,\partial qt} \\ \dfrac{\partial^2 \pi_r^R(q,t)}{\partial t\,\partial q} & \dfrac{\partial^2 \pi_r^R(q,t)}{\partial t^2} \end{vmatrix} = f(q-k(t))\left\{\lambda w\left[k'(t) + \frac{\lambda w^3}{(p-\lambda wt)^3}\right] + (p-\lambda wt)k''(t) - \lambda w k'(t)\right\}$$
$$-\left\{-\lambda w[1-F(q-k(t))] + (p-\lambda wt) f(q-k(t))k'(t)\right\}^2 > 0. \tag{17}$$

Since $(\partial^2 \pi_r^R(q,t)/\partial q^2) > 0$ and the Hessian matrix is positive, it is easy to verify the concavity of profit function. Then, according the firs-order conditions, we can get the optimal solutions.

When the retailer offers warranty, the warranty length only depends on the retailer's own profit margin. While, when the manufacturer offers the warranty, the warranty length also depends on the manufacturer's profit margin. This founding will help us to understand the contrast on the warranty lengths of manufacturer-offering warranty and retailer-offering warranty which will be presented later. □

4.2. Impacts of Product Quality. Since in both the cases, manufacturer offers warranty and retailer offers warranty, there is a unique optimal solution, and we can analyze the impacts of the key parameters. It is interesting that does longer warranty really mean the product has a better quality or the warranty providers just mislead the consumers. In this subsection, we will analyze how does product quality (indicated by the failure rate) affects the warranty length. We have Proposition 3.

Proposition 3. *If $\lambda > 0$, then we have $(\partial t/\partial \lambda) < 0$.*

Proof. We firstly prove the case that manufacturer offers warranty. According to the first-order condition, we have

$$\frac{w-c}{\lambda c} = t + \frac{k(t) + \int_0^{F^{-1}[1-(w/p)]} [1-F(x)]\mathrm{d}x}{k'(t)}. \tag{18}$$

Since, the left-hand side of equation (18) is decreasing in λ, the right-hand side should also be decreasing in λ. $k(t)$ is increasing and concave in t. So, the right-hand side of equation (18) is increasing in t. So, t is certainly decreasing as λ increases (otherwise, the right-hand side is increasing in λ which is obviously impossible). Secondly, we prove it in the case the retailer offers warranty. Similarly, we have the following equation by the first-order condition:

$$\frac{p-w}{\lambda w} = t + \frac{k(t) + \int_0^{F^{-1}[1-(w/(p-\lambda wt))]} [1-F(x)]\mathrm{d}x}{k'(t)}. \tag{19}$$

Similar to the proof of the case manufacturer-offering warranty, we can verify that t is decreasing in λ.

Generally, the manufacturer or retailer has more information about the product quality than customers in most cases. That is, there is information asymmetry between the manufacturer/retailer and the customers. Proposition 3 implies that the customers can judge the product quality through the length of warranty offered by the manufacturer. That means longer warranty reflects better product quality.

In this paper, we just consider a single manufacturer and a single retailer, which means there is no horizontal competition at any echelon. Balachander [32] considers two firms: an established incumbent and a new entrant. He finds that sometimes low-quality firm may offer the longer warranty. So, by integrating the conclusions of Balachander's research and this paper, one can conjecture that the factors that intensify the information asymmetry of product quality between the manufacturer (the retailer) and the consumers are not from the vertical competitions but from the horizontal competitions. □

4.3. Comparison on Warranty Lengths. In this section, we will try to compare the difference of the two models. Denote manufacturer's profit margin $\gamma_m = ((w-c)/c)$, retailer's profit margin $\gamma_r = ((p-w)/w)$. Let t^M and t^R be the optimal warranty length of the case of manufacturer offers warranty and the case of retailer offers warranty. We have the following proposition.

Proposition 4. *(a) If $\gamma_m < \gamma_r$, then we have $t^M < t^R$. (b). If $\gamma_m \gg \gamma_r$, then we have $t^M > t^R$.*

Proof. If $\gamma_m < \gamma_r$, according to (18) and (19), we have

$$t^M + \frac{k(t^M) + \int_0^{F^{-1}(1-(w/p))} [1-F(x)]\mathrm{d}x}{k'(t^M)} < \frac{k(t^R) + \int_0^{F^{-1}\left(1-\left(w/\left(p-w\lambda t^R\right)\right)\right)} [1-F(x)]\mathrm{d}x}{k'(t^R)}. \tag{20}$$

Since $1-(w/p)>1-(w/(p-w\lambda t))$, we have

$$\int_0^{F^{-1}[1-(w/p)]}[1-F(x)]\mathrm{d}x>\int_0^{F^{-1}[1-(w/(p-w\lambda t^R))]}[1-F(x)]\mathrm{d}x. \tag{21}$$

The both sides of (14) is increasing in t^M or t^R, we get $t^M<t^R$.

If $\gamma_m \gg \gamma_r$, according to equation (18), t^M will also be sufficiently large. Note that t^R is independent of γ_m. So, there must exist a sufficient large $\gamma_m \gg \gamma_r$ that results in $t^M>t^R$.

Proposition 4 states that, if the manufacturer's profit margin is less than the retailer's profit margin, then the retailer can offer a longer warranty, ceteris paribus. It means, from the view point of customers, the retailer-offering warranty is better at this situation. If the manufacturer's profit margin is sufficiently higher than the retailer's profit margin, then manufacturer-offering warranty is better (longer warranty period) for the customers.

Generally, the manufacturer has the first choice of offering warranty service to the customers. But in the real business transaction, the retailer sometimes offers extended warranty (prolongs the manufacturer's warranty) to customers for free or at a certain price. One of the explanations may be that sometimes the retailer can offer a longer warranty compared with manufacturer-offering warranty. But, if the product's retail profit margin is very low, then the retailer would not have this motivation. □

4.4. Centralized System. We consider the centralized system case and analyze the supply chain efficiencies of the models manufacturer offers warranty and retailer offers warranty.

In the centralized system, the decision objective is to maximize the sum of manufacturer and retailer's profits. That is

$$\begin{aligned}\operatorname{Max}\pi^C(q,t)=(p-\lambda tc)\Bigg[&\int_0^q D(t)f(D(p,t))d\,D(p,t)\\&+p\int_q^{+\infty}qf(D(t))d\,D(t)\Bigg]-cq.\end{aligned} \tag{22}$$

From equation (22), we can see that the decision of the centralized system is similar to the retailer's decision of the retailer offering warranty case. In both cases, q and t are determined by a single decision maker. But there are some differences. In the centralized system, the decision objective is to maximize the total supply chain profit. In retailer-offering warranty case, the retailer decides to maximize his own profit. Similar to the retailer-offering warranty case, it is easy to verify that $\pi^C(q,t)$ is also concave. Thus, by the first-order conditions, one can get the optimal solution. We detail it with Proposition 5.

Proposition 5. *In the centralized system, the optimal warranty period is found from the following equation:*

$$\frac{p-c}{\lambda c}=t+\frac{k(t)+\int_0^{F^{-1}[1-(c/(p-\lambda tc))]}[1-F(x)]\mathrm{d}x}{k'(t)}. \tag{23}$$

The unique optimal producing quantity for the system is $\mathrm{q}=\mathrm{k}(t)+\int_0^{F^{-1}[1-(c/(p-\lambda tc))]}[1-F(x)]\mathrm{d}x$.

Proof. By comparing the equations (22) and (3), we can see that the only difference between them is in the costs w and c. Thus, by substituting w with c in the proof of Proposition 2, we can get the proof of this proposition.

In the centralized supply chain, there is a unique solution of optimal warranty length and product quantity. The expressions of optimal warranty length and product quantity of the centralized system are similar to the case of retailer-offering warranty service except the costs, because in both cases, the decision marker determines the warranty and quantity simultaneously. □

5. Extension of the Contract

5.1. Price-Dependent Demand. In the above sections, we assume the manufacturer or retailer undertaking after-sales service and assume the demand is independent of the retail price. In this section, we extend our model to the price-dependent demand, where the retail price is decided by the retailer. We denote the demand $D(p,t)$ as a stochastic function of p and t. We assume $D(p,t)$ stochastically decreasing and convex in p and stochastically increasing and concave in t. That is to say, the probability function $\Pr[D(p,t)>x]=1-F(x)$ is decreasing and convex in p for any given t,x and is increasing and concave in t for any given p,x. The symbol x represents the demand level (uncertain variable) of the market. Since we assume the warranty is a free replacement or repair service, we use the symbol s to denote the total warranty cost of per warranty return. By these assumptions, we rewrite the supply chain model as follows.

The retailer's problem is

$$\begin{aligned}\operatorname{Max}\pi_r(q,p)=p\int_0^q D(p,t)f(D(p,t))d\,D(p,t)\\+p\int_q^{+\infty}qf(D(p,t))d\,D(p,t)-wq.\end{aligned} \tag{24}$$

By the above equation, the retailer's problem is an extended newsvendor problem where the customer's demand also depends on the retail price. Similar research has been done by Petruzzi and Dada [33]. But, they assume the demand function has additive or multiplicative form. We relax their restrictions. One can verify that the additive or multiplicative demand case satisfies our assumptions.

The manufacturer's problem is

$$\begin{aligned}&\operatorname{Max}\pi_m(w,t)=(w-c)q-s\lambda t\left[\int_0^q D(p,t)f(D(p,t))dD(p,t)+p\int_q^q qf(D(p,t))dD(p,t)\right],\\&s.t.\,q,p=\arg\max\left\{p\int_0^q D(p,t)f(D\,d(p,t))dD(p,t)+p\int_q^{+\infty} qf(D(p,t))dD(p,t)-wq\right\}.\end{aligned}\tag{25}$$

According to the retailer's problem, the optimal order quantity and the retail price are given by the first-order conditions of equation (24), i.e.,

$$\frac{\partial\pi_r(q,p)}{\partial q}=p[1-F(q-k(t))]-w=0,\tag{26}$$

$$\frac{\partial\pi_r(q,p)}{\partial p}=\int_0^q[1-F(x)]\mathrm{d}x+p\int_0^q[1-F(x)]_p'\,\mathrm{d}x=0.\tag{27}$$

By the equation (26), we can get the optimal order quantity of the retailer, i.e., $q=F^{-1}(1-(w/p))$. And, because $1-F(x)$ is decreasing and convex in p, we get the optimal retail price of the retailer as

$$p=-\frac{\int_0^q[1-F(x)]dx}{p\int_0^q[1-F(x)]_p'dx}.\tag{28}$$

Taking the solutions p and q into the model equation (25), we can get the manufacturer objective function. One can see that, to the manufacturer's problem, the situation is similar to the case of exogenous retail price. If the Proposition 1 holds, then there also be a unique optimal wholesale price and warranty policy for the manufacturer. The unique solution can be got from the first order conditions of manufacturer's problem.

5.2. Warranty Cost Sharing. In this subsection, we assume that the total warranty cost is shared between the manufacturer and retailer. In the real situations, although in most cases the manufacturer establishes the warranty service system and offering warranty service to the final consumers, the retailer also has responsibility for the warranty returns. We use the symbol $\theta\in[0,1]$ to denote the cost-sharing proportion. If the total warranty cost of per warranty return is s, then the manufacturer bears the $(1-\theta)s$ cost and the retailer bears θs cost. Thus, the retailer's problem can be modelled as

$$\begin{aligned}\operatorname{Max}\pi_r(q,p)&=p\int_0^q D(p,t)f(D(p,t))dD(p,t)+p\int_q^{+\infty} qf(D(p,t))dD(p,t)-wq\\&\quad-\theta s\lambda t\left[\int_0^q D(p,t)f(D(p,t))dD(p,t)+\int_q^{+\infty} qf(D(p,t))dD(p,t)\right],\\&=(p-\theta s\lambda t)\left[\int_0^q D(p,t)f(D(p,t))dD(p,t)+\int_q^{+\infty} qf(D(p,t))dD(p,t)\right]-wq.\end{aligned}\tag{29}$$

The manufacturer's problem can be modelled as

$$\operatorname{Max}\pi_m(w,t)=(w-c)q-(1-\theta)s\lambda t\left[\int_0^q D(p,t)f(D(p,t))dD(p,t)+\int_q^{+\infty} qf(D(p,t))dD(p,t)\right].\tag{30}$$

According to equation (29), we can see that, if $\mathrm{p}=\mathrm{p}-\theta s\lambda t$, then the situation is same with equation (24). So, from the first-order conditions of equation (29), we can get the optimal order quantity and retail price to the retailer. Similarly, to the manufacturer's problem, i.e., the equation (30), if the Proposition 1 holds, then there also be a unique optimal wholesale price and warranty policy for the manufacturer.

6. Numerical Analysis

In this section, we conduct a numerical study to investigate how manufacturer's warranty service affects the manufacturer, retailer's profits, and supply chain efficiency. We also study the relationship between the customer's demand distribution and the supply chain performance. In the end, we investigate the impacts of warranty cost-sharing ratio.

We use a specific form of the demand function:

$$d(\mathrm{p},\mathrm{t}) = k_1 p^{-a}(t+k_2)^b + \varepsilon, \tag{31}$$

where $k_1 > 0$, $k_2 \geq 0$, $a > 1$, and $0 < b < 1$ and ε is the stochastic variable.

The base example has the following parameters. The unit producing cost of the product is $c = 5$, and the unit repair cost is $s = 1$. For the demand function, $k_1 = 100$, $k_2 = 2$, $a = 2$, and $b = 0.5$ and ε follows the normal distribution $\varepsilon \sim N(10, 9)$.

6.1. Effects of Failure Rate. We first analyze the manufacturer's and retailer's decisions with different product failure rates λ. A high failure rate reflects low reliability of product and vice versa. The efficiency of supply chain is defined as

$$\rho = \frac{\pi_r + \pi_m}{\pi^C}. \tag{32}$$

The product failure rate in Tables 3–5 varies from 0.1 to 0.9 with step 0.2. Worse product quality (higher failure rate) costs more warranty costs. In Tables 3–5, the wholesale price w is increasing in the failure rate λ. This reflects the manufacturer tends to set a higher price when the product is not reliable. The warranty length t is decreasing in the failure rate. The higher failure rate is associated with shorter warranty service. The retail price and order quantity are also decreasing in the failure rate. That reflects that the retailer tends to make his decisions cautiously when the failure rate is high. The profits of supply chain members and the total supply chain are decreasing in the failure rate. The profit of centralized system π^C is always decreasing in the product failure rate, because worse product quality causes more repairing costs.

6.2. Warranty Cost-Sharing Ratio. The warranty cost-sharing ratio represents the ratio of warranty cost that the retailer should bear. Higher θ represents more costs to the retailer and less to the manufacturer.

The warranty cost-sharing ratio in Tables 6–8 varies from 0 to 1.0 with step 0.2. The higher cost-sharing ratio reflects the retailer sharing more warranty cost and vice versa. In Tables 6, 7, and 8, the wholesale price w is decreasing in the warranty cost-sharing ratio θ. This reflects the manufacturer tends to set a lower price when the retailer shares more warranty cost. The warranty length t is decreasing in the failure rate. The retail price is also decreasing in the failure rate. The order quantity is increasing in the warranty cost-sharing ratio θ. The profits of supply chain members and the total supply chain are decreasing in the failure rate. The profit of centralized system π^C remains unchanged with different sharing ratios.

6.3. Effects of Demand Uncertainty. In practice, demand is affected not only by the warranty service level but also by other factors. These factors may be uncertain and unknown. We use the random variable ε to represent the influence of these unknown factors. We use the normal distribution variable $\varepsilon \sim N(\mu, \delta^2)$ to describe the influence of these factors. Consistent with the above studies, we assume the mean value of the distribution μ is 10. We assume that the standard deviation δ changes from 1 to 3. Higher standard deviation δ indicates a higher level of uncertainty.

Standard deviation δ in Table 9 varies from 1.0 to 3.0 with step 0.5. The higher standard deviation δ indicates a higher level of uncertainty and vice versa. In Table 9, the wholesale price w is decreasing in the standard deviation δ. That reflects the manufacturer tends to set a lower price when the demand uncertainty is high. The warranty length t is increasing in the standard deviation δ. That suggests the manufacturer tends to set a higher warranty service period when the customer demand is more uncertain. The retail price is decreasing in the standard deviation δ. That suggests the retailer tends to set lower retail price when customer demand is more uncertain. The order quantity is decreasing in the standard deviation. It suggests that the retailer tends to order fewer products from the manufacturer when the customer demand has more uncertainty. The profit of the retailer is increasing in the standard deviation. The profit of manufacturer is decreasing in the standard deviation. The profit of the centralized system is increasing in the standard deviation. These results show that higher demand uncertainty benefits retailers and the centralized system but not the manufacturers.

Table 3: Results with different product failure rates ($\theta = 0$).

Failure rate λ	Decision variables				Profits and efficiency			
	w	t	p	q	π_r	π_m	π^C	ρ
0.1	13.5	48.3	35.1	76.6	306.00	321.4	800.3	0.784
0.3	13.6	24.1	34.2	60.3	152.8	215.7	458.9	0.803
0.5	14.1	15.2	35.5	42.9	119.9	174.9	353.9	0.833
0.7	13.6	9.0	31.6	33.2	106.3	146.0	319.0	0.791
0.9	13.0	7.5	30.0	28.3	99.3	124.5	275.6	0.812

Table 4: Results with different product failure rates ($\theta = 0.5$).

Failure rate λ	Decision variables				Profits and efficiency			
	w	t	p	q	π_r	π_m	π^C	ρ
0.1	9.8	45.3	29.4	72.6	300.4	331.2	800.3	0.789
0.3	11.2	23.2	32.5	62.1	145.9	218.6	458.9	0.794
0.5	12.9	14.2	33.1	48.7	118.3	176.9	353.9	0.834
0.7	12.7	8.2	30.7	35.9	105.8	144.2	319.0	0.784
0.9	11.3	6.1	28.3	27.0	97.4	123.1	275.6	0.800

Table 5: Results with different product failure rates ($\theta = 1$).

Failure rate λ	Decision variables				Profits and efficiency			
	w	t	p	q	π_r	π_m	π^C	ρ
0.1	4.1	7.3	12.3	67.3	130.0	536.6	800.3	0.833
0.3	5.7	6.2	14.2	63.1	121.8	268.7	458.9	0.851
0.5	7.3	5.5	18.9	59.1	113.4	204.0	353.9	0.897
0.7	7.9	4.0	21.3	55.0	100.1	190.8	319.0	0.912
0.9	8.3	3.5	22.8	53.7	89.5	153.9	275.6	0.883

Table 6: Results with different warranty cost-sharing ratios ($\lambda = 0.1$).

Sharing ratio θ	Decision variables				Profits and efficiency			
	w	t	p	q	π_r	π_m	π^C	ρ
0.0	13.5	48.3	35.1	76.6	306.0	321.4	800.3	0.784
0.4	11.3	45.0	31.7	73.1	301.4	330.8	800.3	0.790
0.6	8.9	40.3	25.4	70.6	280.4	359.1	800.3	0.799
0.8	8.3	20.1	20.4	69.0	200.7	449.1	800.3	0.812
1.0	4.1	7.3	12.3	67.3	130.0	536.6	800.3	0.833

Table 7: Results with different warranty cost-sharing ratios ($\lambda = 0.5$).

Sharing ratio θ	Decision variables				Profits and efficiency			
	w	t	p	q	π_r	π_m	π^C	ρ
0.0	14.1	15.2	35.5	42.9	119.9	174.9	353.9	0.833
0.4	13.0	13.9	33.0	46.8	117.1	180.5	353.9	0.841
0.6	11.7	12.0	29.7	50.3	115.2	187.0	353.9	0.854
0.8	8.9	7.7	23.8	55.5	114.4	199.1	353.9	0.886
1.0	7.3	5.5	18.9	59.1	113.4	204.0	353.9	0.897

Table 8: Results with different warranty cost-sharing ratios ($\lambda = 0.9$).

Sharing ratio θ	Decision variables				Profits and efficiency			
	w	t	p	q	π_r	π_m	π^C	ρ
0.0	13.0	7.5	30.0	28.3	99.3	124.5	275.6	0.812
0.4	12.7	7.1	29.1	29.8	96.0	130.8	275.6	0.823
0.6	12.0	6.6	28.9	31.3	95.4	133.3	275.6	0.831
0.8	10.6	5.1	26.4	44.2	91.7	149.7	275.6	0.876
1.0	8.3	3.5	22.8	53.7	89.5	153.9	275.6	0.883

Table 9: Results with different standard deviations δ.

Standard deviation δ	Decision variables				Profits and efficiency			
	w	t	p	q	π_r	π_m	π^C	ρ
1.0	14.2	12.7	35.5	54.4	99.7	191.0	333.7	0.871
1.5	13.8	13.3	34.4	52.1	104.9	186.4	339.1	0.859
2.0	13.4	13.8	33.7	50.8	111.6	182.8	347.1	0.848
2.5	13.1	14.1	33.4	49.3	115.2	179.1	350.7	0.839
3.0	12.9	14.2	33.1	48.7	118.3	176.9	353.9	0.834

7. Conclusion

Manufacturer-offering warranty and retailer-offering warranty are two common types of warranty service. In this paper, we develop the two-echelon supply chain model in which the manufacturer is selling to a retailer. In the first model, the manufacturer offers warranty to the customers directly. In the second model, the retailer offers warranty instead. We analyze and compare these models as well as compare them with the centralized system. We prove that unique optimal solutions exist in these models. The warranty period is decreasing in the product quality. Since the customers have less product quality information than the manufacturer or the retailer, this conclusion suggests that the warranty partially reduces the information asymmetry between manufacturer/retailer and customers. We also find that, when retailer's profit margin is not too low, retailer-offering warranty will generate a longer warranty period. Otherwise, the manufacturer offers a longer warranty.

By a numerical analysis, we find that the retailer tends to make his decisions cautiously when the failure rate is high. The profits of supply chain members and the total supply chain are decreasing in the failure rate. The profit of centralized system is always decreasing in the product failure rate. The manufacturer tends to set a lower price when the retailer shares more warranty cost. The warranty length is decreasing in the failure rate. The retail price is also decreasing in the failure rate. The manufacturer tends to set a higher warranty service period when the customer demand is more uncertain. The retailer tends to set lower retail price when customer demand is more uncertain. Higher demand uncertainty benefits the retailers and centralized system but not the manufacturers.

In this paper, we consider the free replacement warranty service. An extension would be to consider much more complicated warranty services such as the extended warranty service, the pro rata warranty service, etc. It would also be very interesting to extend our model to other types of supply chain structure such as the dual-channel supply chain, multi-echelon supply chain, etc.

Data Availability

The data used in this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] M. Shafiee and S. Chukova, "Maintenance models in warranty: a literature review," *European Journal of Operational Research*, vol. 229, no. 3, pp. 561–572, 2013.

[2] Y. Lei, Q. Liu, and S. Shum, "Warranty pricing with consumer learning," *European Journal of Operational Research*, vol. 263, no. 2, pp. 596–610, 2017.

[3] S. Mo, J. Zeng, and W. Xu, "A new warranty policy based on a buyer's preventive maintenance investment," *Computers & Industrial Engineering*, vol. 111, pp. 433–444, 2017.

[4] A. Y. Alqahtani and S. M. Gupta, "Warranty as a marketing strategy for remanufactured products," *Journal of Cleaner Production*, vol. 161, pp. 1294–1307, 2017.

[5] C.-W. Yeh and C.-C. Fang, "Optimal decision for warranty with consideration of marketing and production capacity,"

International Journal of Production Research, vol. 53, no. 18, pp. 5456–5471, 2015.

[6] J. A. Guajardo and M. A. Cohen, "Service competition and product quality in the U.S. Automobile industry," *Management Science*, vol. 62, no. 7, pp. 1860–1877, 2016.

[7] B. Jiang and X. Zhang, "How does a retailer's service plan affect a manufacturer's warranty?" *Management Science*, vol. 57, no. 4, pp. 727–740, 2011.

[8] B. Radej, J. Drnovsek, and G. Beges, "An overview and evaluation of quality-improvement methods from the manufacturing and supply-chain perspective," *Advances in Production Engineering & Management*, vol. 12, no. 4, pp. 388–400, 2017.

[9] S. Ebrahimi, S.-M. Hosseini-Motlagh, and M. Nematollahi, "Proposing a delay in payment contract for coordinating a two-echelon periodic review supply chain with stochastic promotional effort dependent demand," *International Journal of Machine Learning and Cybernetics*, vol. 10, no. 5, pp. 1037–1050, 2019.

[10] D. N. P. Murthy and I. Djamaludin, "New product warranty: a literature review," *International Journal of Production Economics*, vol. 79, no. 3, pp. 231–260, 2002.

[11] D. Murthy, O. Solem, and T. Roren, "Product warranty logistics: issues and challenges," *European Journal of Operational Research*, vol. 156, no. 1, pp. 110–126, 2004.

[12] A. Rahman and G. Chattopadhyay, "Review of long-term warranty policies," *Asia-Pacific Journal of Operational Research*, vol. 23, no. 4, pp. 453–472, 2006.

[13] M. Luo and S. Wu, "A mean-variance optimisation approach to collectively pricing warranty policies," *International Journal of Production Economics*, vol. 196, pp. 101–112, 2018.

[14] Y.-S. Huang, C.-D. Huang, and J.-W. Ho, "A customized two-dimensional extended warranty with preventive maintenance," *European Journal of Operational Research*, vol. 257, no. 3, pp. 971–978, 2017.

[15] N. Zhang, M. Fouladirad, and A. Barros, "Evaluation of the warranty cost of a product with type III stochastic dependence between components," *Applied Mathematical Modelling*, vol. 59, pp. 39–53, 2018.

[16] C.-K. Chen, C.-C. Lo, and T.-C. Weng, "Optimal production run length and warranty period for an imperfect production system under selling price dependent on warranty period," *European Journal of Operational Research*, vol. 259, no. 2, pp. 401–412, 2017.

[17] M. N. Darghouth, D. Ait-kadi, and A. Chelbi, "Joint optimization of design, warranty and price for products sold with maintenance service contracts," *Reliability Engineering & System Safety*, vol. 165, pp. 197–208, 2017.

[18] M. Park and H. Pham, "Altered quasi-renewal concepts for modeling renewable warranty costs with imperfect repairs," *Mathematical and Computer Modelling*, vol. 52, no. 9-10, pp. 1435–1450, 2010.

[19] A. Mitra and S. I. Resnick, "Modeling total expenditure on warranty claims," *Stochastic Models*, vol. 27, no. 3, pp. 431–473, 2011.

[20] M. Park, K. M. Jung, and D. H. Park, "Optimization of periodic preventive maintenance policy following the expiration of two-dimensional warranty," *Reliability Engineering & System Safety*, vol. 170, pp. 1–9, 2018.

[21] S. He, Z. Zhang, G. Zhang, and Z. He, "Two-dimensional base warranty design based on a new demand function considering heterogeneous usage rate," *International Journal of Production Research*, vol. 55, no. 23, pp. 7058–7072, 2017.

[22] K. Li, S. Mallik, and D. Chhajed, "Design of extended warranties in supply chains under additive demand," *Production and Operations Management*, vol. 21, no. 4, pp. 730–746, 2012.

[23] P. Tong, X. Song, and L. Zixian, "A maintenance strategy for two-dimensional extended warranty based on dynamic usage rate," *International Journal of Production Research*, vol. 55, no. 19, pp. 5743–5759, 2017.

[24] C. Zhou, R. B. Chinnam, E. Dalkiran, and A. Korostelev, "Bayesian approach to hazard rate models for early detection of warranty and reliability problems using upstream supply chain information," *International Journal of Production Economics*, vol. 193, pp. 316–331, 2017.

[25] C.-H. Chen and W.-L. Chang, "Optimal design of expected lifetime and warranty period for product with quality loss and inspection error," *Expert Systems with Applications*, vol. 37, no. 4, pp. 3521–3526, 2010.

[26] W. M. Yeo and X.-M. Yuan, "Optimal warranty policies for systems with imperfect repair," *European Journal of Operational Research*, vol. 199, no. 1, pp. 187–197, 2009.

[27] Y.-H. Chien, "Optimal age for preventive replacement under a combined fully renewable free replacement with a pro-rata warranty," *International Journal of Production Economics*, vol. 124, no. 1, pp. 198–205, 2010.

[28] K. M. Jung, S. S. Han, and D. H. Park, "Optimization of cost and downtime for replacement model following the expiration of warranty," *Reliability Engineering & System Safety*, vol. 93, no. 7, pp. 995–1003, 2008.

[29] L. Xu, Y. Li, K. Govindan, and X. Yue, "Return policy and supply chain coordination with network-externality effect," *International Journal of Production Research*, vol. 56, no. 10, pp. 3714–3732, 2018.

[30] S.-M. Hosseini-Motlagh, M. Nouri-Harzvili, M. Johari, and B. R. Sarker, "Coordinating economic incentives, customer service and pricing decisions in a competitive closed-loop supply chain," *Journal of Cleaner Production*, vol. 255, Article ID 120241, 2020.

[31] B. Dan, S. Zhang, and M. Zhou, "Strategies for warranty service in a dual-channel supply chain with value-added service competition," *International Journal of Production Research*, vol. 56, no. 17, pp. 5677–5699, 2018.

[32] S. Balachander, "Warranty signalling and reputation," *Management Science*, vol. 47, no. 9, pp. 1282–1289, 2001.

[33] N. C. Petruzzi and M. Dada, "Pricing and the newsvendor problem: a review with extensions," *Operations Research*, vol. 47, no. 2, pp. 183–194, 1999.