Research Article

Robust Time-Consistent Portfolio Selection for an Investor under CEV Model with Inflation Influence

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A robust time-consistent optimal investment strategy selection problem under inflation influence is investigated in this article. The investor may invest his wealth in a financial market, with the aim of increasing wealth. The financial market includes one risk-free asset, one risky asset, and one inflation-indexed bond. The price process of the risky asset is governed by a constant elasticity of variance (CEV) model. The investor is ambiguity-averse; he doubts about the model setting under the original probability measure. To dispel this concern, he seeks a set of alternative probability measures, which are absolutely continuous to the original probability measure. The objective of the investor is to seek a time-consistent strategy so as to maximize his expected terminal wealth meanwhile minimizing his variance of the terminal wealth in the worst-case scenario. By using the stochastic optimal control technique, we derive closed-form solutions for the optimal time-consistent investment strategy, the probability scenario, and the value function. Finally, the influences of model parameters on the optimal investment strategy and utility loss function are examined through numerical experiments.

1. Introduction

Nowadays, portfolio selection is a very important research topic in mathematical finance. As we all know, Markowitz [1] pioneered this research topic. He measured the investment gain and risk by expectation and variance, respectively. Nowadays, scholars call this method as the mean-variance (MV) criterion. Li and Ng [2] and Zhou and Li [3], respectively, pioneered the multiperiod and the continuous-time MV problem, where the explicit solutions were obtained. Dai et al. [4] studied the continuous-time MV problem with transaction costs. Recently, Sun et al. [5, 6] studied the MV problem for an insurer.

The aforementioned work under the MV criterion is time inconsistency. That is to say, the optimal strategy made at time t may not be optimal at time s, $s > t$. However, time consistency of strategy is an important and a rational requirement in many practice scenarios. For example, a few years ago, the Chinese government decided to get rid of poverty by 2020. Naturally, they hope to get rid of poverty

at 2020. To the best of our knowledge, Strotz [7] first formally treated time inconsistency. He proposed a game theoretic approach and sought an equilibrium strategy. Thus, the equilibrium strategy is time-consistent. Nowadays, this method becomes a mainstream way to deal with time-inconsistency. Ekeland et al. [8] first proposed the rigorous definition of the equilibrium strategy in a continuous-time framework. Björk and Murgoci [9] studied the time-consistent investment strategy, where the model parameters are controlled by a common Markov process. Li et al. [10] considered the optimal time-consistent MV problem under the Heston model. Yang [11] studied the similar problem, where the aggregate claim and the price are shocked by a common Poisson process. Zeng et al. [12] studied the robust time-consistent strategy under the MV framework. Furthermore, Yang et al. [13] extended the time-consistent MV problem to a new interaction mechanism. For more other detailed and related studies, one can refer to Björk et al. [14], Czichowsky [15], and Kronborg and Steffensen [16].

However, most of the aforementioned papers ignore the inflation risk and assume that the risky asset follows a geometric Brownian motion, which is usually contrary to practice investment activities according to many empirical studies. Brennan and Xia [17] studied the asset distribution problem under inflation influence. Han and Huang [18] considered the defined-contribution (DC) pension problem under inflation influence. Kwak and Lim [19] studied the optimal consumption-investment problem under inflation influence. Li et al. [20] studied the time-consistent investment strategy selection problem for a DC pension under partial information and inflation influence.

On the contrary, the CEV model has become popular among academia and practitioners since proposed by Cox and Ross [21]. Gao [22] investigated the DC pension investment problem under the CEV model. Gu et al. [23], Lin and Li [24], and Liang et al. [25] investigated the optimal reinsurance and investment strategy selection problem under the CEV model. Lin and Qian [26] studied the similar problem, where the strategy is time-consistent. For more other detailed and related studies about the CEV model, readers can refer to Li et al. [27] and Zheng et al. [28].

Bearing in mind the aforementioned state of the art, in this article, we will investigate the time-consistent investment strategy under the CEV model with inflation influence. The financial market includes one risk-free asset, one risky asset, and one inflation-indexed bond. The price process of the risky asset is governed by the CEV model. We assume that the investor is ambiguity-averse and worries about uncertainty in model setting. To dispel this concern, he seeks a set of alternative probability measures, which are absolutely continuous to the original probability measure. Then, the new model setting is obtained under the new probability measure. The objective of the investor is to find a time-consistent strategy so as to maximize his expected terminal wealth meanwhile minimizing his variance of the terminal wealth in the worst-case scenario. The problem's solving steps are as follows: first, we provide an extended HJB equation and a verification theorem. Second, by using the stochastic optimal control technique, we derive analytically the optimal time-consistent investment strategy, the probability scenario, and the value function. Finally, the influences of model parameters on the optimal investment strategy and utility loss function are examined through numerical experiments.

Compared with some related current research studies, our main contributions are given as follows:

(i) We consider a general financial market, which includes one risk-free asset, one risky asset, and one inflation-indexed bond. The price process of the risky asset is governed by the CEV model.

(ii) We study the time-consistent investment problem under inflation influence.

(iii) Closed-form solutions for the optimal time-consistent robust investment strategy, optimal probability scenario, and value function are derived.

(iv) The influences of model parameters on the optimal time-consistent investment strategy and utility loss

function are systematically examined through numerical experiments.

The remainder of this article is arranged as follows. In Section 2, we describe the model setup. In Section 3, an optimal robust time-consistent investment strategy selection problem is formulated. In Section 4, the closed-form solutions for the time-consistent optimal investment strategy and optimal probability scenario are derived. In Section 5, the influences of model parameters on the optimal time-consistent investment strategy and utility loss function are examined through numerical experiments. The final section summarizes this article.

2. Model Setup

We construct a financial model and present some basic assumptions in this section. All stochastic processes and random variables, mentioned later, are defined on a filtered complete probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$. We assume that $\mathscr{F} := \{\mathscr{F}_t, t \geq 0\}$ is right-continuous and is complete with respect to P. $\mathscr{F}_t$ stands for the information acquired by the investor up to time t. We assume that there are no market frictions in trading.

2.1. Financial Market. A financial market with the inflation risk is given in this section. To the best of our knowledge, consumer price index (CPI) is often used to describe the inflation rate in economics and academic research. CPI can be seen as a price level process. In this article, the inflation price level $P_0(t)$ satisfies the following process:

$$dP_0(t) = P_0(t)[\mu_1 dt + \sigma_1 dW_1(t)], \tag{1}$$

where $\mu_1 > 0$ represents the instantaneous expected rate of the inflation, $\sigma_1 > 0$ stands for the volatility of the inflation, and $W_1(t)$ stands for a standard Brownian motion.

We assume that there are three assets available for the investor: one risk-free asset, one risky asset, and one inflation-indexed bond. The price process $P_1(t)$ of the risk-free asset is given by

$$dP_1(t) = r_1 P_1(t)dt. \tag{2}$$

Here, $r_1 > 0$ represents the interest rate of the risk-free asset. The price process $S(t)$ of the risky asset satisfies the following CEV model:

$$dS(t) = S(t)[\mu_2 dt + \sigma_2 S^\beta(t)dW_2(t)], \; S(0) = s_0. \tag{3}$$

Here, $\mu_2 \geq r_1$ stands for the appreciation rate, $\sigma_2 > 0$ represents the price volatility, $W_2(t)$ represents a standard Brownian motion, and β is the constant elasticity parameter. We assume $W_1(t)$ and $W_2(t)$ are mutually independent.

Remark 1. The parameter β can take all real numbers. Obviously, if $\beta = 0$, the CEV model will degenerate to a geometric Brownian motion. When the CEV model is proposed, β is assumed to be negative. Emanuel and Macbeth [29] proved that β can also be assumed to be

positive. The CEV model has no jump, i.e., we do not consider the jump risk in this article. We can also consider the price process with the jump. Then, the exponential Lévy model is a good choice. For the exponential Lévy model, readers can refer to Yu et al. [30] and Zhang et al. [31].

The third asset to invest in is the inflation-indexed bond. Similar to Kwak and Lim [19], the price $P(t)$ of the inflation-indexed bond is given by

$$\frac{dP(t)}{P(t)} = \left[r_2 dt + \frac{dP_0(t)}{P_0(t)} \right] = (r_2 + \mu_1)dt + \sigma_1 dW_1(t). \quad (4)$$

Here, $r_2 > 0$ represents the real interest rate, and $r_2 + \mu_1 > r_1$ stands for the appreciation rate.

2.2. Inflation-Adjusted Wealth Process. Let $u_1(t)$ and $u_2(t)$, respectively, denote the proportions of the wealth invested in the risky asset and the inflation-indexed bond at time t, and the remainder of the proportion is invested in the risk-free asset. At time t, the investor (here, we only consider an investor. To reflect the general situation, one can consider n investors. For example, Yang et al. [13], Espinosa and Touzi [32], and Yu et al. [33] studied the optimal control problem for n agents) can choose the proportions $u_1(t)$ and $u_2(t)$ as control strategies; we denote them as $u(t) = (u_1(t), u_2(t))$. For each strategy $u(t)$, the wealth process R_t^u can be described as

$$\frac{dR_t^u}{R_t^u} = [1 - u_1(t) - u_2(t)] \frac{dP_1(t)}{P_1(t)} + u_1(t) \frac{dS(t)}{S(t)} + u_2(t) \frac{dP(t)}{P(t)}$$

$$= [r_1 + u_1(t)(\mu_2 - r_1) + u_2(t)(r_2 + \mu_1 - r_1)]dt$$

$$+ u_1(t)\sigma_2 S^\beta(t)dW_2(t) + u_2(t)\sigma_1 dW_1(t). \quad (5)$$

Denote

$$X_t^u = \frac{R_t^u}{P_0(t)}. \quad (6)$$

By applying Itô's formula, X_t^u can be described as

$$dX_t^u = \frac{1}{P_0(t)} dR_t^u + R_t^u d\frac{1}{P_0(t)} + d\left\langle R_t^u, \frac{1}{P_0(t)} \right\rangle$$

$$= X_t^u \left\{ \left[r_1 + \sigma_1^2 - \mu_1 + u_1(t)(\mu_2 - r_1) \right. \right.$$

$$+ u_2(t)\left(r_2 + \mu_1 - r_1 - \sigma_1^2\right) \bigg] dt$$

$$+ u_1(t)\sigma_2 S^\beta(t)dW_2(t) + (u_2(t) - 1)\sigma_1(t)dW_1(t) \bigg\}. \quad (7)$$

Here, X_t^u is the real wealth with stripping out inflation.

To facilitate solving the optimization problem in Section 3, we define the following notations:

$$\begin{cases} \pi_1(t) = u_1(t)X_t^u, \pi_2(t) = [u_2(t) - 1]X_t^u, & \pi(t) = (\pi_1(t), \pi_2(t)), \\ \widetilde{r}_1 = r_1 + \sigma_1^2 - \mu_1, \widetilde{r}_2 = r_2 + \mu_1 - r_1 - \sigma_1^2, & \widetilde{r} = \widetilde{r}_1 + \widetilde{r}_2. \end{cases} \quad (8)$$

Then, the real wealth X_t^π can be described as

$$dX_t^\pi = [\widetilde{r}X_t^\pi + \pi_1(t)(\mu_2 - r_1) + \pi_2(t)\widetilde{r}_2]dt$$

$$+ \pi_1(t)\sigma_2 S^\beta(t)dW_2(t) + \pi_2(t)\sigma_1 dW_1(t). \quad (9)$$

3. Problem Formulation

MV investment strategy selection problem as a classic optimization problem can be given as follows:

$$\sup_\pi \left\{ E_{0,x_0,s_0}[X_T^\pi], -\text{Var}_{0,x_0,s_0}[X_T^\pi] \right\}. \quad (10)$$

Here, $E_{0,x_0,s_0}[\cdot] = E[X_0^\pi = x_0, S(0) = s_0]$, $\text{Var}_{0,x_0,s_0}[\cdot] = \text{Var}[X_0^\pi = x_0, S(0) = s_0]$, x_0 is the investor's initial wealth, and T is the termination time of the investment. Obviously, the objective of the investor is to find an investment strategy so as to maximize his expected terminal wealth meanwhile minimizing his variance of the terminal wealth. It is well known that this problem is equivalent to the following problem:

$$\sup_\pi \left\{ E_{0,x_0,s_0}[X_T^\pi] - \frac{\gamma}{2}\text{Var}_{0,x_0,s_0}[X_T^\pi] \right\}, \quad (11)$$

where $\gamma > 0$ represents the risk-aversion parameter. According to Björk and Murgoci [9] and Kronborg and

Steffensen [16], we know that the optimal strategy of (11) is a precommitment strategy, which is time-inconsistent.

As we explained in the introduction, one usually takes into account the time-consistent strategy. Similar to Yang [11], Lin and Qian [26], and other related papers, we consider the following problem:

$$\sup_\pi \left\{ E_{t,x,s}[X_T^\pi] - \frac{\gamma}{2}\text{Var}_{t,x,s}[X_T^\pi] \right\}, \quad (12)$$

where $E_{t,x,s}[\cdot] = E[X_t^\pi = x, S(t) = s]$ and $\text{Var}_{t,x,s}[\cdot] = \text{Var}[X_t^\pi = x, S(t) = s]$. The target is to develop the corresponding equilibrium investment strategy, which is time-consistent.

Problem (12) is the traditional MV problem, where the investor is ambiguity-neutral. That is, he fully believes in the model defined under the probability measure P. In the actual economic activities, the investor often is ambiguity-averse and wants to protect himself against worst-case scenarios. Similar to Zeng et al. [12], Maenhout [34, 35], Chen and Yang [36], and other papers, we incorporate ambiguity aversion into MV problem (12). Since the investor is ambiguity-averse, he may doubt the model defined under the probability measure P. Hence, we defined an alternative probability measure Q, which is absolutely continuous to P. All such Q are denoted by $\mathcal{Q}$, that is,

$$\mathbb{Q} \coloneqq \{Q \mid Q \sim P\}. \tag{13}$$

Now, we define an admissible strategy.

Definition 1. For any fixed $t \in [0, T]$, an investment strategy $\pi(t) = (\pi_1(t), \pi_2(t))$ is called an admissible strategy for the ambiguity-averse investor (AAI) if it satisfies the following:

  (i) $\pi_1(t)$ and $\pi_2(t)$ are progressively measurable with respect to $\mathcal{F}_t$

  (ii) $E_{t,x,s}^{Q^*}[\int_0^T \pi_1^2(t)S^{2\beta}dt] < \infty$ and $E_{t,x,s}^{Q^*}[\int_0^T \pi_2^2(t)dt] < \infty$, where Q^* is the probability measure under the worst-case scenario and $E_{t,x,s}^{Q^*}[\cdot] = E_{t,x,s}^{Q^*}[\cdot \mid X_t^\pi = x, S(t) = s]$

  (iii) Equation (9) with respect to $\pi(t)$ has a unique strong solution

We denote all such admissible investment strategies on the time interval $[0, T]$ as Π.

To transform model (9) from the probability measure P to the probability measure Q, we define a process $\{\theta(t) = (\theta_1(t), \theta_2(t)) \mid t \in [0, T]\}$, which satisfies the following:

  (i) $\theta_1(t)$ and $\theta_2(t)$ are progressively measurable with respect to $\mathcal{F}_t$

  (ii) $E[\exp\{(1/2)\int_0^T [\theta_1^2(t) + \theta_2^2(t)]dt\}] < \infty$

All such $\theta(t)$ on the time interval $[0, T]$ are denoted by Θ. For any $\theta \in \Theta$, we define a real-valued process $\{Z^\theta(t) \mid t \in [0, T]\}$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ as

$$Z^\theta(t) \coloneqq \exp\left\{\int_0^t \theta_1(s)dW_1(s) + \int_0^t \theta_2(s)dW_2(s) \right. \\ \left. - \frac{1}{2}\int_0^t \theta_1^2(s)ds - \frac{1}{2}\int_0^t \theta_2^2(s)ds\right\}. \tag{14}$$

Then, $Z^\theta(t)$ is a martingale with respect to P. In the following, we define a new probability measure Q, which is given by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} \coloneqq Z^\theta(T). \tag{15}$$

From the definition of Q, it is clear that Q is absolutely continuous to P.

For any $\theta \in \Theta$, we, respectively, define two new processes $W_1^Q(t)$ and $W_2^Q(t)$ by

$$dW_1^Q(t) = dW_1(t) - \theta_1(t)dt, \\ dW_2^Q(t) = dW_2(t) - \theta_2(t)dt. \tag{16}$$

According to Girsanov's theorem, $W_1^Q(t)$ and $W_2^Q(t)$ are standard Brownian motions with respect to Q. Furthermore, the wealth process (9) under Q can be rewritten as

$$dX_t^\pi = [\tilde{r}X_t^\pi + \pi_1(t)(\mu_2 - r_1) + \pi_2(t)\tilde{r}_2 \\ + \pi_1(t)\theta_2(t)\sigma_2 S^\beta + \pi_2(t)\theta_1(t)\sigma_1]dt \\ + \pi_1(t)\sigma_2 S^\beta dW_2^Q(t) + \pi_2(t)\sigma_1 dW_1^Q(t), \tag{17}$$

and corresponding CEV model (3) becomes

$$dS(t) = S(t)\left[(\mu_2 + \theta_2(t)\sigma_2 S^\beta(t))dt + \sigma_2 S^\beta(t)dW_2^Q(t)\right]. \tag{18}$$

In the following, we modify MV problem (12) under the worst-case scenario. Through modifying MV problem (12), we shall deal with a robust time-consistent MV strategy selection problem as follows:

$$\sup_{\pi \in \Pi} V^\pi(t, x, s) = \sup_{\pi \in \Pi}\left\{\inf_{Q \in \mathbb{Q}} \widehat{V}^{\pi,Q}(t, x, s)\right\}. \tag{19}$$

Here,

$$\widehat{V}^{\pi,Q}(t, x, s) = E_{t,x,s}^Q\left\{\int_t^T \left[\frac{(\theta_1(v))^2}{2\phi_1(v)} + \frac{(\theta_2(v))^2}{2\phi_2(v)}\right]dv\right\} \\ + E_{t,x,s}^Q[X_T^\pi] - \frac{\gamma}{2}\mathrm{Var}_{t,x,s}^Q[X_T^\pi], \tag{20}$$

$$V^\pi(t, x, s) = \inf_{Q \in \mathbb{Q}} \widehat{V}^{\pi,Q}(t, x, s), \tag{21}$$

where $\phi_1(t)$ and $\phi_2(t)$ are nonnegative and stand for the investor's ambiguity aversion with respect to the model under P. The larger $\phi_1(t)$ and $\phi_2(t)$ are, the more the model under Q will deviate from the model under P. Therefore, AAI's ambiguity aversion is an increasing function of $\phi_1(t)$ and $\phi_2(t)$. To embody some good properties of the model under P, AAI deviation from P is penalized by the first two terms in (20). The penalty terms depend on the relative entropy. The increase in relative entropy from t to $t + dt$ equals

$$\frac{1}{2}\left[(\theta_1(t))^2 + (\theta_2(t))^2\right]dt. \tag{22}$$

The proof of (22) is similar to that in Appendix A in Zeng et al. [12]; we omit it here.

To obtain the time-consistent investment strategy, we present the definition of the equilibrium strategy.

Definition 2. For $\forall(t, x, s) \in [0, T] \times R \times R^+$, we choose an admissible strategy $\pi^*(t, x, s) \in \Pi$. Then, through choosing three real numbers $\tilde{a} \in R^+$, $\tilde{b} \in R$, and $\varsigma > 0$, we define a new strategy by

$$\pi^\varsigma(l, \tilde{x}, \tilde{s}) = \begin{cases} (\tilde{a}, \tilde{b}) & \text{for } (l, \tilde{x}, s) \in [t, t + \varsigma) \times R \times R^+, \\ \pi^*(l, \tilde{x}, s), & \text{for } (l, \tilde{x}, s) \in [t + \varsigma, T] \times R \times R^+. \end{cases} \tag{23}$$

If for any $(\tilde{a}, \tilde{b}) \in R^+ \times R$ and $(t, x, s) \in [0, T] \times R \times R^+$, we have

$$\liminf_{\varsigma \longrightarrow 0} \frac{V^{\pi^*}(t, x, s) - V^{\pi^\varsigma}(t, x, s)}{\varsigma} \geq 0, \tag{24}$$

which holds; then, $\pi^*(t, x, s)$ is called an equilibrium strategy, and the equilibrium value function $V(t, x, s)$ is given by

$$V(t, x, s) = V^{\pi^*}(t, x, s). \tag{25}$$

As evidenced by many references such as Björk and Murgoci [9] and Yang et al. [13], the equilibrium strategy defined by Definition 2 is time-consistent. Therefore, in the following, we call the equilibrium strategy and the equilibrium value function as the optimal time-consistent strategy and the optimal value function. Thus, the AAI's aim is to find an optimal time-consistent investment strategy to solve robust optimization problem (19).

For convenience, we define two notations:

$$\mathscr{A}^{\pi,\theta}(\varphi(t,x,s)) = \varphi_t(t,x,s) + \left[\tilde{r}x + \pi_1(t)(\mu_2 - r_1) + \pi_2(t)\tilde{r}_2 + \pi_2(t)\theta_1(t)\sigma_1 + \pi_1(t)\theta_2(t)\sigma_2 s^{\beta}\right]\varphi_x(t,x,s)$$

$$+ \frac{1}{2}\left[\pi_1^2(t)\sigma_2^2 s^{2\beta} + \pi_2^2(t)\sigma_1^2\right]\varphi_{xx}(t,x,s) + \left[\mu_2 s + \theta_2(t)\sigma_2 s^{\beta+1}\right]\varphi_s(t,x,s) + \frac{1}{2}\sigma_2^2 s^{2\beta+2}\varphi_{ss} + \pi_1(t)\sigma_2^2(t)s^{2\beta+1}\varphi_{sx}(t,x,s),$$

(27)

where $\varphi(t,x,s) \in C^{1,2,2}([0,T] \times R \times R^+)$, $\pi \in \Pi$, and $\theta \in \Theta$.

To ensure the strategy obtained from robust MV problem (19) is optimal, we present the following verification theorem. The proof of this theorem is similar to that in Theorem 7.1 of Björk and Murgoci [9], Theorem 4.1 of Yang et al. [13], and other papers, so we omit it here.

$$C^{1,2,2}([0,T] \times R \times R^+) = \{\varphi(t,x,s) \mid \text{for } \forall [0,T] \times R \times R^+,$$
$$\varphi_x(t,x,s), \varphi_{xx}(t,x,s), \varphi_s(t,x,s),$$
$$\varphi_{ss}(t,x,s) \text{ are continuous}\},$$

(26)

and the usual infinitesimal generator $\mathscr{A}^{\pi,\theta}$ for the wealth process (17) is given by

Theorem 1. *For robust MV problem (19), if there exist two functions $W(t,x,s)$ and $g(t,x,s)$ satisfying the following extended HJB equation*

$$\sup_{\pi\in\Pi} \inf_{Q\in\mathcal{Q}} \left\{\mathscr{A}^{\pi,\theta}(W(t,x,s)) - \mathscr{A}^{\pi,\theta}\left(\frac{\gamma}{2}g^2(t,x,s)\right) + \gamma g(t,x,s)\mathscr{A}^{\pi,\theta}(g(t,x,s)) + \frac{(\theta_1(t))^2}{2\phi_1(t)} + \frac{(\theta_2(t))^2}{2\phi_2(t)}\right\} = 0, \ W(T,x,s) = x,$$

(28)

$$\mathscr{A}^{\pi^*,\theta^*}(g(t,x,s)) = 0, \ g(T,x,s) = x,$$

(29)

$$(\pi^*, \theta^*) = \arg\sup_{\pi\in\Pi} \inf_{Q\in\mathcal{Q}} \left\{\mathscr{A}^{\pi,\theta}(W(t,x,s)) - \mathscr{A}^{\pi,\theta}\left(\frac{\gamma}{2}g^2(t,x,s)\right) + \gamma g(t,x,s)\mathscr{A}^{\pi,\theta}(g(t,x,s)) + \frac{(\theta_1(t))^2}{2\phi_1(t)} + \frac{(\theta_2(t))^2}{2\phi_2(t)}\right\},$$

(30)

then $V(t,x,s) = W(t,x,s)$, $E_{t,x,s}^Q[X_T^{\pi^*}] = g(t,x,s)$, π^* is the optimal robust time-consistent investment strategy for AAI, and θ^* is the optimal probability scenario for the market.

4. The Solution to the Robust MV Problem

This section is devoted to solve robust MV problem (19). By solving HJB equations (28) and (29), we can derive the solution to robust MV problem (19). To solve robust MV problem (19), similar to Maenhout [34, 35] and Chen and Yang [36], this article assumes that $\phi_1(t) = \alpha_1$ and $\phi_2(t) = \alpha_2$, where α_1 and α_1 are nonnegative.

Now, we give an explicit expression for HJB equation (28).

According to (27), we obtain

$$\mathscr{A}^{\pi,\theta}\left(\frac{\gamma}{2}g^2(t,x,s)\right) = \gamma g g_t + \left[\tilde{r}x + \pi_1(t)(\mu_2 - r_1) + \pi_2(t)\tilde{r}_2 + \pi_2(t)\theta_1(t)\sigma_1 + \pi_1(t)\theta_2(t)\sigma_2 s^{\beta}\right]\gamma g g_x$$

$$+ \frac{1}{2}\left[\pi_1^2(t)\sigma_2^2 s^{2\beta} + \pi_2^2(t)\sigma_1^2\right]\left(\gamma g_x^2 + \gamma g g_{xx}\right) + \mu_2 s\gamma g g_s + \frac{1}{2}\sigma_2^2 s^{2\beta+2}\left[\gamma g_s^2 + \gamma g g_{ss}\right]$$

$$+ \pi_1(t)\sigma_2^2 s^{2\beta+1}\left[\gamma g_s g_x + \gamma g g_{sx}\right],$$

(31)

$$\gamma g(t,x,s)\mathscr{A}^{\pi,\theta}(g(t,x,s)) = \gamma g g_t + \left[\tilde{r}x + \pi_1(t)(\mu_2 - r_1) + \pi_2(t)\tilde{r}_2 + \pi_2(t)\theta_1(t)\sigma_1 + \pi_1(t)\theta_2(t)\sigma_2 s^{\beta}\right]\gamma g g_x$$

$$+ \frac{1}{2}\left[\pi_1^2(t)\sigma_2^2 s^{2\beta} + \pi_2^2(t)\sigma_1^2\right]\gamma g g_{xx} + \mu_2 s\gamma g g_s + \frac{1}{2}\sigma_2^2 s^{2\beta+2}\gamma g g_{ss} + \pi_1(t)\sigma_2^2(t)s^{2\beta+1}\gamma g g_{sx},$$

where $g(t, x, s)$ is abbreviated as g and g_t, g_x, g_{xx}, g_s, g_{ss}, and g_{sx} are the partial derivatives of g with respect to the corresponding variables.

Then, HJB equation (28) can be more explicitly expressed as

$$\sup_{\pi \in \Pi} \inf_{Q \in \mathcal{Q}} \left\{ W_t + \left[\tilde{r}x + \pi_1(t)(\mu_2 - r_1) + \pi_2(t)\tilde{r}_2 + \pi_2(t)\theta_1(t)\sigma + \pi_1(t)\theta_2(t)\sigma_2 s^\beta \right] W_x \right.$$
$$+ \frac{1}{2} \left[\pi_1^2(t)\sigma_2^2 s^{2\beta} + \pi_2^2(t)\sigma_1^2 \right] \left(W_{xx} - \gamma g_x^2 \right) + \left[\mu_2 s + \theta_2(t)\sigma_2 s^{\beta+1} \right] W_s \tag{32}$$
$$\left. + \frac{1}{2}\sigma_2^2 s^{2\beta+2} \left(W_{ss} - \gamma g_s^2 \right) + \pi_1(t)\sigma_2^2 s^{2\beta+1} \left(W_{sx} - \gamma g_s g_x \right) + \frac{\theta_1^2(t)}{2\alpha_1} + \frac{\theta_2^2(t)}{2\alpha_2} \right\} = 0,$$

where $W(t, x, s)$ is abbreviated as W and W_t, W_x, W_{xx}, W_s, W_{ss}, and W_{sx} are the partial derivatives of W with respect to the corresponding variables.

Theorem 2. *For robust MV problem (19), the optimal robust time-consistent investment strategies are given by*

$$\pi_1^*(t) = \frac{(\mu_2 - r_1) + 2\gamma\beta\sigma_2^2\overline{B}(t) + 2\alpha_2\beta\sigma_2^2 B(t)}{(\alpha_2 + \gamma)\sigma_2^2 s^{2\beta}} e^{-\tilde{r}(T-t)}, \tag{33}$$

$$\pi_2^*(t) = \frac{\tilde{r}_2}{(\alpha_1 + \gamma)\sigma_1^2} e^{-\tilde{r}(T-t)}. \tag{34}$$

The optimal probability scenarios for the market are given by

$$\theta_1^*(t) = -\frac{\alpha_1 \tilde{r}_2}{(\alpha_1 + \gamma)\sigma_1}, \tag{35}$$

$$\theta_2^*(t) = \frac{2\beta\alpha_2\sigma_2 B(t)}{s^\beta} - \frac{\alpha_2 \left[(\mu_2 - r_1) + 2\gamma\beta\sigma_2^2\overline{B}(t) + 2\alpha_2\beta\sigma_2^2 B(t) \right]}{(\alpha_2 + \gamma)\sigma_2 s^\beta}. \tag{36}$$

The optimal value function is given by

$$W(t, x, s) = xe^{\tilde{r}(T-t)} + B(t)s^{-2\beta} + C(t), \tag{37}$$

where $C(t)$ is given by (60), and $\overline{B}(t)$ and $B(t)$ are determined by the following ordinary differential equations (ODEs):

$$\begin{cases} B'(t) - 2\mu_2\beta B(t) - 2\sigma_2^2\gamma\beta^2\overline{B}^2 - 2\alpha_2\beta^2\sigma_2^2 B^2(t) + \dfrac{\left[\mu_2 - r_1 + 2\beta\sigma_2^2(\gamma\overline{B}(t) + \alpha_2 B(t)) \right]^2}{2(\alpha_2 + \gamma)\sigma_2^2} = 0, \\[4mm] \overline{B}'(t) - 2\overline{B}(t)\left[\mu_2\beta + 4\alpha_2\beta^2\sigma_2^2 B(t) \right] + \dfrac{1}{(\alpha_2 + \gamma)\sigma_2^2} \left\{ \left[\mu_2 - r_1 + 2\sigma_2^2\alpha_2\beta(\overline{B}(t) + B(t)) \right] \times \left[\mu_2 - r_1 + 2\beta\sigma_2^2(\gamma\overline{B}(t) + \alpha_2 B(t)) \right] \right\} \\[4mm] \quad - \dfrac{\alpha_2 \left[\mu_2 - r_1 + 2\beta\sigma_2^2(\gamma\overline{B}(t) + \alpha_2 B(t)) \right]^2}{(\alpha_2 + \gamma)^2\sigma_2^2} = 0, \end{cases} \tag{38}$$

with boundary conditions $B(T) = 0$ and $\overline{B}(T) = 0$.

Proof. According to the structure of the wealth dynamics (17) and the boundary conditions $W(T, x, s) = g(T, x, s) = x$, we will seek the solution to HJB equations (28) and (29) with the following parametric form:

$$W(t, x, s) = A(t)x + B(t)s^{-2\beta} + C(t), \tag{39}$$

with boundary conditions

$$A(T) = 1, B(T) = 0, C(T) = 0, \tag{40}$$

$$g(t, x, s) = \overline{A}(t)x + \overline{B}(t)s^{-2\beta} + \overline{C}(t),$$

with boundary conditions

$$\overline{A}(T) = 1, \overline{B}(T) = 0, \overline{C}(T) = 0. \tag{41}$$

Then, the partial derivatives of $W(t, x, s)$ and $g(t, x, s)$ are given by

$$\begin{cases} W_t = A'(t)x + B'(t)s^{-2\beta} + C'(t), W_x = A(t), W_{xx} = 0, \\[2mm] W_s = -2\beta B(t)s^{-2\beta-1}, W_{ss} = 2\beta(2\beta + 1)B(t)s^{-2\beta-2}, W_{sx} = 0, \\[2mm] g_t = \overline{A}'(t)x + \overline{B}'(t)s^{-2\beta} + \overline{C}'(t), g_x = \overline{A}(t), g_{xx} = 0, \\[2mm] g_s = -2\beta\overline{B}(t)s^{-2\beta-1}, g_{ss} = 2\beta(2\beta + 1)\overline{B}(t)s^{-2\beta-2}, g_{sx} = 0. \end{cases} \tag{42}$$

Substituting equations (39)–(42) into equation (32), we obtain

$$\left[A'(t) + \tilde{r}A(t)\right]x + \left[B'(t) - 2\mu_2\beta B(t) - 2\sigma_2^2\gamma\beta^2\overline{B}^2\right]s^{-2\beta} + C'(t)$$

$$+ \sigma_2^2\beta(2\beta+1)B(t) + \sup_{\pi\in\Pi}\inf_{Q\in\mathcal{Q}}\left\{\frac{\theta_2^2(t)}{2\alpha_2} + \theta_2(t)\left[\pi_1(t)\sigma_2 s^\beta A(t) - 2\sigma_2\beta B(t)s^{-\beta}\right]\right.$$

(43)

$$+ \frac{\theta_1^2(t)}{2\alpha_1} + \pi_2\theta_1(t)\sigma_1 A(t) - \frac{1}{2}\pi_1^2(t)\gamma\sigma_2^2 s^{2\beta}\overline{A}^2(t) + \pi_1(t)\left[(\mu_2 - r_1)A(t) + 2\gamma\sigma_2^2\beta\overline{A}(t)\overline{B}(t)\right]$$

$$\left.- \frac{1}{2}\pi_2^2(t)\gamma\sigma_1^2\overline{A}^2(t) + \pi_2(t)\tilde{r}_2 A(t)\right\} = 0.$$

According to the first-order optimality condition, $\theta_1(t)$ and $\theta_2(t)$, solving the minimization problem in equation (43), we obtain

$$\theta_1^*(t) = -\sigma_1\alpha_1\pi_2(t)A(t), \tag{44}$$

$$\theta_2^*(t) = -\sigma_2\alpha_2\pi_1(t)s^\beta A(t) + 2\alpha_2\beta\sigma_2 s^{-\beta}B(t). \tag{45}$$

Plugging (44) and (45) into equation (43) yields

$$\left[A'(t) + \tilde{r}A(t)\right]x + \left[B'(t) - 2\mu_2\beta B(t) - 2\sigma_2^2\gamma\beta^2\overline{B}^2 - 2\alpha_2\beta^2\sigma_2^2 B^2(t)\right]s^{-2\beta} + C'(t) + \sigma_2^2\beta(2\beta+1)B(t) + \sup_{\pi\in\Pi}$$

$$\left\{-\frac{1}{2}\pi_1^2\left[\alpha_2 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_2^2 s^{2\beta} + \pi_1(t)\left[(\mu_2 - r_1)A(t) + 2\gamma\sigma_2^2\beta\overline{A}(t)\overline{B}(t) + 2\alpha_2\sigma_2^2\beta A(t)B(t)\right]\right.$$

(46)

$$\left.- \frac{1}{2}\pi_2^2\left[\alpha_1 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_1^2 + \pi_2(t)\tilde{r}_2 A(t)\right\} = 0.$$

According to the first-order optimality condition, $\pi_1^*(t)$ and $\pi_2^*(t)$, solving the maximization problem in equation (46), we obtain

$$\pi_1^*(t) = \frac{(\mu_2 - r_1)A(t) + 2\gamma\beta\sigma_2^2\overline{A}(t)\overline{B}(t) + 2\alpha_2\sigma_2^2\beta A(t)B(t)}{\left[\alpha_2 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_2^2(t)s^{2\beta}},$$

(47)

$$\pi_2^*(t) = \frac{\tilde{r}_2 A(t)}{\left[\alpha_1 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_1^2(t)}. \tag{48}$$

Inserting (47) and (48) into equation (46) and after simplifying give

$$\left[A'(t) + \tilde{r}A(t)\right]x + \left\{B'(t) - 2\mu_2\beta B(t) - 2\sigma_2^2\gamma\beta^2\overline{B}^2 - 2\alpha_2\beta^2\sigma_2^2 B^2(t) + \frac{\left[(\mu_2 - r_1)A(t) + 2\gamma\beta\sigma_2^2\overline{A}(t)\overline{B}(t) + 2\alpha_2\sigma_2^2\beta A(t)B(t)\right]^2}{2\left[\alpha_2 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_2^2}\right\}s^{-2\beta}$$

$$+ \left\{C'(t) + \sigma_2^2\beta(2\beta+1)B(t) + \frac{\tilde{r}_2^2 A^2(t)}{2\left[\alpha_1 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_1^2}\right\} = 0.$$

(49)

By substituting (40), (42), (44), (45), (47), and (48) into equation (29) yields

$$
\begin{aligned}
&\left[\overline{A}'(t) + \widetilde{r}\overline{A}(t)\right]x + \Big\{\overline{B}'(t) - 2\mu_2\beta\overline{B}(t) - 4\alpha_2\beta^2\sigma_2^2 B(t)\overline{B}(t) \\
&+ \frac{1}{\left[\alpha_2 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_2^2}\Big[\big[(\mu_2 - r_1)\overline{A}(t) + 2\sigma_2^2\alpha_2\beta(A(t)\overline{B}(t) + \overline{A}(t)B(t))\big] \\
&\times\big[(\mu_2 - r_1)A(t) + 2\gamma\beta\sigma_2^2\overline{A}(t)\overline{B}(t) + 2\alpha_2\sigma_2^2\beta A(t)B(t)\big]\Big] \\
&- \frac{\alpha_2 A(t)\overline{A}(t)\big[(\mu_2 - r_1)A(t) + 2\gamma\beta\sigma_2^2\overline{A}(t)\overline{B}(t) + 2\alpha_2\sigma_2^2\beta A(t)B(t)\big]^2}{\left[\alpha_2 A^2(t) + \gamma\overline{A}^2(t)\right]^2\sigma_2^2}\Big\}s^{-2\beta} \\
&+ \left\{\overline{C}'(t) + \sigma_2^2\beta(2\beta + 1)\overline{B}(t) + \frac{\widetilde{r}_2^2 A(t)\overline{A}(t)}{\left[\alpha_1 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_1^2} - \frac{\alpha_1\widetilde{r}_2^2 A^3(t)\overline{A}(t)}{\left[\alpha_1 A^2(t) + \gamma\overline{A}^2(t)\right]^2\sigma_1^2}\right\} = 0.
\end{aligned}
\tag{50}
$$

Thus, we must solve the following ODEs:

$$
A'(t) + \widetilde{r}A(t) = 0, \; A(T) = 1,
\tag{51}
$$

$$
\begin{aligned}
&B'(t) - 2\mu_2\beta B(t) - 2\sigma_2^2\gamma\beta^2\overline{B}^2 - 2\alpha_2\beta^2\sigma_2^2 B^2(t) \\
&+ \frac{\big[(\mu_2 - r_1)A(t) + 2\gamma\beta\sigma_2^2\overline{A}(t)\overline{B}(t) + 2\alpha_2\sigma_2^2\beta A(t)B(t)\big]^2}{2\left[\alpha_2 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_2^2} = 0, \; B(T) = 0,
\end{aligned}
\tag{52}
$$

$$
C'(t) + \sigma_2^2\beta(2\beta + 1)B(t) + \frac{\widetilde{r}_2^2 A^2(t)}{2\left[\alpha_1 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_1^2} = 0, \; C(T) = 0,
\tag{53}
$$

$$
\overline{A}'(t) + \widetilde{r}\overline{A}(t) = 0, \; \overline{A}(T) = 1,
\tag{54}
$$

$$
\begin{aligned}
&\overline{B}'(t) - 2\mu_2\beta\overline{B}(t) - 4\alpha_2\beta^2\sigma_2^2 B(t)\overline{B}(t) \\
&+ \frac{1}{\left[\alpha_2 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_2^2}\Big[\big[(\mu_2 - r_1)\overline{A}(t) + 2\sigma_2^2\alpha_2\beta(A(t)\overline{B}(t) + \overline{A}(t)B(t))\big] \\
&\times\big[(\mu_2 - r_1)A(t) + 2\gamma\beta\sigma_2^2\overline{A}(t)\overline{B}(t) + 2\alpha_2\sigma_2^2\beta A(t)B(t)\big]\Big] \\
&- \frac{\alpha_2 A(t)\overline{A}(t)\big[(\mu_2 - r_1)A(t) + 2\gamma\beta\sigma_2^2\overline{A}(t)\overline{B}(t) + 2\alpha_2\sigma_2^2\beta A(t)B(t)\big]^2}{\left[\alpha_2 A^2(t) + \gamma\overline{A}^2(t)\right]^2\sigma_2^2} = 0, \; \overline{B}(T) = 0,
\end{aligned}
\tag{55}
$$

$$
\overline{C}'(t) + \sigma_2^2\beta(2\beta + 1)\overline{B}(t) + \frac{\widetilde{r}_2^2 A(t)\overline{A}(t)}{\left[\alpha_1 A^2(t) + \gamma\overline{A}^2(t)\right]\sigma_1^2} - \frac{\alpha_1\widetilde{r}_2^2 A^3(t)\overline{A}(t)}{\left[\alpha_1 A^2(t) + \gamma\overline{A}^2(t)\right]^2\sigma_1^2} = 0, \; \overline{C}(T) = 0.
\tag{56}
$$

First, solving equations (51) and (54), respectively, we obtain

$$
A(t) = e^{\widetilde{r}(T-t)},
\tag{57}
$$

$$
\overline{A}(t) = e^{\widetilde{r}(T-t)}.
\tag{58}
$$

By substituting (57) and (58) into (52) and (55), we can obtain $B(t)$ and $\overline{B}(t)$ which are the solutions of ODEs (38).

Inserting (57), (58), and $\overline{B}(t)$ into equation (56), we obtain the solution to equation (56) which is given by

$$\overline{C}(t) = \frac{\gamma \widetilde{r}_2^2 (T - t)}{(\alpha_1 + \gamma)^2 \sigma_1^2} + \sigma_2^2 \beta (2\beta + 1) \int_t^T \overline{B}(s) \mathrm{d}s. \quad (59)$$

By substituting (57), (58), and $B(t)$ into equation (53), and solving it, we obtain

$$C(t) = \frac{\widetilde{r}_2^2}{2(\alpha_1 + \gamma)\sigma_1^2}(T - t) + \sigma_2^2 \beta (2\beta + 1) \int_t^T B(s) \mathrm{d}s. \quad (60)$$

Inserting (57), (58), $B(t)$, and $\overline{B}(t)$ into (47), we can obtain (33). By substituting (57) and (58) into (48), we can obtain (34). Plugging (33) into (45) and (34) into (44), we yield (36) and (35), respectively.

Based on Theorem 2 and the notation setting (8), we have the following theorem. $\square$

Theorem 3. *For the wealth process (7), the optimal robust time-consistent investment strategies to robust MV problem (19) are given by*

$$u_1^*(t) = \frac{(\mu_2 - r_1) + 2\gamma\beta\sigma_2^2 \overline{B}(t) + 2\alpha_2 \beta \sigma_2^2 B(t)}{x(\alpha_2 + \gamma)\sigma_2^2 s^{2\beta}} e^{-\widetilde{r}(T-t)}, \quad (61)$$

$$u_2^*(t) = \frac{\widetilde{r}_2}{x(\alpha_1 + \gamma)\sigma_1^2} e^{-\widetilde{r}(T-t)} + 1. \quad (62)$$

Remark 2. From the expressions of optimal robust time-consistent investment strategies (61) and (62), we note that $u_1^*(t)$ is related to the appreciation rate μ_2 and price volatility σ_2 of the risky asset and the current wealth x, whereas $u_2^*(t)$ is related to the parameters μ_1 and σ_1 of the inflation level and the current wealth x. This difference between $u_1^*(t)$ and $u_2^*(t)$ is mainly because the inflation price process and the risky asset price process are independent in our model assumption.

5. Numerical Experiments

This section is devoted to providing the influences of model parameters on the optimal time-consistent investment strategy and utility loss function by numerical experiments. In the following, unless otherwise stated, we choose the basic parameter setting as given in Table 1.

5.1. The Influences of the Parameters on $u_1^*(t)$. We first present the effects of model parameters on $u_1^*(t)$, where $u_1^*(t)$ is determined by (61).

Let $s \in [2, 10]$; Figure 1 shows the effect of s on $u_1^*(t)$. From Figure 1, it is evident that $u_1^*(t)$ is a decreasing function of s, which describes the price of the risky asset. Form the meaning of s, we know that larger s implies that the investment risk becomes greater. Therefore, as s increases, the AAI will wish to invest less in the risky asset.

Figure 2 presents two curves about the effect of β on $u_1^*(t)$, where parameter $\beta \in [0.1, 0.6]$. From Figure 2, we can see that when $s = 2$, $u_1^*(t)$ is a decreasing function in β, while

TABLE 1: Values of model parameters.

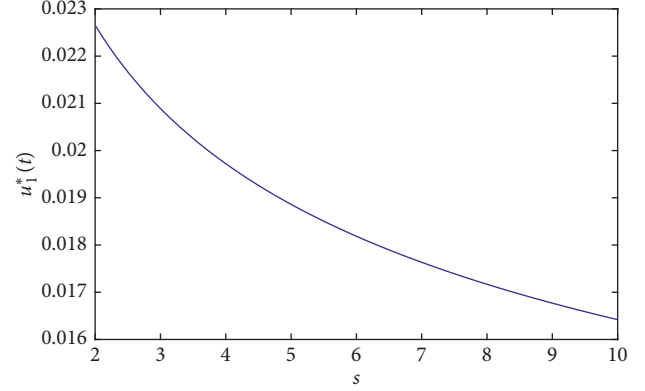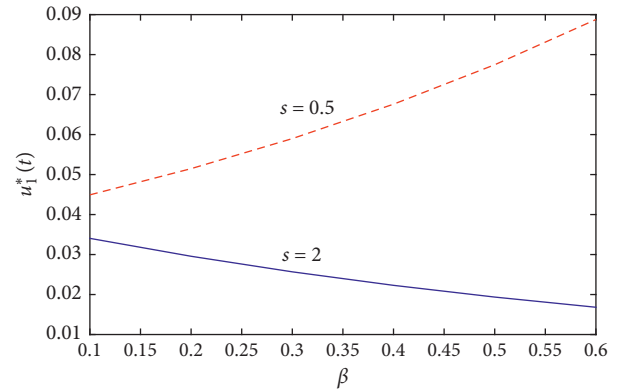| μ_1	σ_1	r_1	μ_2	σ_2	r_2	γ	x	T	t	β	s	α_1	α_2
0.05	0.3	0.03	0.08	0.25	0.02	2	10	10	9	0.1	2	2	1

FIGURE 1: Effect of s on $u_1^*(t)$.

FIGURE 2: Effect of β on $u_1^*(t)$.

if $s = 0.5$, $u_1^*(t)$ is an increasing function in β. When $s = 2$ ($s = 0.5$), the larger β is, the greater (less) s^β will be and hence the greater (less) the risk will be; thus, the AAI will wish to invest less (greater) in the risky asset.

Let $\mu_2 \in [0.05, 0.1]$; we present the effect of μ_2 on $u_1^*(t)$ in Figure 3. From Figure 3, it is clear that $u_1^*(t)$ is increasing with respect to μ_2, which is the appreciation rate of the risky asset. Larger μ_2 implies that the expected income of the risky asset becomes greater. As a result, the AAI will wish to invest more in the risky asset.

Let $\sigma_2 \in [0.2, 0.6]$; Figure 4 displays the effect of σ_2 on $u_1^*(t)$. From Figure 4, we can observe that $u_1^*(t)$ is a decreasing function of σ_2. Recall that σ_2 is the price volatility; therefore, with the increase of σ_2, the investment risk will increase. Thus, the AAI will wish to invest less in the risky asset.

Figure 5 illustrates the effect of the risk-free interest rate r_1 on $u_1^*(t)$, where $r_1 \in [0.02, 0.06]$. This figure shows that $u_1^*(t)$ is a decreasing function with respect to r_1. Larger r_1 implies that the expected income of the risk-free asset
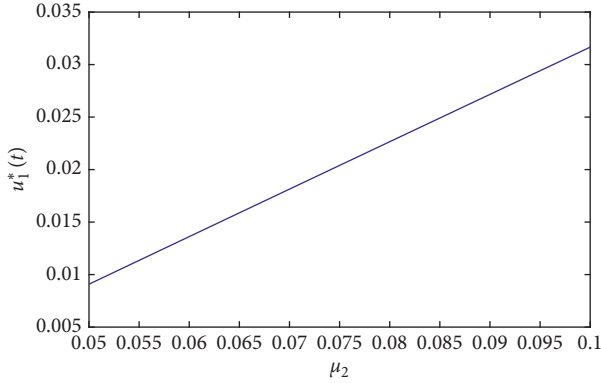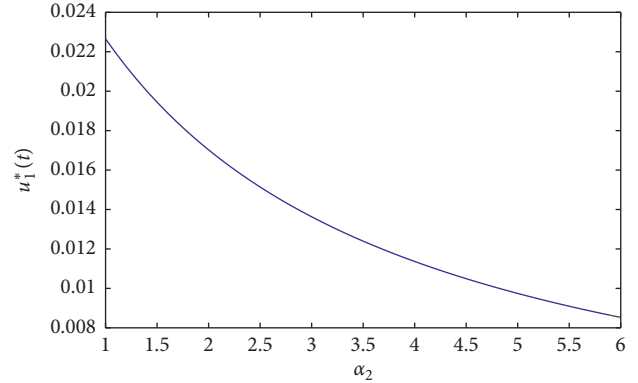
FIGURE 3: Effect of μ_2 on $u_1^*(t)$.
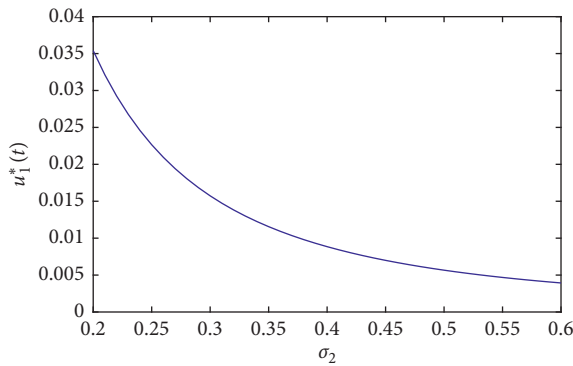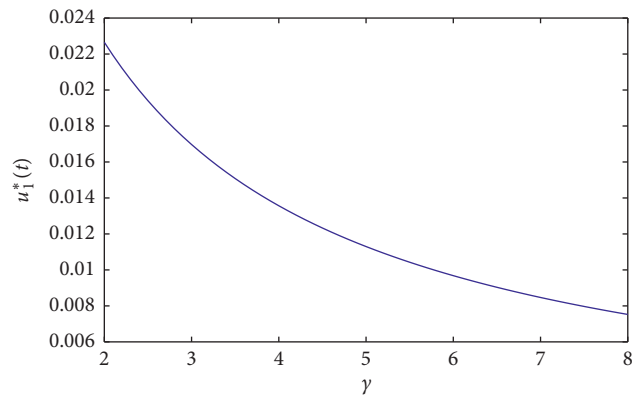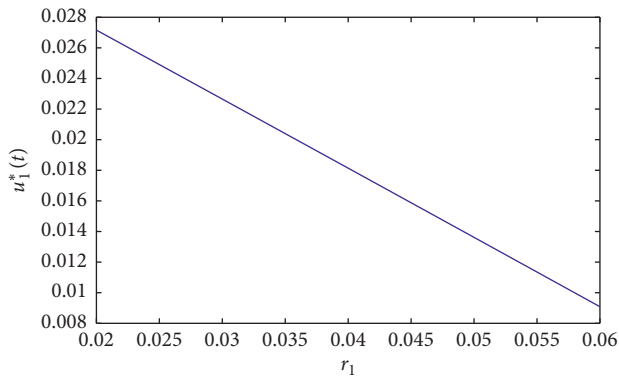

FIGURE 4: Effect of σ_2 on $u_1^*(t)$.

FIGURE 5: Effect of r_1 on $u_1^*(t)$.

becomes greater. A natural result is that the AAI will invest more in the risk-free asset. That is to say, the AAI will invest less in the risky asset.

Figure 6 discloses that $u_1^*(t)$ is a decreasing function of α_2, where $\alpha_2 \in [1, 6]$. Recall that α_2 stands for the AAI's ambiguity aversion. Larger α_2 implies that the model uncertainty becomes greater. Hence, the AAI will invest less in the risky asset.

Figure 7 shows the effect of γ on $u_1^*(t)$, where $\gamma \in [2, 8]$. From Figure 7, we see that $u_1^*(t)$ is a decreasing function with respect to γ. This is obvious. Since γ is the risk-aversion parameter, larger γ implies that the less aggressive the AAI will be. Therefore, the AAI will invest less in the risky asset.

FIGURE 6: Effect of α_2 on $u_1^*(t)$.

FIGURE 7: Effect of γ on $u_1^*(t)$.

Figure 8 reveals that $u_1^*(t)$ is increasing with respect to t, where $t \in [2, 9]$. This implies that, as the termination time of the investment approaches, the AAI will invest more in the risky assets.

5.2. The Influences of the Parameters on $u_2^*(t)$.
Now, we illustrate the effects of model parameters on the optimal robust time-consistent investment strategy $u_2^*(t)$, where $u_2^*(t)$ is determined by (62).

Differentiating (62) with respect to μ_1, we obtain

$$\frac{\partial u_2^*(t)}{\partial \mu_1} = \frac{1}{x(\alpha_1 + \gamma)\sigma_1^2} e^{-\widetilde{r}(T-t)} > 0, \tag{63}$$

that is, $u_2^*(t)$ is increasing with respect to μ_1. This implies that, as the interest rate of the inflation-indexed bond increases, the AAI should retain more investments in the inflation-indexed bond.

Differentiating (62) with respect to σ_1, we have

$$\frac{\partial u_2^*(t)}{\partial \sigma_1} = -\frac{2(r_2 + \mu_1 - r_1)}{x(\alpha_1 + \gamma)\sigma_1^3} e^{-\widetilde{r}(T-t)} < 0, \tag{64}$$

so we know that $u_2^*(t)$ is decreasing with respect to σ_1. Recall that σ_1 is the volatility of the inflation-indexed bond. Therefore, larger σ_1 implies that the investment risk in the inflation-indexed bond becomes larger. Hence, the AAI retains less investments in the inflation-indexed bond.
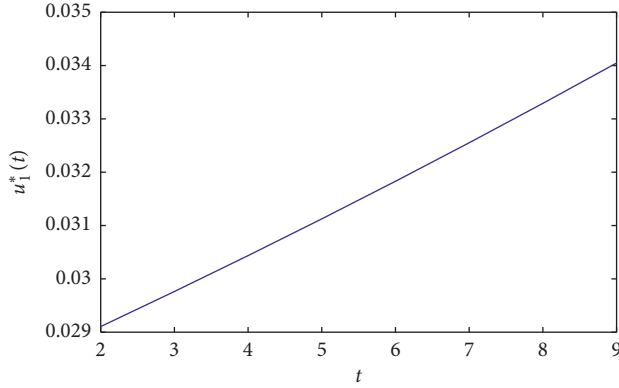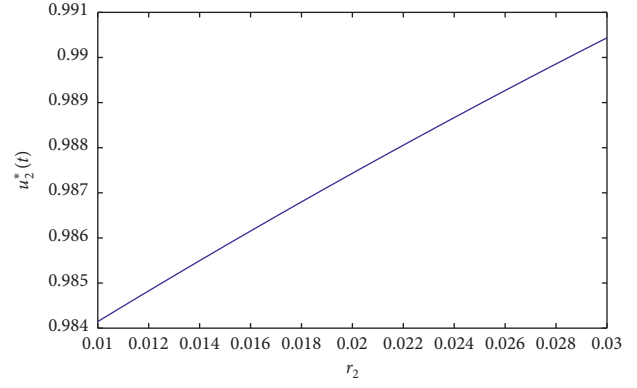
FIGURE 8: Effect of t on $u_1^*(t)$.

FIGURE 9: Effect of r_2 on $u_2^*(t)$.

Differentiating (62) with respect to r_1, we obtain

$$\frac{\partial u_2^*(t)}{\partial r_1} = -\frac{1}{x(\alpha_1 + \gamma)\sigma_1^2}e^{-\widetilde{r}(T-t)} < 0, \tag{65}$$

which implies that $u_2^*(t)$ is decreasing with respect to r_1. The reason is similar to what we have explained in Figure 5.

In the following, we analyze the effect of r_2 on $u_2^*(t)$ by the numerical experiment. We set $\mu_1 = 0.02, \sigma_1 = 0.04, \alpha_1 = 10, \quad \gamma = 2, x = 10, T = 10, t = 5, r_2 \in [0.01, 0.03]$, and the result is shown in Figure 9. From Figure 9, we can observe that $u_2^*(t)$ is an increasing function with respect to r_2. Recall that r_2 stands for the real interest rate of the inflation-indexed bond. Hence, the larger r_2 is, the more the AAI will invest in the inflation-indexed bond.

5.3. The Influences of Model Parameters on Utility Loss Function.
In Sections 5.1 and 5.2, we studied the case under the robust model. This part uses numerical experiments to investigate the influences of model parameters on utility loss function.

First, we give some results under the ambiguity-neutral case.

For an ambiguity-neutral optimization problem, all admissible investment strategies $\overline{\pi}(t)$ are denoted by $\overline{\Pi}$. Then, the wealth process becomes

$$dX_t^{\overline{\pi}} = \left[\widetilde{r}X_t^{\overline{\pi}} + \overline{\pi}_1(t)(\mu_2 - r_1) + \overline{\pi}_2(t)\widetilde{r}_2\right]dt$$
$$+ \overline{\pi}_1(t)\sigma_2 S^\beta(t)dW_2(t) + \overline{\pi}_2(t)\sigma_1 dW_1(t), \tag{66}$$

and the objective function is as follows:

$$\sup_{\overline{\pi} \in \overline{\Pi}}\left\{E_{t,x,s}\left[X_T^{\overline{\pi}}\right] - \frac{\gamma}{2}\text{Var}_{t,x,s}\left[X_T^{\overline{\pi}}\right]\right\}. \tag{67}$$

Similar to what we have proved in Theorem 2, we can obtain the following theorem.

Theorem 4. *For problem (67), the optimal time-consistent investment strategies are given by*

$$\overline{\pi}_1^*(t) = \frac{(\mu_2 - r_1) + 2\gamma\beta\sigma_2^2 B_2(t)}{\gamma\sigma_2^2 s^{2\beta}}e^{-\widetilde{r}(T-t)}, \tag{68}$$

$$\overline{\pi}_2^*(t) = \frac{\widetilde{r}_2}{\gamma\sigma_1^2}e^{-\widetilde{r}(T-t)}. \tag{69}$$

Furthermore, the optimal value function is given by

$$W_1(t,x,s) = xe^{\widetilde{r}(T-t)} + B_1(t)s^{-2\beta} + C_1(t), \tag{70}$$

where

$$B_1(t) = e^{-2\mu_2\beta(T-t)}\int_t^T f_1(s)e^{2\mu_2\beta(T-s)}ds,$$

$$B_2(t) = \frac{(\mu_2 - r_1)^2}{2r_1\beta\gamma\sigma_2^2}\left[1 - e^{-2r_1\beta(T-t)}\right],$$

$$f_1(t) = 2\beta(\mu_2 - r_1)B_2(t) + \frac{(\mu_2 - r_1)^2}{2\gamma\sigma_2^2}, \tag{71}$$

$$C_1(t) = \frac{\widetilde{r}_2^2}{2\gamma\sigma_1^2}(T-t) + \sigma_2^2\beta(2\beta + 1)\int_t^T B_1(s)ds.$$

Based on Theorems 2 and 4, we can define the utility loss function as

$$L(t) = 1 - \frac{W_1(t,x,s)}{W(t,x,s)}. \tag{72}$$

In the following, we investigate the effects of model parameters on $L(t)$.

(i) *The influence of ambiguity aversion on the utility loss:*

Let $\alpha_2 = 4$ and $\alpha_1 = 2, 3, 4, 5, 6, 7, 8$; we can obtain $W_1(t,x,s) = 10.2370$. The values of $W(t,x,s)$ and $L(t)$ are presented in Table 2. From Table 2, we know that the utility loss function is a decreasing function in α_1, where α_1 stands for the ambiguity aversion which arises from the inflation-indexed bond. By comparing the AAI's utility loss at different α_1, we find that the more ambiguity averse the AAI is, the less conservative investment strategy the AAI will be (because $W(t,x,s)$ is decreasing in α_1). Because AAI's

TABLE 2: Effect of α_1 on $W(t, x, s)$ and $L(t)$.

α_1	2	3	4	5	6	7	8
$W(t, x, s)$	11.7217	11.7212	11.7209	11.7207	11.7205	11.7204	11.7202
$L(t)$	0.12666	0.12663	0.1266	0.12658	0.12657	0.12656	0.12655

TABLE 3: Effect of α_2 on $W(t, x, s)$ and $L(t)$.

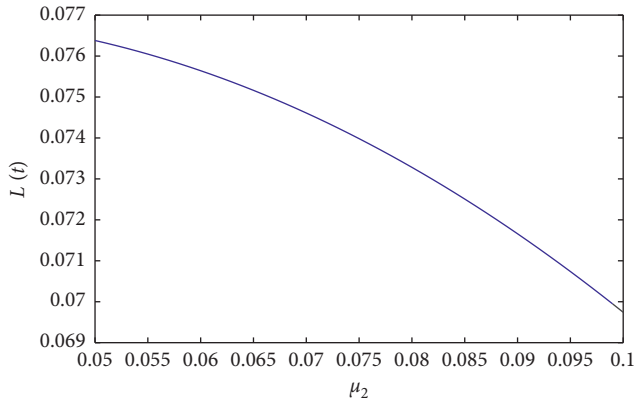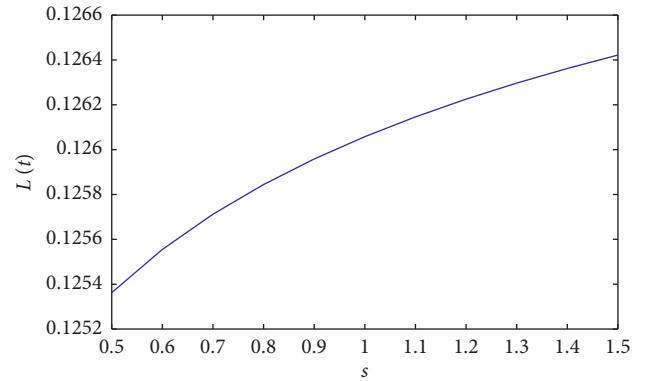| α_2	2	3	4	5	6	7	8
$W(t, x, s)$	11.0307	11.0386	11.0429	11.0455	11.0472	11.0484	11.0492
$L(t)$	0.0719	0.0726	0.0730	0.0732	0.0733	0.0734	0.0735

FIGURE 10: Effect of μ_2 on $L(t)$.
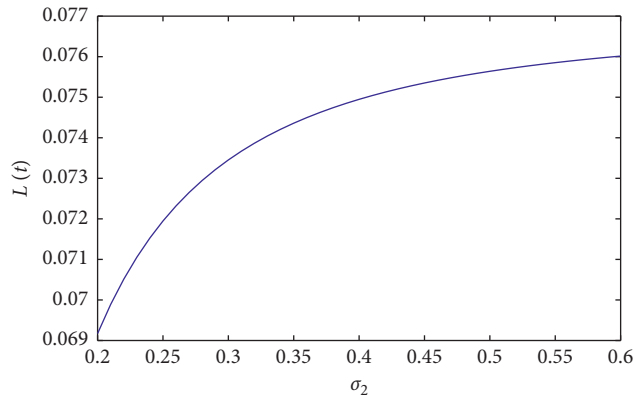

FIGURE 12: Effect of s on $L(t)$.
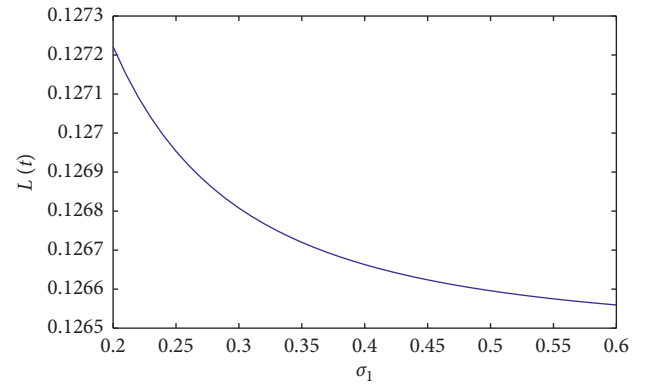

FIGURE 11: Effect of σ_2 on $L(t)$.

FIGURE 13: Effect of σ_1 on $L(t)$.

investment is robust, he will invest more money in the risk-free asset, so AAI will suffer less utility loss.

Let $\alpha_1 = 4$ and $\alpha_2 = 2, 3, 4, 5, 6, 7, 8$; we have $W_1(t, x, s) = 10.2370$. The values of $W(t, x, s)$ and $L(t)$ are presented in Table 3. From Table 3, it is evident that the utility loss function is an increasing function in α_2, where α_2 stands for the ambiguity aversion which arises from the risky asset.

(ii) The influence of market parameters on the utility loss:

Figure 10 discloses that $L(t)$ is a decreasing function with respect to μ_2, where $\mu_2 \in [0.05, 0.1]$. This indicates that the utility loss increases as the appreciation rate of the risky asset diminishes.

Figure 11 reveals that $L(t)$ is an increasing function with regard to σ_2, where $\sigma_2 \in [0.2, 0.6]$. Larger σ_2 implies that the investment risk becomes greater, and thus, the AAI will suffer from more utility losses.

Let $s \in [0.5, 1.5]$; Figure 12 gives the influence of parameter s on the utility loss. From Figure 12, we can observe that $L(t)$ is an increasing function of s. The larger the s is, the larger the price volatility will be, that is, the greater the risk will be, and hence, the AAI will suffer from more utility losses.

Let $\sigma_1 \in [0.2, 0.6]$; Figure 13 shows that the utility loss is a decreasing function of σ_1. Comparing Figures 11 with 13, it is clear that the effect of the volatilities of

the risky asset and the inflation on the utility loss is opposite.

6. Conclusions

We have investigated a robust time-consistent MV strategy selection under the inflation risk for an AAI. The AAI doubts the model setting under the original probability measure. Then, he seeks a set of alternative probability measures, which are absolutely continuous to the original probability measure. Furthermore, we modify the model setting under the alternative probability measures. We form the AAI's objective under the framework of the robust MV criterion. As we all know, this problem is time-inconsistent. To solve this time-inconsistent problem, we seek the equilibrium strategy from the game framework. The equilibrium strategy is time-consistent. By using the stochastic optimal control technique, we derive the closed-form solutions for the optimal time-consistent investment strategy and the optimal value function. Finally, the influences of model parameters on the optimal time-consistent investment strategy and utility loss function are examined through numerical experiments.

Compared with Zeng et al. [12], Zheng et al. [28], Maenhout [34, 35], and other studies about the robust MV problem, the inflation risk and the CEV model are simultaneously introduced in this article. In addition, the influence of the model parameters of the inflation on the optimal time-consistent investment strategy and the utility loss are systematic analysis by numerical experiments.

It may be interesting to extend the current framework to that with dividend and state-dependent risk-aversion function or that in a partial information market in the future research. For these future research topics, we may need to adopt the methods used in Bjök et al. [14], Yu et al. [37], and Raychaudhury and Pal [38].

Data Availability

The data used to support the findings of this study are available upon request to the author.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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