Research Article

Extremal Solutions for a Class of Tempered Fractional Turbulent Flow Equations in a Porous Medium

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In this paper, we are concerned with the existence of the maximum and minimum iterative solutions for a tempered fractional turbulent flow model in a porous medium with nonlocal boundary conditions. By introducing a new growth condition and developing an iterative technique, we establish new results on the existence of the maximum and minimum solutions for the considered equation; at the same time, the iterative sequences for approximating the extremal solutions are performed, and the asymptotic estimates of solutions are also derived.

1. Introduction

Tempered stable laws were introduced to model turbulent velocity fluctuations of physics [1]. Normally, tempered stable laws retain their signature power-law behaviour at infinity and infinite divisibility [2]. By multiplying by an exponential factor for the usual second derivative, one can obtain tempered fractional derivatives and integrals. In [3], an exponential tempering factor was applied to the particle jump density in random walk and stochastic model for turbulence in the inertial range, which is the fractional derivative of Brownian motion exhibiting semi-long range dependence with a power law at moderate time scales.

Tempered stable laws are useful in statistical physics and provide a basic physical model such as turbulent flow for the underlying physical phenomena. Motivated by these physical backgrounds and the sources, in this paper, we focus on the existence of the maximum and minimum iterative solutions for the following tempered fractional turbulent flow equation with nonlocal boundary conditions:

$$\begin{cases} {}^R_0\mathbb{D}_t^{\alpha,\lambda}\left(\varphi_p\left({}^R_0\mathbb{D}_t^{\beta,\lambda}x(t)\right)\right)=h(t)f(x(t)), \\ x(0)=x'(0)=\dots=x^{(n-2)}(0)={}_0^R\mathbb{D}_t^{\beta,\lambda}x(0)=0, \\ {}^R_0\mathbb{D}_t^{\beta,\lambda}x(1)=\int_0^1e^{-\lambda(1-t)}x(t)dt, \\ {}^R_0\mathbb{D}_t^{\gamma,\lambda}\left(\varphi_p\left({}^R_0\mathbb{D}_t^{\beta,\lambda}x\right)\right)(1)=\int_0^1a(t){}_0^R\mathbb{D}_t^{\gamma,\lambda}\left(\varphi_p\left({}^R_0\mathbb{D}_t^{\beta,\lambda}x(t)\right)\right)dA(t), \end{cases} \quad (1)$$

where $1 < \alpha \leq 2, n-1 < \beta \leq n, n \geq 4, 0 < \gamma < \alpha-1, \lambda > 0$ is a constant, φ_p is the p -Laplacian operator defined by $\varphi_p(s) = |s|^{p-2}s, p > 1, {}^R_0\mathbb{D}_t^{\alpha,\lambda}$ is the tempered fractional derivative, $\int_0^1 a(t){}_0^R\mathbb{D}_t^{\gamma,\lambda}\left(\varphi_p\left({}^R_0\mathbb{D}_t^{\beta,\lambda}x(t)\right)\right)dA(t)$ denotes a Riemann-Stieltjes integral, A is a function of bounded variation, $f: [0, +\infty) \rightarrow (0, +\infty)$ is continuous, and $h \in [L(0, 1), [0, +\infty)]$.

Turbulent flow is a fundamental fluid mechanics problem which can be described by a p -Laplacian equation with a suitable boundary condition; for details, see [4]. Particularly, if the model is of fractional order, then it

can describe turbulent flow in a porous medium [5–10]. On the contrary, fractional-order derivative has nonlocal characteristics; based on this property, the fractional differential equation can also interpret many abnormal phenomena that occur in applied science and engineering, such as viscoelastic dynamical phenomena [11–29], advection-dispersion process in anomalous diffusion [30–34], and bioprocesses with genetic attribute [35, 36]. As a powerful tool of modeling the above phenomena, in recent years, the fractional calculus theory has been perfected gradually by many researchers, and various different types of fractional derivatives were studied, such as Riemann–Liouville derivatives [16, 37–62], Hadamard-type derivatives [63–71], Katugampola–Caputo derivatives [72], conformable derivatives [73–76], Caputo–Fabrizio derivatives [77, 78], Hilfer derivatives [79–82], and tempered fractional derivatives [83]. These works also enlarged and enriched the application of the fractional calculus in impulsive theories [84–89], chaotic system [90–93], and resonance phenomena [94–96]. Among them, by using the fixed point theorem of the mixed monotone operator, Zhang et al. [9] established the result of uniqueness of the positive solution for the Riemann–Liouville-type turbulent flow in a porous medium:

$$\begin{cases} \mathcal{D}_t^\beta(\varphi_p(-\mathcal{D}_t^\alpha z))(t) = -f(z(t), \mathcal{D}_t^\gamma z(t)), & t \in (0, 1), \\ \mathcal{D}_t^\alpha z(0) = \mathcal{D}_t^{\alpha+1} z(0) = \mathcal{D}_t^\alpha z(1) = 0, \\ \mathcal{D}_t^\gamma z(0) = 0, \mathcal{D}_t^\gamma z(1) = \int_0^1 \mathcal{D}_t^\gamma z(s) dA(s), \end{cases} \quad (2)$$

where $0 < \gamma \leq 1 < \alpha \leq 2 < \beta < 3$, $\alpha - \gamma > 1$, $\mathcal{D}_t^\alpha$, $\mathcal{D}_t^\beta$, and $\mathcal{D}_t^\gamma$ denote the Riemann–Liouville derivatives, and $\int_0^1 z(s) dA(s)$ indicates the Riemann–Stieltjes integral, and A is a function of bounded variation; the nonlinear term may be singular at

both first variable and second variable. Recently, Zhou et al. [83] investigated a class of tempered fractional differential equations with Riemann–Stieltjes integral boundary conditions; by using the fixed point theorem of the sum-type mixed monotone operator, the existence and uniqueness of positive solutions were established, and iterative sequences for approximating the unique positive solution were also constructed.

However, to the best of our knowledge, there are relatively few results on fractional turbulent flow in a porous medium with nonlocal Riemann–Stieltjes integral boundary conditions, and no work has been reported on the maximal and minimal solutions for the tempered-type fractional turbulent flow equation. Thus, following the previous work, this paper will pay attention to the extremal solutions for the tempered fractional turbulent flow equation in a porous medium with nonlocal Riemann–Stieltjes integral boundary conditions by developing iterative technique, also see [97–100]. Different from [9, 83], in this paper, we will give a new type of growth condition for the nonlinear term to guarantee equation (1) has the extremal solutions. At the same time, the iterative sequences for approximating the extremal solutions are performed, and the asymptotic estimates of solutions are also obtained.

2. Preliminaries and Lemmas

Before starting our work, we firstly recall the definition of the tempered fractional derivative which is an extension of the Riemann–Liouville derivative and integral.

Let $\lambda > 0$; the α -order left tempered fractional derivative is defined by

$${}_0^R \mathbb{D}_t^{\alpha, \lambda} x(t) = e^{-\lambda t} {}_0^R \mathcal{D}_t^\alpha (e^{\lambda t} x(t)), \quad (3)$$

where ${}_0^R \mathcal{D}_t^\alpha$ denotes the standard Riemann–Liouville fractional derivative which can be found in [101].

Let

$$H(t, s) = \begin{cases} \frac{[\beta(1-s)^{\beta-1}(\beta-1+s)e^{\lambda s}t^{\beta-1} - \beta(\beta-1)e^{\lambda s}(t-s)^{\beta-1}]e^{-\lambda t}}{(\beta-1)\Gamma(\beta+1)}, & 0 \leq s \leq t \leq 1, \\ \frac{[\beta(1-s)^{\beta-1}(\beta-1+s)]e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)}t^{\beta-1}e^{-\lambda t}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (4)$$

$$\Delta = \frac{e^{-\lambda} - \delta}{\Gamma(\alpha - \gamma)},$$

$$\delta = \int_0^1 e^{-\lambda s} s^{\alpha-\gamma-1} a(s) dA(s).$$

The following results have been proven in [83].

Lemma 1. Given $k \in C[0, 1]$; then, the boundary value problem,

$$\begin{cases} {}^R_0\mathbb{D}_t^{\alpha,\lambda} \left(\varphi_p \left({}^R_0\mathbb{D}_t^{\beta,\lambda} x(t) \right) \right) = k(t), \\ x(0) = x'(0) = \dots = x^{(n-2)}(0) = {}^R_0\mathbb{D}_t^{\beta,\lambda} x(0) = 0, \\ {}^R_0\mathbb{D}_t^{\beta,\lambda} x(1) = \int_0^1 e^{-\lambda(1-t)} x(t) dt, \\ {}^R_0\mathbb{D}_t^{\gamma,\lambda} \left(\varphi_p \left({}^R_0\mathbb{D}_t^{\beta,\lambda} x \right) \right) (1) = \int_0^1 a(t) {}^R_0\mathbb{D}_t^{\gamma,\lambda} \left(\varphi_p \left({}^R_0\mathbb{D}_t^{\beta,\lambda} x(t) \right) \right) dA(t), \end{cases} \quad (5)$$

has the unique solution

$$x(t) = \int_0^1 H(t, s) \varphi_q \left(\int_0^1 G(t, \tau) k(\tau) d\tau \right) ds, \quad (6)$$

where $H(t, s)$ is defined by (4) and $G(t, s)$ denotes the Green function as follows:

$$\begin{aligned} G(t, s) &= G_1(t, s) + \frac{t^{\alpha-1} e^{-\lambda t}}{\Delta \Gamma(\alpha - \gamma)} \int_0^1 a(t) G_2(t, s) dA(t), \\ G_1(t, s) &= \frac{e^{\lambda(s-t)}}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-\gamma-1} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ (1-s)^{\alpha-2} t^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases} \\ G_2(t, s) &= \frac{e^{\lambda(s-t)}}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-\gamma-1} t^{\alpha-\gamma-1} - (t-s)^{\alpha-\gamma-1}, & 0 \leq s \leq t \leq 1, \\ (1-s)^{\alpha-\gamma-1} t^{\alpha-\gamma-1}, & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned} \quad (7)$$

In order to obtain the positive extremal solutions of tempered fractional turbulent flow equation (1), it is necessary to preserve nonnegativity of the Green function.

(H0):

$$\int_0^1 e^{-\lambda s} s^{\alpha-\gamma-1} a(s) dA(s) < e^{-\lambda}. \quad (8)$$

Lemma 2. Assume (H0) holds; then, functions $G(t, s)$ and $H(t, s)$ have the following properties:

- (1) $G(t, s)$ and $H(t, s)$ are nonnegative and continuous for $(t, s) \in [0, 1] \times [0, 1]$.
- (2) For any $t, s \in [0, 1]$, $H(t, s)$ satisfies

$$m_1(s) e^{-\lambda t} t^{\beta-1} \leq H(t, s) \leq M_1(s) e^{-\lambda t} t^{\beta-1}, \quad (9)$$

where

$$\begin{aligned} M_1(s) &= \frac{\beta(1-s)^{\beta-1}(\beta-1+s)e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)}, \\ m_1(s) &= \frac{\beta s(1-s)^{\beta-1}e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)}. \end{aligned} \quad (10)$$

(3)

$$m_2(s) e^{-\lambda t} t^{\alpha-1} \leq G(t, s) \leq M_2(s) e^{-\lambda t} t^{\alpha-1}, \quad s, t \in [0, 1], \quad (11)$$

where

$$\begin{aligned} M_2(s) &= \left[\frac{1}{\Gamma(\alpha)} + \frac{\delta}{\Delta \Gamma(\alpha) \Gamma(\alpha - \gamma)} \right] e^{\lambda s} (1-s)^{\alpha-\gamma-1}, \\ m_2(s) &= \frac{e^{\lambda s} [(1-s)^{\alpha-\gamma-1} - (1-s)^{\alpha-1}]}{\Gamma(\alpha)}. \end{aligned} \quad (12)$$

Let

$$M_1^* = \frac{\beta^2 e^{\lambda}}{(\beta-1)\Gamma(\beta+1)}. \quad (13)$$

In order to obtain the existence of positive extremal solutions of tempered fractional turbulent flow equation (1), we introduce the following new control conditions.

(H1): $f: [0, +\infty) \rightarrow (0, +\infty)$ is continuous and nondecreasing, and there exists a positive constant $\epsilon > 3/(q-1)$ such that

$$0 < d =: \sup_{s \geq 0} \frac{f(s)}{(s+2)^\epsilon} < +\infty. \quad (14)$$

(H2):

$$0 < \int_0^1 M_2(\tau) h(\tau) d\tau < \left(\frac{M_1^*}{3^{\epsilon(q-1)}} \right)^{1/q-1} d^{-1}. \quad (15)$$

Remark 1. Assumption (14) we introduced is a new type of growth condition, which includes a large number of basic functions such as

- (1) $f(t) = b_0 + \sum_{i=1}^n b_i (t+2)^{\mu_i}$, where $b_0 > 0, b_i, \mu_i > 3/q-1, i = 1, 2, \dots, n$.
- (2) $f(t) = b_0 + [\sum_{i=1}^n b_i (t+2)^{\mu_i}]^{1/\mu}$, where $b_0 > 0, \mu > 0, b_i > 0, \mu_i > 0 (i = 1, 2, \dots, n)$ and

$$\min_{i=1, \dots, n} \{\mu_i\} > \frac{3\mu}{q-1}. \quad (16)$$

- (3) $f(t) = (t+1)^{\mu-1} \ln(1 + (1/2 + t)) + (2+t)^{\mu-2} + b$, $b > 0, \mu > 2 + (3/q-1)$.
- (4) $f: [0, +\infty) \rightarrow (0, +\infty)$ is continuous and nondecreasing, and there exists a positive constant $\epsilon > 3/q-1$ such that $f(x)/(x+2)^\epsilon$ is increasing on x , and

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{(x+2)^\epsilon} = M > 0. \quad (17)$$

- (5) $f: [0, +\infty) \rightarrow (0, +\infty)$ is continuous and nondecreasing, and there exists a positive constant $\epsilon > 3/q - 1$ such that $f(x)/(x+2)^\epsilon$ is nonincreasing on x .

Proof. For cases (1)–(3), take

$$\begin{aligned}\epsilon &= \max_{i=1,2,\dots,n} \{\mu_i\}, \\ \epsilon &= \frac{\max_{i=1,2,\dots,n} \{\mu_i\}}{\mu}, \\ \epsilon &= \mu - 2,\end{aligned}\quad (18)$$

respectively; obviously,

$$0 < d := \sup_{s \geq 0} \frac{f(s)}{(s+2)^\epsilon} < +\infty. \quad (19)$$

For cases (4) and (5), it is clear; we omit the proof. $\square$

Denote $E = C[0, 1]$ as all continuous functions equipped the maximum norm

$$\|x\| = \max\{x(t) : t \in [0, 1]\}. \quad (20)$$

Define a cone P ,

$$P = \{x \in E : \text{there exists a number } 0 < l_x < 1 \text{ such that } 0 \leq x(t) \leq l_x^{-1} e^{-\lambda t} t^{\beta-1}, \quad t \in [0, 1]\},$$

(21)

and an operator T in E :

$$T(x)(t) = \int_0^1 H(t, s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) f(x(\tau)) d\tau \right) ds. \quad (22)$$

Then, the fixed point of operator T in E is the solution of tempered fractional turbulent flow equation (1). $\square$

Lemma 3. Assume that (H0)–(H2) hold. Then, $T: P \rightarrow P$ is a continuous, compact operator.

Proof. It follows from the definition of P that, for any $x \in P$, there exists a number $0 < l_x < 1$ such that

$$0 \leq x(t) \leq l_x^{-1} e^{-\lambda t} t^{\beta-1}, \quad t \in [0, 1]. \quad (23)$$

Since T is increasing with respect to x , by (14), (23), and Lemma 2, we have

$$\begin{aligned}T(x)(t) &= \int_0^1 H(t, s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) f(x(\tau)) d\tau \right) ds \\ &\leq e^{-\lambda t} t^{\beta-1} \int_0^1 M_1(s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) f(x(\tau)) d\tau \right) ds \\ &\leq e^{-\lambda t} t^{\beta-1} \int_0^1 M_1(s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) \frac{f(x(\tau))}{(x(\tau)+2)^\epsilon} (x(\tau)+2)^\epsilon d\tau \right) ds \\ &\leq \varphi_q(d) e^{-\lambda t} t^{\beta-1} \int_0^1 M_1(s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) (x(\tau)+2)^\epsilon d\tau \right) ds \\ &\leq \varphi_q(d) e^{-\lambda t} t^{\beta-1} \int_0^1 M_1(s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) (l_x^{-1} e^{-\lambda \tau} \tau^{\beta-1} + 2)^\epsilon d\tau \right) ds \\ &\leq M_1^* \varphi_q \left(d(l_x^{-1} + 2)^\epsilon \int_0^1 M_2(\tau) h(\tau) d\tau \right) e^{-\lambda t} t^{\beta-1} \\ &\leq \frac{1}{l_x^*} e^{-\lambda t} t^{\beta-1},\end{aligned}\quad (24)$$

where

$$l_x^* = \min \left\{ \frac{1}{2}, M_1^{*-1} \left(d(l_x^{-1} + 2)^\epsilon \int_0^1 M_2(\tau) h(\tau) d\tau \right)^{1-q} \right\}. \quad (25)$$

Thus, it follows from (24) that

$$0 \leq T(x)(t) \leq \frac{1}{l_x^*} e^{-\lambda t} t^{\beta-1}, \quad (26)$$

which implies that T is well defined and uniformly bounded, and $T(P) \subset P$.

On the contrary, according to the Arzela–Ascoli theorem and the Lebesgue dominated convergence theorem, it is easy to know that $T: P \rightarrow P$ is completely continuous. $\square$

3. Main Results

Before we begin to state our main result, we first give the following lemma.

Lemma 4. Suppose $\epsilon(q-1) > 3$ and (H2) hold; then, the equation

$$M_1^* \left(d \int_0^1 M_2(\tau) h(\tau) d\tau \right)^{q-1} (x+3)^{\epsilon(q-1)} (x+1)^{-1} = 1, \quad (27)$$

has unique solution δ^* in $(0, \infty)$.

Proof. Let

$$\begin{aligned} \phi'(x) &= -\epsilon(q-1)M_1^* \left(d \int_0^1 M_2(\tau) h(\tau) d\tau \right)^{q-1} (x+3)^{\epsilon(q-1)-1} (x+1)^{-1} + M_1^* \left(d \int_0^1 M_2(\tau) h(\tau) d\tau \right)^{q-1} (x+3)^{\epsilon(q-1)} (x+1)^{-2} \\ &= -M_1^* \left(d \int_0^1 M_2(\tau) h(\tau) d\tau \right)^{q-1} (x+3)^{\epsilon(q-1)-1} (x+1)^{-1} \left(\epsilon(q-1) - \frac{x+3}{x+1} \right) < 0, \quad x \in (0, \infty). \end{aligned} \quad (31)$$

Thus, (29)–(31) imply equation (27) has unique zero point δ^* in $(0, \infty)$. $\square$

Theorem 1. Suppose (H0)–(H2) hold. Then, the following is obtained:

- (i) Existence: equation (1) has a positive minimal solution $\underline{x}$ and a positive maximal solution $\bar{x}$.
- (ii) Asymptotic estimates: there exist positive numbers $n_i > 0$, $i = 1, 2$, such that

$$\begin{aligned} \frac{\underline{x}(t)}{e^{-\lambda t} t^{\beta-1}} &\in [0, n_1], \\ \frac{\bar{x}(t)}{e^{-\lambda t} t^{\beta-1}} &\in [0, n_2], \quad t \in (0, 1]. \end{aligned} \quad (32)$$

- (iii) Iterative sequences: for initial values $x^{(0)}(t) = 0$ and $y^{(0)}(t) = \delta^* + 1$, construct the iterative sequences

$$\begin{aligned} x^{(n)}(t) &= \int_0^1 H(t, s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) f(x^{(n-1)}(\tau)) d\tau \right) ds, \\ y^{(n)}(t) &= \int_0^1 H(t, s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) f(y^{(n-1)}(\tau)) d\tau \right) ds. \end{aligned} \quad (33)$$

Then,

$$\begin{aligned} \lim_{n \rightarrow +\infty} x^{(n)}(t) &= \underline{x}(t), \\ \lim_{n \rightarrow +\infty} y^{(n)}(t) &= \bar{x}(t), \end{aligned} \quad (34)$$

uniformly, for $t \in [0, 1]$, where δ^* is the unique solution of equation (27) in $(0, \infty)$.

$$\phi(x) = 1 - M_1^* \left(d \int_0^1 M_2(\tau) h(\tau) d\tau \right)^{q-1} (x+3)^{\epsilon(q-1)} (x+1)^{-1}. \quad (28)$$

It follows from $\epsilon(q-1) > 3$ and (H2) that

$$\phi(0) = 1 - M_1^* \left(d \int_0^1 M_2(\tau) h(\tau) d\tau \right)^{q-1} 3^{\epsilon(q-1)} > 0, \quad (29)$$

$$\phi(+\infty) = -\infty. \quad (30)$$

On the contrary, $\phi(x)$ is a continuous function in $[0, \infty)$ satisfying

Proof. Firstly, let $P_{\delta^*} = \{x \in P: 0 \leq \|x\| \leq \delta^* + 1\}$; we shall show $T(P_{\delta^*}) \subset P_{\delta^*}$.

For any $x \in P_{\delta^*}$ and for any $t \in (0, 1)$, we have

$$0 \leq x(t) \leq \max_{t \in [0, 1]} x(t) \leq \delta^* + 1. \quad (35)$$

Consequently, it follows from (H1) and Lemma 4 that

$$\begin{aligned} \|T(x)\| &= \max_{t \in [0, 1]} \left\{ \int_0^1 H(t, s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) f(x(\tau)) d\tau \right) ds \right\} \\ &\leq \int_0^1 M_1(s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) f(x(\tau)) d\tau \right) ds \\ &\leq \int_0^1 M_1(s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) \frac{f(x(\tau))}{(x(\tau) + 2)^\epsilon} (x(\tau) + 2)^\epsilon d\tau \right) ds \\ &\leq \varphi_q(d) \int_0^1 M_1(s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) (x(\tau) + 2)^\epsilon d\tau \right) ds \\ &\leq \varphi_q(d) \int_0^1 M_1(s) \varphi_q \left(\int_0^1 G(s, \tau) h(\tau) (\delta^* + 3)^\epsilon d\tau \right) ds \\ &\leq M_1^* \left(d \int_0^1 M_2(\tau) h(\tau) d\tau \right)^{q-1} (\delta^* + 3)^{\epsilon(q-1)} \\ &= \delta^* + 1, \end{aligned} \quad (36)$$

which implies that $T(P_{\delta^*}) \subset P_{\delta^*}$.

Next, take the initial value $x^{(0)}(t) = 0$, and let

$$x^{(1)}(t) = T(x^{(0)})(t) = T(0)(t), \quad t \in [0, 1]. \quad (37)$$

It follows from $x^{(0)}(t) \in P_{\delta^*}$ that $x^{(1)}(t) \in T(P_{\delta^*}) \subset P_{\delta^*}$. Denote

$$x^{(n+1)}(t) = T x^{(n)}(t) = T^{n+1} x^{(0)}(t), \quad n = 1, 2, \dots \quad (38)$$

By $T(P_{\delta^*}) \subset P_{\delta^*}$, we have $x_n \in P_{\delta^*}$ for $n \geq 1$. It follows from the fact of T being a compact operator that $\{x^{(n)}\}$ is a sequentially compact set.

On the contrary, since $x^{(1)}(t) \geq 0 = x^{(0)}(t)$ and T is increasing on x , we have

$$x^{(2)}(t) = (Tx^{(1)})(t) \geq (Tx^{(0)})(t) = x^{(1)}(t), \quad t \in [0, 1]. \quad (39)$$

By induction, one has

$$0 \leq x^{(n)}(t) \leq x^{(n+1)}(t) \leq \delta^* + 1, \quad n = 1, 2, \dots \quad (40)$$

Consequently, there exists $\underline{x} \in P_{\delta^*}$ such that $x^{(n)} \rightarrow \underline{x}$. Noticing that $Tx^{(n)} = x^{(n+1)}$ and letting $n \rightarrow +\infty$, by the continuity of T , we have $T\underline{x} = \underline{x}$, which implies that $\underline{x}$ is a nonnegative solution of equation (1), and then $\underline{x}$ is a positive solution of equation (1) since $f(0) > 0$.

Now, we take $y^{(0)}(t) = \delta^* + 1$ as the initial value and let

$$y^{(1)}(t) = (Ty^{(0)})(t), \quad t \in [0, 1]. \quad (41)$$

It follows from $y^{(0)}(t) = \delta^* + 1 \in P_{\delta^*}$ that $y^{(1)} \in P_{\delta^*}$. Thus, construct the iterative sequence

$$y^{(n+1)}(t) = Ty^{(n)}(t) = T^{n+1}y^{(0)}(t), \quad n = 1, 2, \dots \quad (42)$$

We have

$$y^{(n)}(t) \in P_{\delta^*}, \quad n = 0, 1, 2, \dots, \quad (43)$$

since $T(P_{\delta^*}) \subset P_{\delta^*}$. It follows from Lemma 3 that $\{y^{(n)}\}$ is a sequentially compact set.

Now, since $y^{(1)} \in P_{\delta^*}$ and T is increasing, one has

$$0 \leq y^{(1)}(t) \leq \|y^{(1)}\| \leq \delta^* + 1 = y^{(0)}(t), \quad (44)$$

and then

$$y^{(2)}(t) = Ty^{(1)}(t) \leq Ty^{(0)}(t) = y^{(1)}(t). \quad (45)$$

It follows from induction that

$$0 \leq y^{(n+1)}(t) \leq y^{(n)}(t) \leq \delta^* + 1, \quad n = 0, 1, 2, \dots, \quad (46)$$

which implies that there exists $\bar{x} \in P_{\delta^*}$ such that $y^{(n)} \rightarrow \bar{x}$. Letting $n \rightarrow +\infty$, from the continuity of T and $Ty^{(n)} = y^{(n+1)}$, we have $T\bar{x} = \bar{x}$, which implies that $\bar{x}$ is another positive solution of equation (1).

Next, we prove that $\underline{x}$ and $\bar{x}$ are the maximum and minimum positive solutions of equation (1). In fact, suppose $\tilde{x}$ is any positive solution of equation (1); then, we have

$$x^{(0)}(t) = 0 \leq \tilde{x}(t) \leq \delta^* + 1 = y^{(0)}(t),$$

$$x^{(1)}(t) = Tx^{(0)}(t) \leq T\tilde{x}(t) = \tilde{x}(t) \leq Ty^{(0)}(t) = y^{(1)}(t). \quad (47)$$

Thus, it follows from induction that

$$x^{(n)}(t) \leq \tilde{x}(t) \leq y^{(n)}(t), \quad n = 1, 2, 3, \dots \quad (48)$$

Taking the limit, we have

$$\underline{x} \leq \tilde{x} \leq \bar{x}, \quad (49)$$

which implies that $\underline{x}$ and $\bar{x}$ are the maximal and minimal positive solutions of equation (1), respectively.

In the end, since $\underline{x}, \bar{x} \in P_{\delta^*} \subset P$, there exist constants $n_1 > 0, n_2 > 0$ such that

$$\frac{\underline{x}(t)}{e^{-\lambda t} t^{\beta-1}} \in [0, n_1], \quad (50)$$

$$\frac{\bar{x}(t)}{e^{-\lambda t} t^{\beta-1}} \in [0, n_2], \quad t \in (0, 1].$$

□

4. Example

Since the fractional-order derivative possesses long-memory characteristics, in fluid mechanics, equation (1) can describe a turbulent flow in a porous medium. Here, we give a specific example to illustrate the main results.

Example: consider the following nonlocal tempered fractional turbulent flow equation:

$$\left\{ \begin{array}{l} {}^R_0\mathbb{D}_t^{(3/2),1} \left(\varphi_{(3/2)} \left({}^R_0\mathbb{D}_t^{(7/2),1} x(t) \right) \right) = \frac{e^{-t}(1-t)^{-(1/4)}}{400} \left((x(t)+1)^3 \ln \left(1 + \frac{1}{2+x(t)} \right) + (2+x(t))^2 + 2 \right), \\ x(0) = x'(0) = x''(0) = {}^R_0\mathbb{D}_t^{(7/2),1} x(0) = 0, \\ {}^R_0\mathbb{D}_t^{(7/2),1} x(1) = \int_0^1 e^{-(1-t)} x(t) dt, \\ {}^R_0\mathbb{D}_t^{(1/4),1} \left(\varphi_{(3/2)} \left({}^R_0\mathbb{D}_t^{(7/2),1} x \right) \right) (1) = \int_0^1 {}^R_0\mathbb{D}_t^{(1/4),1} \left(\varphi_{(3/2)} \left({}^R_0\mathbb{D}_t^{(7/2),1} x(t) \right) \right) dA(t), \end{array} \right. \quad (51)$$

where

$$A(t) = \begin{cases} 0, & t \in \left[0, \frac{1}{2}\right), \\ 1, & t \in \left[\frac{1}{2}, \frac{3}{4}\right), \\ \frac{1}{2}, & t \in \left[\frac{3}{4}, 1\right]. \end{cases} \quad (52)$$

Then, equation (51) has the positive minimal and maximal solutions $\underline{x}$ and $\bar{x}$, and there exist constants $n_1 > n_2 > 0$ such that

$$\begin{aligned} \frac{\underline{x}(t)}{e^{-t}t^{5/2}} &\in [0, n_1], \\ \frac{\bar{x}(t)}{e^{-t}t^{5/2}} &\in [0, n_2], \end{aligned} \quad (53)$$

$$t \in (0, 1].$$

Let

$$\begin{aligned} \alpha &= \frac{3}{2}, \\ \beta &= \frac{7}{2}, \\ \lambda &= 1, \\ \gamma &= \frac{1}{4}, \\ p &= \frac{3}{2}, \\ a(t) &= 1, \\ h(t) &= \frac{1}{400}e^{-t}(1-t)^{-1/4}, \end{aligned} \quad (54)$$

$$f(x) = (x+1)^3 \ln\left(1 + \frac{1}{2+x}\right) + (2+x)^2 + 2.$$

Firstly, we have

$$\begin{aligned} \delta &= \int_0^1 e^{-\lambda s} s^{\alpha-\gamma-1} a(s) dA(s) = \int_0^1 e^{-s} s^{1/4} dA(s) \\ &= 0.2902 < e^{-1} = 0.3679. \end{aligned} \quad (55)$$

Thus, (H0) holds.

Obviously, $f: [0, +\infty) \rightarrow (0, +\infty)$ is continuous and nondecreasing. Take $\epsilon = 2 > 3/q - 1 = 3/2$; then, we have

$$\begin{aligned} 0 < d &= \sup_{x \geq 0} \frac{f(x)}{(x+2)^\epsilon} = \sup_{x \geq 0} \left(\left(\frac{x+1}{x+2} \right)^3 \ln \left(1 + \frac{1}{2+x} \right) \right)^{x+2} \\ &\quad + 1 + \frac{2}{(x+2)^2} = 2 < +\infty, \end{aligned} \quad (56)$$

which implies that (H1) is satisfied.

Now, we compute M_1^* and Δ :

$$\begin{aligned} M_1^* &= \frac{\beta^2 e^\lambda}{(\beta-1)\Gamma(\beta+1)} = \frac{3.5^2 \times e}{2.5 \times \Gamma(4.5)} = 0.1550, \\ \Delta &= \frac{e^{-\lambda} - \delta}{\Gamma(\alpha - \gamma)} = \frac{e^{-1} - 0.3679}{\Gamma(5/4)} = 0.0857. \end{aligned} \quad (57)$$

Thus, we have

$$\begin{aligned} M_2(s) &= \left[\frac{1}{\Gamma(3/2)} + \frac{0.3679}{0.0857 \times \Gamma(3/2)\Gamma(5/4)} \right] e^s (1-s)^{1/4} \\ &= 5.3439e^s (1-s)^{1/4}. \end{aligned} \quad (58)$$

Consequently,

$$0 < \int_0^1 M_2(\tau) h(\tau) d\tau = 0.013358 < \left(\frac{M_1^*}{3^{\epsilon(q-1)}} \right)^{1/q-1} d^{-1} = 0.0219. \quad (59)$$

So, condition (H3) holds.

Thus, by Theorem 1, equation (51) has a positive minimal solution $\underline{x}$ and a positive maximal solution $\bar{x}$, and there exist constants $n_1 > n_2 > 0$ such that

$$\begin{aligned} \frac{\underline{x}(t)}{e^{-t}t^{5/2}} &\in [0, n_1], \\ \frac{\bar{x}(t)}{e^{-t}t^{5/2}} &\in [0, n_2], \end{aligned} \quad (60)$$

$$t \in (0, 1].$$

5. Conclusion

In this work, we establish a new result on the existence of the maximum and minimum solutions for a class of tempered fractional-order differential equations with nonlocal boundary conditions. This type of equation can describe a turbulent flow of a porous medium in fluid mechanics and diffusive interaction. In order to obtain the extremal solutions of the equation, a new type of growth condition is introduced, and the iterative sequences with explicit initial values are constructed which converge uniformly to the maximum and minimum solutions; in addition, the estimations of the upper bounds of the maximum and minimum solutions are also derived.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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