Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 2539196, 9 pages
https://doi.org/10.1155/2020/2539196

Research Article

Convergence Analysis of a Trust-Region Multidimensional Filter Method for Nonlinear Complementarity Problems

C. W. Wu[1], **J. P. Cao**[2], **and L. F. Wang**[2]

[1]*Department of Math, Xi'an High-Tech Institute, Xi'an 710025, Shaanxi Province, China*
[2]*Xi'an High-Tech Institute, Xi'an 710025, Shaanxi Province, China*

Correspondence should be addressed to C. W. Wu; wucongweihome@hotmail.com

Received 15 March 2020; Revised 16 May 2020; Accepted 20 May 2020; Published 19 June 2020

Guest Editor: Cuimei Jiang

For solving nonlinear complementarity problems, a new algorithm is proposed by using multidimensional filter techniques and a trust-region method. The algorithm is shown to be globally convergent under the reasonable assumptions and does not depend on any extra restoration procedure. In particular, it shows that the subproblem is a convex quadratic programming problem, which is easier to be solved. The results of numerical experiments show its efficiency.

1. Introduction

Let $F(x)\colon \mathfrak{R}^n \longmapsto \mathfrak{R}^n$ be a continuous differentiable function. The nonlinear complementarity problem (NCP) is to find a vector $x \in \mathfrak{R}^n$ such that

$$\begin{aligned} x &\geq 0, \\ F(x) &\geq 0, \\ x^T F(x) &= 0. \end{aligned} \tag{1}$$

For convenience, denote $\mathcal{I} = \{1, 2, \ldots, n\}$. Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm.

The traditional approach for NCP involves reformulating the problem as an optimization problem [1–12] or a nonlinear differentiable function [13–18]. In this paper, a new method for solving this optimization problem is based on the class of trust-region methods and also filter methods introduced by Fletcher and Leyffer in 1997 and subsequently published as [19]. This technique has important reference value for many nonlinear system problems [20–22]. The idea of filter methods is that trial points are accepted as long as they could reduce the value of objective function or improve the feasibility, which is different from the conventional approach of combining these two measures by a penalty function. Filter approaches play an important role to balance the objective function and constraints and have advantages over penalty function methods. Numerical experiments have shown the impressing efficiency of filter methods [19, 23]. The global convergence proof of filter-SQP algorithm is given by Fletcher et al. [24], and relevant superlinear local convergence is achieved by Ulbrich [25].

Because of good numerical results, filter techniques are extensively studied to handle the nonlinear complementarity problem [7–9, 26, 27]. Most of the contributions to NCP of filter algorithms rely on an external "restoration procedure" [19, 27–29] whose purpose is to reduce constraint infeasibilities, since the filter idea introduced by Fletcher and Leyffer [19] is based on constrained optimization problems, and the constrained optimization problems transformed by NCP may be infeasible. Gould et al. [30, 31] proposed a multidimensional filter algorithm for nonlinear unconstrained optimization problems instead of a two-dimensional filter [19] for constrained optimization problems. This motivates us to consider the possibility of reformulating NCP as an unconstrained optimization problem and solving the optimization by a multidimensional filter method. In spite of the fact, we suggest a reformulation of NCP as an optimization problem with nonnegativity constraints because Fischer [2] points out that stationary points with negative components can be avoided in contrast to the reformulation as unconstrained minimization problem. Unfortunately, the multidimensional filter method is proposed on unconstrained optimization [30, 31].

There are two main motivations of this paper. One is to find an effective method to overcome the influence of "restoration procedure" on the efficiency of the algorithm. The other is to find a technology to solve the constrained optimization problem with nonnegative constraints by using a multidimensional filter method, just like solving the unconstrained optimization problem, so as to solve the NCP problem. Our proposal is to reformulate NCP as an optimization problem with nonnegative constraints and solve the optimization by a multidimensional filter method. The gradient-projection method [32] shows that the first-order optimality condition of the nonnegativity constrained optimization problem is equivalent to the fact that the projected gradient is zero, which is similar to the optimality condition of the unconstrained optimization problem. The characteristics of NCP make sure that the trust-region subproblem of the equivalent reformulation is a convex quadratic programming problem because the matrix B in the subproblem is a positive semidefinite symmetric matrix. Even we find that the twice continuously differentiability of $F(x)$ implies that B is uniformly bounded. It is very important for the global convergent analysis of our algorithm. Especially, the new algorithm does not depend on any extra restoration procedure because it always remains the compatibility of the trust-region subproblem.

This paper is organized as follows: the algorithm and preliminaries are introduced in Section 2 and the global convergence analysis for the proposed method is given in Section 3. Some numerical results are reported in Section 4. The final section gives some conclusions.

2. The Algorithm and Preliminaries

In this section, we will describe the specific strategies and motivations for solving problem (1) and finally present a multidimensional filter algorithm for the nonlinear complementarity problem.

2.1. Equivalent Model and Gradient-Projection Method. We consider using the Fischer–Burmeister function [33] to reformulate NCP as the following optimization problem and solve the optimization by a multidimensional filter method:

$$\min_{x\geq 0} \quad f_\mu(x) := \frac{1}{2}\Phi_\mu(x)^T\Phi_\mu(x), \tag{2}$$

where $\mu \geq 0$ is a smooth parameter, and $\Phi_\mu(x) = \left(\sqrt{x_1^2 + F_1^2(x) + \mu^2} - x_1 - F_1(x), \dots, \sqrt{x_n^2 + F_n^2(x) + \mu^2} - x_n - F_n(x)\right)^T$. For $\mu = 0$, we get the following equivalence relation [34]: x^* solves (1) $\Longleftrightarrow x^*$ solves (2). Nonnegative constraints in optimization problem (2) can help to avoid stationary points with negative components [2], but it brings some troubles to the multidimensional filter method [30, 31]. Therefore, we use a gradient-projection method [32] and define the "projected" gradient of $f_\mu(x)$ into the feasible set of problem (2) as follows:

$$\overline{g}_{\mu,i}(x) = \begin{cases} g_{\mu,i}(x), & x_i \geq g_{\mu,i}(x), \\ x_i, & x_i < g_{\mu,i}(x), \end{cases} \tag{3}$$

where $g_{\mu,i}(x)$ is the i-th component of the $g_\mu(x)$, $g_\mu(x) := \nabla_x f_\mu(x)$ and $i \in \mathcal{I}$.

The advantage of this strategy is obvious: x_μ^* is a KKT point of problem (2) if and only if $\overline{g}_\mu(x_\mu^*) = 0$. So, we can use a multidimensional filter method to solve problem (2) when μ approaches to zero just as we can solve unconstrained optimization problem.

2.2. Trust-Region Subproblem. To solve problem (2), we compute a trial step d_k by finding an approximation to the solution of the trust-region subproblem:

$$\begin{aligned} \min \quad & Q_k(d) = f_{\mu_k}(x_k) + \nabla_x f_{\mu_k}(x_k)^T d + \frac{1}{2}d^T B_k d, \\ \text{s.t.} \quad & x_k + d \geq 0, \quad \|d\|_\infty \leq \Delta_k, \end{aligned} \tag{4}$$

where x_k is the current iteration point, Δ_k is the trust-region radius, and $B_k = \nabla_x\Phi_{\mu_k}(x_k)\nabla_x\Phi_{\mu_k}(x_k)^T$, and $\nabla_x\Phi_\mu(x) := (b_{ij})_{nn}$, $b_{ij} = (\partial\Phi_{\mu,j}(x)/\partial x_i)$. The positive parameters μ_k tend to zero during the iterate of algorithm. A trial point x_k^+ is then computed by the trial step d_k; denote $x_k^+ = x_k + d_k$.

Subproblem (4) obviously has a solution $d = 0$ at least. Therefore, our new algorithm does not depend on any extra restoration procedure [19, 27–29] because it always remains the compatibility of the trust-region subproblem. Especially, we note that the characteristics of NCP make sure that the trust-region subproblem of the equivalent reformulation is a convex quadratic programming problem, which is comparatively easy to solve.

2.3. The Multidimensional Filter Mechanism. Whether the new trial point x_k^+ can be considered as a successful point requires the following multidimensional filter mechanism [30, 32] to assist in judgment. This mechanism helps the components of function $\overline{g}_\mu(x)$ to approach to zero evenly and provides an effective rule to judge whether x_k^+ is accepted. In practical computation, a iterate point x_k is said to dominate another point x_l if and only if $|\overline{g}_{\mu_{k-1},i}(x_k)| \leq |\overline{g}_{\mu_{l-1},i}(x_l)|$, $\forall i \in \mathcal{I}$. Besides, a filter set $\mathcal{F}$ is a set of points such that no pair dominates any other.

Acceptability rule: a new trial point x_k^+ is acceptable for the filter $\mathcal{F}_k$ if and only if

$$\forall x_l \in \mathcal{F}_k, \exists j \in \mathcal{I}, \left|\overline{g}_{\mu_k,j}(x_k^+)\right| \leq \left|\overline{g}_{\mu_{l-1},j}(x_l)\right| - \gamma_g\left\|\overline{g}_{\mu_{l-1}}(x_l)\right\|, \tag{5}$$

where $\gamma_g \in (0, (1/\sqrt{n}))$. If an iterate x_k is acceptable for the filter $\mathcal{F}_k$, we add it to the filter and remove from it every $x_l \in \mathcal{F}_k$ such that $|\overline{g}_{\mu_k,i}(x_k^+)| \leq |\overline{g}_{\mu_{l-1},i}(x)_l|$ for all $i \in \mathcal{I}$, i.e.,

$$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{x_k^+\} \backslash \mathscr{D}_k, \tag{6}$$

where

$$\mathscr{D}_k = \left\{ x_l \in \mathscr{F}_k \middle| \overline{g}_{\mu_k,i}(x_k^+) \le \overline{g}_{\mu_{l-1},i}(x_l) \right|, \forall i \in \mathscr{I} \right\}. \tag{7}$$

2.4. A Multidimensional Filter Algorithm for NCPs. In this part, we will present a multidimensional filter algorithm for nonlinear complementarity problem (1).

Algorithm 1. A multidimensional filter algorithm for NCPs.

Step 0: initialization. An initial point and an initial trust-region radius Δ_0 are given. Let an initial point x_0, an initial trust-region radius $\Delta_0 > 0$, and an initial filter set $\mathscr{F}_0 = (10^5, \ldots, 10^5)^T$ be given, as well as constants ε, $\gamma_g \in (0, (1/\sqrt{n}))$, $0 < \gamma_1 < \gamma_2 < 1 < \gamma_3$, $0 < \eta_1 < \eta_2 < 1$, $0 < \Delta_0 \le \Delta_{max}$. Compute $f_{\mu_0}(x_0)$, $g_{\mu_0}(x_0)$, $\overline{g}_{\mu_0}(x_0)$, B_0, set $k := 0$.

Step 1: test for optimality. If $\|\overline{g}_{\mu_k}(x_k)\| + \mu_k < \varepsilon$, stop.

Step 2: determine a trial step. Compute a solution d_k of subproblem (4).

Step 3: if $d_k = 0$, set $x_{k+1} = x_k, \mu_{k+1} = \theta\mu_k, B_{k+1} = B_k$, $k := k+1$, and go to Step 1; otherwise, set $x_k^+ = x_k + d_k$, and compute $f_{\mu_k}(x_k^+)$, $\overline{g}_{\mu_k}(x_k^+)$.

Step 4: test for optimality. If $\|\overline{g}_{\mu_k}(x_k^+)\| + \mu_k < \varepsilon$, stop; otherwise, compute

$$\rho_k = \frac{f_{\mu_k}(x_k) - f_{\mu_k}(x_k^+)}{Q_k(0) - Q_k(d_k)}. \tag{8}$$

Step 5: tests to accept the trial step.

(i) If $\rho_k \ge \eta_1$, set $x_{k+1} = x_k^+$, $\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{x_k^+\}\backslash\mathscr{D}_k$;
(ii) Besides, if $\rho_k < \eta_1$ and x_k^+ is acceptable for the filter $\mathscr{F}_k$, set $x_{k+1} = x_k^+$, $\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{x_k^+\}\backslash\mathscr{D}_k$,
(iii) Otherwise, set $x_{k+1} = x_k$, $\mathscr{F}_{k+1} = \mathscr{F}_k$.

Step 6: update the trust-region radius and the smooth parameter.

$$\Delta_{k+1} = \begin{cases} [\gamma_1\Delta_k, \gamma_2\Delta_k], & \text{if } \rho_k < \eta_1, \\ (\gamma_2\Delta_k, \Delta_k], & \text{if } \rho_k \in [\eta_1, \eta_2), \\ \min\{\Delta_{\max}, \gamma_3\Delta_k\}, & \text{if } \rho_k \ge \eta_2. \end{cases} \tag{9}$$

$$\mu_{k+1} = \begin{cases} \theta\mu_k, & \text{if } \mu_k > 0.1\left\|\overline{g}_{\mu_k}(x_{k+1})\right\|, \\ \mu_k, & \text{otherwise.} \end{cases} \tag{10}$$

Step 7: compute $f_{\mu_{k+1}}(x_{k+1})$, $g_{\mu_{k+1}}(x_{k+1})$, $\overline{g}_{\mu_{k+1}}(x_{k+1})$, B_{k+1}, set $k := k+1$, and go to Step 1.

Note that, there is an advantage to choosing a large $\Delta_{k+1} = \gamma_3\Delta_k$ when $\rho_k \ge \eta_2$, but it may be unwise to choose it to be too large; hence, we give a upper bound $\Delta_{\max}$ and set $\Delta_{k+1} = \min\{\Delta_{\max}, \gamma_3\Delta_k\}$ when $\rho_k \ge \eta_2$. Specifically, $B_{k+1} = \nabla_x\Phi_{\mu_{k+1}}(x_{k+1})\nabla_x\Phi_{\mu_{k+1}}(x_{k+1})^T$ can be computed easily instead of updating B_{k+1} with higher numerical expenditure (e.g., BFGS). By the way, from (8), we are surprised to find that the smooth parameter μ_k does not tend to zero before $\|\overline{g}_{\mu_k}(x_k)\|$ during our algorithm. In other words, $f_{\mu_k}(x)$ is twice continuously differentiable form beginning to end; this is a critical condition to our algorithm.

3. Analysis for Global Convergence

Global convergence properties of Algorithm 1 will be proved under the following assumptions.

A1. $F(x)$: $\mathfrak{R}^n \longrightarrow \mathfrak{R}^n$ is a twice continuously differentiable function.

A2. The iterates x_k remain in a closed, bounded convex domain Ω of $\mathfrak{R}^n$.

Note that, for subproblem (4), A1 and A2 together imply that B_k is uniformly bounded on Ω. In other words, there exist constants $\kappa_{\text{umb}} > 0$ such that $|B_k| \le \kappa_{\text{umb}}$, $\forall k$.

3.1. Well Definedness. Let d_k denote the solution of (4), then we have some technical lemmas which are very important for the global convergence of Algorithm 1.

Lemma 1. *If $d_k = 0$, we have $\overline{g}_{\mu_k}(x_k) = 0$.*

Proof. We first change the constraint $\|d\|_\infty \le \Delta_k$ to another form $d + \Delta_k e \ge 0$ and $d - \Delta_k e \le 0$, then we have the Lagrange function of (4):

$$\begin{aligned} L(d, \lambda, \overline{\mu}, \widehat{\mu}) = {} & Q_k(d) - \lambda^T(x_k + d) - \overline{\mu}^T(d + \Delta_k e) \\ & + \widehat{\mu}^T(d - \Delta_k e), \end{aligned} \tag{11}$$

where $\lambda, \overline{\mu}, \widehat{\mu} \in R^n$ are Lagrange multipliers, $e = (1, 1, \ldots, 1)^T \in \mathfrak{R}^n$. Since d_k is the solution of (4), we then obtain that

$$\begin{cases} \nabla_d L(d_k, \lambda, \overline{\mu}, \widehat{\mu}) = 0, \quad \lambda, \overline{\mu}, \widehat{\mu} \ge 0, \\ x_k + d_k \ge 0, \\ d_k + \Delta_k e \ge 0, \\ d_k - \Delta_k e \le 0, \\ \lambda^T(x_k + d_k) = \overline{\mu}^T(d_k + \Delta_k e) = \widehat{\mu}^T(d_k - \Delta_k e) = 0. \end{cases} \tag{12}$$

Observe that $d_k = 0$ and $\Delta_k > 0$ ensure that

$$\begin{cases} g_{\mu_k}(x_k) = \lambda + \overline{\mu} - \widehat{\mu}, \\ \lambda, x_k \ge 0, \\ \lambda^T x_k = 0, \\ \overline{\mu} = \widehat{\mu} = 0. \end{cases} \tag{13}$$

Thus, $x_k^T g_{\mu_k}(x_k) = 0$, i.e., $\overline{g}_{\mu_k}(x_k) = 0$. □

Lemma 2 (see [32]). *There exists a constant $\kappa_{mdc} \in (0, 1)$ such that*

$$Q_k(0) - Q_k(d_k) \geq \kappa_{\text{mdc}} \left\| \overline{g}_{\mu_k}(x_k) \right\| \min \left\{ \frac{\left\| \overline{g}_{\mu_k}(x_k) \right\|}{1 + [B_k]}, \Delta_k \right\}. \tag{14}$$

Lemma 3 (see [35]). *Suppose that A1 and A2 hold. If* $\overline{g}_{\mu_k}(x_k) \neq 0$ *and*

$$\Delta_k \leq \min \left\{ \frac{\kappa_{\text{mdc}}(1-\eta_2)}{\kappa_{\text{ubh}}}, \frac{1}{1+\kappa_{\text{umb}}} \right\} \cdot \left\| \overline{g}_{\mu_k}(x_k) \right\|, \tag{15}$$

we have $\rho_k \geq \eta_2$, *and* $\Delta_{k+1} \geq \Delta_k$.

Consequently, we may now obtain that the trust-region radius cannot become arbitrarily small if the iterates stay away from first-order critical points.

Lemma 4. *Suppose that A1 and A2 hold. Suppose furthermore that there exists a constant* $\kappa_{lpg} > 0$ *such that* $\|\overline{g}_{\mu_k}(x_k)\| \geq \kappa_{lpg}$ *for all k. Then, there is a constant* $\kappa_{lbd} > 0$ *such that* $\Delta_k \geq \kappa_{lbd}$ *for all k.*

Proof. Assume that iteration k_0 is the first such that

$$\Delta_{k_0} \leq \gamma_1 \kappa_{\text{lpg}} \cdot \min \left\{ \frac{\kappa_{\text{mdc}}(1-\eta_2)}{\kappa_{\text{ubh}}}, \frac{1}{1+\kappa_{\text{umb}}} \right\}. \tag{16}$$

Then, we have from our assumption $\|\overline{g}_{\mu_k}(x_k)\| \geq \kappa_{\text{lpg}}$ and (9) that $\gamma_1 \Delta_{k-1} \leq \Delta_k$, $\forall k$, and hence, $\Delta_{k_0-1} \leq \min\{(\kappa_{\text{mdc}}(1-\eta_2)/\kappa_{\text{ubh}}), (1/1+\kappa_{\text{umb}})\} \cdot \|\overline{g}_{\mu_{k_0-1}}(x_{k_0-1})\|$. It implies that (15) holds and thus $\Delta_{k_0-1} \leq \Delta_{k_0}$. But this contradicts the fact that iteration k_0 is the first such that (16) holds, and our initial assumption is therefore impossible. □

3.2. Convergence to Stationary Points. For convenience of discussion, we shall denote $\mathcal{T} = \{k \mid d_k = 0\}$ as the set of zero-solution of (4) iterations, denote $\mathcal{S} = \{k \mid x_{k+1} \neq x_k\}$ as the set of successful iterations, denote $\mathcal{U} = \{k \mid \rho_k < \eta_1,\ x_k^+ \text{ is not accepted by } \mathcal{F}_k\}$ as the set of unsuccessful iterations, and denote $\mathcal{P} = \{k \mid \rho_k < \eta_1,\ x_k^+ \text{ is accepted by } \mathcal{F}_k\}$ as the set of iterations which is not accepted by trust-region rule but filter rule (5). Now, we consider the first-order global convergent conclusion of our algorithm in the following three cases: $|\mathcal{T}| = +\infty$, $|\mathcal{U}| = +\infty$, and $|\mathcal{S}| = +\infty$.

Theorem 1. *Suppose that A1 and A2 hold and that* $|\mathcal{T}| = +\infty$, *then* $\lim_{k\in\mathcal{T}} (\|\overline{g}_{\mu_k}(x_k)\| + \mu_k) = 0$.

Proof. Note that, if $|\mathcal{T}| = +\infty$, Lemma 1 implies that $\overline{g}_{\mu_k}(x_k) = 0$. Moreover, by the mechanism of updating μ_k (10), we have $\mu_{k+1} = \theta \mu_k$. Then, we get the first-order global convergence of our Algorithm 1 $\lim_{k\in\mathcal{T}} (\|\overline{g}_{\mu_k}(x_k)\| + \mu_k) = 0$, i.e., Algorithm 1 can be terminated with finite steps. □

Theorem 2. *Suppose that A1 and A2 hold and that* $|\mathcal{U}| = +\infty$, *then* $\lim_{k\in\mathcal{U}} (\|\overline{g}_{\mu_k}(x_k)\| + \mu_k) = 0$.

Proof. Assume, to arrive at a contradiction, that there exist $\varepsilon > 0$ and k_0 such that

$$\left\| \overline{g}_{\mu_k}(x_k) \right\| + \mu_k \geq \varepsilon, \quad \forall k > k_0. \tag{17}$$

Since $|\mathcal{U}| = +\infty$, there exists a positive integer $k_1 > k_0$ such that

$$x_k = x_{k_1}, \quad \forall k \geq k_1. \tag{18}$$

By the mechanism of updating μ_k (10), we can consider two cases. □

Case 1. $\lim_{k\longrightarrow\infty} \mu_k \neq 0$, i.e., there exists a positive integer $k_2 \geq k_1$ such that $\mu_k = \mu_{k_2}$ for all $k \geq k_2$. By Algorithm 1, we obtain that

$$\begin{aligned} &\Delta_k \leq \gamma_2^{k-k_2} \Delta_{k_2}, \\ &\mu_k \leq 0.1 \left\| \overline{g}_{\mu_k}(x_{k+1}) \right\|, \end{aligned} \tag{19}$$

for all $k \geq k_2$. Hence, we can deduce from (18) and (19) that $\|\overline{g}_{\mu_k}(x_k)\| \geq \kappa_{\text{lpg}}$ for all $k \geq k_2$, where $\kappa_{\text{lpg}} = 10\mu_{k_2}$. Note that $\gamma_2 \in (0,1)$ in Algorithm 1, so we have $\lim_{k\longrightarrow\infty} \Delta_k = 0$. Applying Lemma 4, there is a constant $\kappa_{\text{lbd}} > 0$ such that $\Delta_k \geq \kappa_{\text{lbd}}$ for all k, which contradicts the fact $\lim_{k\longrightarrow\infty} \Delta_k = 0$, as stated in (19). Therefore, Case 1 cannot hold.

Case 2. $\lim_{k\longrightarrow\infty} \mu_k = 0$. We deduce from (17) that there is a integer $k_3 \geq k_1$ such that $\|\overline{g}_{\mu_k}(x_k)\| \geq 0.5\varepsilon \geq 10\mu_k$ for all $k \geq k_3$, i.e., $\mu_k \leq 0.1\|\overline{g}_{\mu_k}(x_k)\| = 0.1\|\overline{g}_{\mu_k}(x_{k+1})\|$, for all $k \geq k_3$. By the mechanism of updating μ_k, then we have $\mu_k = \mu_{k_3}$ for all $k \geq k_3$, which contradicts $\lim_{k\longrightarrow\infty} \mu_k = 0$. Thus, Case 2 cannot hold. Our initial assumption must then be false.

Theorem 3. *Suppose that A1 and A2 hold and that* $|\mathcal{S}| = +\infty$, *then* $\lim_{k\in\mathcal{S}} (\|\overline{g}_{\mu_k}(x_k)\| + \mu_k) = 0$.

Proof. Assume, to arrive at a contradiction, that there exist $\varepsilon > 0$ and k_0 such that

$$\left\| \overline{g}_{\mu_k}(x_k) \right\| + \mu_k \geq \varepsilon, \quad \forall k > k_0. \tag{20}$$

By the mechanism of updating μ_k (10), we can consider two cases. □

Case 3. $\lim_{k\longrightarrow\infty} \mu_k \neq 0$, i.e., there exists a positive integer $k_1 > k_0$ such that $\mu_k = \mu_{k_1}$ for all $k \geq k_1$. Observe first that (10) implies that $\mu_k \leq 0.1\|\overline{g}_{\mu_k}(x_{k+1})\| = 0.1\|\overline{g}_{\mu_{k+1}}(x_{k+1})\|$, $\forall k \geq k_1$. Let $\kappa_{\text{lpg}} = 10\mu_{k_1}$, then

$$\left\| \overline{g}_{\mu_k}(x_k) \right\| \geq \kappa_{\text{lpg}}, \quad \forall k > k_1. \tag{21}$$

(i) $|\mathcal{P}| < +\infty$ (i.e. $|\mathcal{S} \backslash \mathcal{P}| = +\infty$). That means there exists a positive integer $k_2 \geq k_1$ such that either $k \in \mathcal{S} \backslash \mathcal{P}$ or $k \in \overline{\mathcal{S}}$ for all $k \geq k_2$. Besides, we can deduce from $|\mathcal{S}| = +\infty$ that $\rho_k \geq \eta_1$ for all $k \in \mathcal{S} \backslash \mathcal{P}$. Then, by (9), we have that for all $k > k_2$,

$$\begin{aligned} & f_{\mu_{k_2}}(x_{k_2}) - f_{\mu_k}(x_k) \\ & = \sum_{j=k_2, j\in\mathcal{S}\setminus\mathcal{P}}^{k-1} \left(f_{\mu_j}(x_j) - f_{\mu_{j+1}}(x_{j+1}) \right) + \sum_{j=k_2, j\in\overline{\mathcal{S}}}^{k-1} \left(f_{\mu_j}(x_j) - f_{\mu_{j+1}}(x_{j+1}) \right) \\ & = \sum_{j=k_2, j\in\mathcal{S}\setminus\mathcal{P}}^{k-1} \left(f_{\mu_j}(x_j) - f_{\mu_{j+1}}(x_{j+1}) \right) \geq \sum_{j=k_2, j\in\mathcal{S}/\mathcal{P}}^{k-1} \eta_1 \left(Q_j(0) - Q_j(d_j) \right) \\ & \geq \eta_1 \kappa_{\text{mdc}} \kappa_{\text{lpg}} \cdot \sum_{j=k_2, j\in\mathcal{S}/\mathcal{P}}^{k-1} \min\left\{ \frac{\kappa_{\text{lpg}}}{1+\kappa_{\text{umb}}}, \Delta_j \right\}. \end{aligned} \tag{22}$$

Hence, we deduce from the boundedness of $\{f_{\mu_k}(x_k)\}$ that $\lim_{k>k_2}\Delta_k = 0$. But it derives a contradiction from (21) and Lemma 3.

(ii) $|\mathcal{P}| = +\infty$. Assumptions A1 and A2 show that $\{\|\overline{g}_{\mu_k}(x_{k+1})\|\}$ is a bounded subsequence. Then, there exists an infinite subsequence $\{k_i\}_{k_i>k_1} \subseteq \mathcal{P}$ such that

$$\lim_{i\longrightarrow\infty} \left\| \overline{g}_{\mu_{k_i}}(x_{k_i+1}) \right\| = \|\overline{g}_\infty\| \geq \kappa_{\text{lpg}}. \tag{23}$$

By definition of $k_i \in \mathcal{P}$, x_{k_i+1} is acceptable for the current filter $\mathcal{F}_{k_i}$. This implies, by (21) and filter mechanism (5), that, for each $x_k \in \mathcal{F}_{k_i}$, there exists an index $j(k) \in \mathcal{I}$ such that

$$\begin{aligned} \left| \overline{g}_{\mu_{k_i}, j(k)}(x_{k_i+1}) \right| & \leq \left| \overline{g}_{\mu_{k-1}, j(k)}(x_k) \right| - \gamma_g \left\| \overline{g}_{\mu_{k-1}}(x_k) \right\| \\ & = \left| \overline{g}_{\mu_{k-1}, j(k)}(x_k) \right| - \gamma_g \left\| \overline{g}_{\mu_k}(x_k) \right\| \\ & \leq \left| \overline{g}_{\mu_{k-1}, j(k)}(x_k) \right| - \gamma_g \kappa_{\text{lpg}}. \end{aligned} \tag{24}$$

If $x_{k_{i-1}+1} \notin \mathcal{F}_{k_i}$, we obtain from (6) and (7) that there exists $x_{k_s} \in \mathcal{F}_{k_i}$ such that $|\overline{g}_{\mu_{k_s-1}, j}(x_{k_s})| \leq |\overline{g}_{\mu_{k_{i-1}}, j}(x_{k_{i-1}+1})|$, $\forall j \in \mathcal{I}$. Thus, (24) ensures that there exists an index $j(k_s) \in \mathcal{I}$ such that

$$\left| \overline{g}_{\mu_{k_i}, j(k_s)}(x_{k_i+1}) \right| - \left| \overline{g}_{\mu_{k_{i-1}}, j(k_s)}(x_{k_{i-1}+1}) \right| \leq -\gamma_g \kappa_{\text{lpg}}. \tag{25}$$

If $x_{k_{i-1}+1} \in \mathcal{F}_{k_i}$, (23) implies that there exists an index $j(k_{i-1}+1) \in \mathcal{I}$ such that

$$\left| \overline{g}_{\mu_{k_i}, j(k_{i-1}+1)}(x_{k_i+1}) \right| - \left| \overline{g}_{\mu_{k_{i-1}}, j(k_{i-1}+1)}(x_{k_{i-1}+1}) \right| \leq -\gamma_g \kappa_{\text{lpg}}. \tag{26}$$

Since the finite possibility of $j(k_{i-1}+1)$, let $j(k_{i-1}+1) = j_0$. Then, the left-hand side of inequality (25) and (26) tends to zero when i tends to infinity because of (23), which is impossible. In a word, Case 3 does not happen.

Case 4. $\lim_{k\longrightarrow\infty} \mu_k = 0$. The infiniteness of $|\mathcal{S}|$ and (10) implies that there exists a infinite index $\mathcal{K} \subseteq \mathcal{S}$ such that $\lim_{k\in\mathcal{K}}\|\overline{g}_{\mu_k}(x_{k+1})\| = \lim_{k\in\mathcal{K}}\|\overline{g}_{\mu_k}(x_k^+)\| = 0$. We then obtain that $\lim_{k\in\mathcal{K}}(\|\overline{g}_{\mu_k}(x_k^+)\| + \mu_k) = 0$, which contradicts (20), and therefore, Case 4 cannot hold, yielding the desired result.

4. Numerical Experiments

Now, we give some numerical results for the following 10 complementarity test problems in Table 1. The values for the constants used in our tests are $\mu_0 = 10^{-5}$, $\gamma_g = 10^{-3}$, $\gamma_1 = 0.25$, $\gamma_3 = 2$, $\eta_1 = 0.25$, $\eta_2 = 0.95$, $\Delta_0 = 2$, $\theta = 0.1$, $\Delta_{\max} = 10^3$, $\varepsilon = 10^{-5}$, $\mathcal{F}_0 = (10^5, \ldots, 10^5)^T$. The iteration is terminated once $\|f_{\mu_k}(x_k)\| \leq 10^{-5}$.

In addition, we use the nonlinear equations to carry out large-scale data experiments. Let $p(x) = 0$ be a (large-scale) differentiable system of nonlinear equations and let $x^* \in R^n$ be defined by $x^* = (1, 0, 1, 0, \ldots)^T$. For all $i \in \mathcal{I}$, set

$$F_i(x) = \begin{cases} p_i(x) - p_i(x^*), & \text{if } i \text{ odd or, } i > r, \\ p_i(x) - p_i(x^*) + 1, & \text{otherwise,} \end{cases} \tag{27}$$

where $r \geq 0$ is a given integer. In this way, x^* is a solution of the nonlinear complementarity (but not necessarily its unique solution). As done in [36], we used the collection of 6 large-scale problems (Examples 5–10) form Lukšan [37]. Some numerical results for these test problems are presented in Table 1.

Example 1. (Kojima–Shindo nonlinear complementarity test problem)

(i) Degenerate example [7, 10, 11, 27, 38, 39]:

$$F(x) = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{pmatrix}. \tag{28}$$

(ii) Nondegenerate example [10, 38]:

Table 1: Numerical result of test problems.

Example	Start point x_0	n	iter	Resf
1.1	$(0,0,0,0)^T$	4	7	$4.43e-12$
1.2	$(0,0,0,0)^T$	4	9	$7.98e-15$
2	$(3,2,1,2,3)^T$	5	2	$1.43e-13$
3.1	$(2,\dots,2)^T$	10^3	8	$1.66e-11$
3.2	$(2,\dots,2)^T$	10^3	5	$1.05e-14$
4.1	$(0.5,\dots,0.5)^T$	10^3	5	$6.60e-14$
4.2	$(0,\dots,0)^T$	500	5	$5.78e-10$
4.3	$(0,\dots,0)^T$	80	9	$1.92e-08$
4.4	$(1,\dots,1)^T$	10^3	3	$8.00e-16$
5	—	100	16	$1.65e-08$
5	—	10^3	72	$1.63e-10$
6	$(1,0,1,0,\dots)^T$	100	5	$4.03e-15$
6	$(1,0,1,0,\dots)^T$	10^3	6	$6.23e-14$
7	$(0,\dots,0)^T$	100	14	$1.49e-11$
7	$(0,\dots,0)^T$	10^3	25	$1.94e-10$
8	$(-1.2,1,-1.2,1,\dots)^T$	100	1	$3.97e-24$
8	$(-1.2,1,-1.2,1,\dots)^T$	10^3	3	$9.25e-15$
9	—	100	15	$3.61e-12$
9	—	10^3	15	$3.04e-11$
10	$(-1,\dots,-1)^T$	100	7	$5.53e-10$
10	$(-1,\dots,-1)^T$	10^3	7	$4.35e-19$

$$F(x)=\begin{pmatrix} 3x_1^2+2x_1x_2+2x_2^2+x_3+3x_4-6 \\ 2x_1^2+x_1+x_2^2+3x_3+2x_4-2 \\ 3x_1^2+x_1x_2+2x_2^2+2x_3+3x_4-1 \\ x_1^2+3x_2^2+2x_3+3x_4-3 \end{pmatrix}. \tag{29}$$

Example 1 (i) has a degenerate solution $((\sqrt{6}/2),0,0,(1/2))^T$ and a nondegenerate solution $(1,0,3,0)^T$. Example 1 (ii) has only one solution $((\sqrt{6}/2),0,0,(1/2))^T$ which is nondegenerate.

Example 2 (Kanzow nonlinear complementarity test problem [10, 40]). We choose $F=(F_1,\dots,F_n)^T\colon \mathfrak{R}^n \longmapsto \mathfrak{R}^n$ with

$$F_i(x)=2(x_i-i+2)\cdot \exp\left\{\sum_{i=1}^{5}(x_i-i+2)^2\right\},\quad 1\le i\le 5. \tag{30}$$

Example 2 has only one solution $(0,0,1,2,3)^T$.

Example 3 (see [41]). *We choose* $F=(F_1,\dots,F_n)^T$: $\mathfrak{R}^n \longmapsto \mathfrak{R}^n$ *with*

$$F_i(x)=-x_{i-1}+2x_i-x_{i+1}+\frac{1}{3}x_i^3-b_i,\quad 1\le i\le n, \tag{31}$$

where n is a positive integer and $x_0=x_{n+1}=0$. We choose the constant $b=((-1)^1,\dots,(-1)^i,\dots,(-1)^n)^T$ and $b=((-1)^1\sqrt{1},\dots,(-1)^i\sqrt{i},\dots,(-1)^n\sqrt{n})^T$, respectively.

Example 4. We consider the following four linear complementarity problem [7, 11, 27, 39, 42–44]: $F(x)=M_ix+q$, $i=1,2,3,4$, where $q=(-1,\dots,-1)^T$, and M_1,M_2,M_3,M_4 are given as follows, respectively:

$$\begin{pmatrix} 4 & -2 & & \\ 1 & 4 & \ddots & \\ & \ddots & \ddots & -2 \\ & & 1 & 4 \end{pmatrix}, \quad \begin{pmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{pmatrix}, \quad \begin{pmatrix} (1/n) & & & \\ & (2/n) & & \\ & & \ddots & \\ & & & (n/n) \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ & 1 & 2 & \dots & 2 \\ & & 1 & \dots & 2 \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}. \tag{32}$$

Example 5. Countercurrent reader problem [37]:

$$p_k(x)=\begin{cases} \alpha-(1-\alpha)x_{k+2}-x_k(1+4x_{k+1}), & \text{if } k=1, \\ -(2-\alpha)x_{k+2}-x_k(1+4x_{k-1}), & \text{if } k=2, \\ \alpha x_{k-2}-(1-\alpha)x_{k+2}-x_k(1+4x_{k+1}), & \text{if } k\in(2,n-1) \text{ and } \operatorname{mod}(k,2)=1, \\ \alpha x_{k-2}-(2-\alpha)x_{k+2}-x_k(1+4x_{k-1}), & \text{if } k\in(2,n-1) \text{ and } \operatorname{mod}(k,2)=1, \\ \alpha x_{k-2}-x_k(1+4x_{k+1}), & \text{if } k=n-1, \\ \alpha x_{k-2}-(2-\alpha)x_k(1+4x_{k-1}), & \text{if } k=n, \end{cases} \tag{33}$$

where $\alpha = (1/2)$. We choose the following initial point $\overline{x}$:

$$\overline{x}_k = \begin{cases} 0.1, & \text{if } \bmod(k,8) = 1, \\ 0.2, & \text{if } \bmod(k,8) = 2 \text{ and } \bmod(k,8) = 0, \\ 0.3, & \text{if } \bmod(k,8) = 3 \text{ and } \bmod(k,8) = 7, \\ 0.4, & \text{if } \bmod(k,8) = 4 \text{ and } \bmod(k,8) = 6, \\ 0.5, & \text{if } \bmod(k,8) = 5. \end{cases} \quad (34)$$

Example 6. Extended Powell badly scaled function [37]:

$$p_k(x) = \begin{cases} 10000 x_k x_{k+1}, & \text{if } \bmod(k,2) = 1, \\ e^{-x_{k-1}} + e^{-x_k} - 1.0001, & \text{if } \bmod(k,2) = 0. \end{cases} \quad (35)$$

Example 7. Trigonometric exponential system [37]:

$$p_k(x) = \begin{cases} 3x_k^3 + 2x_{k+1} - 5 + \sin(x_k - x_{k+1})\sin(x_k + x_{k+1}), & \text{if } k = 1, \\ 3x_k^3 + 2x_{k+1} - 5 + \sin(x_k - x_{k+1})\sin(x_k + x_{k+1}), & \\ +4x_k - x_{k-1}e^{x_{k-1} - x_k}, & \text{if } k \in (1,n), \\ 4x_k - x_{k-1}e^{x_{k-1} - x_k}, & \text{if } k = n. \end{cases} \quad (36)$$

Example 8. Extended Rosenbrock function [37]:

$$p_k(x) = \begin{cases} 10(x_{k+1} - x_k^2), & \text{if } \bmod(k,2) = 1, \\ 1 - x_{k-1}, & \text{if } \bmod(k,2) = 0. \end{cases} \quad (37)$$

Example 9. Extended Cragg and Levy Function [37].

$$p_k = \begin{cases} (e^{x_k} - x_{k+1})^2, & \text{if } \bmod(k,4) = 1, \\ 10(x_k - x_{k+1})^3, & \text{if } \bmod(k,4) = 2, \\ \tan^2(x_k - x_{k+1}), & \text{if } \bmod(k,4) = 3, \\ x_k - 1, & \text{if } \bmod(k,4) = 0. \end{cases} \quad (38)$$

We choose the following initial point $\overline{x}$ as $\overline{x}_k = \begin{cases} 1, & \text{if } \bmod(k,4) = 1, \\ 2, & \text{otherwise.} \end{cases}$

Example 10. Broyden tridiagonal problem [37]:

$$p_k(x) = \begin{cases} (3 - 2x_k)x_k - 2x_{k+1} + 1, & \text{if } k = 1, \\ (3 - 2x_k)x_k - x_{k-1} - 2x_{k+1} + 1, & \text{if } k \in (1,n), \\ (3 - 2x_k)x_k - x_{k-1} + 1, & \text{if } k = n. \end{cases} \quad (39)$$

The computational results are listed in Table 1, in which iter denotes the number of iterations, and Resf stands for the computing accuracy, i.e., Resf $= f_{\mu_k}(x_k)$. The numerical results show that Algorithm 1 is robust and efficient. The number of iterations and computing accuracy for most problems are satisfactory.

5. Conclusions

In this work, we have proposed multidimensional filter techniques for solving nonlinear complementarity problem (1) and have shown this algorithm to be globally convergent under a weaker assumption because assumptions A1 and A2 imply that the matrix sequence $\{B_k\}$ in subproblem (4) is uniformly bounded. Moreover, we are surprised to find that $B_{k+1} = \nabla_x \Phi_{\mu_k}(x_k) \nabla_x \Phi_{\mu_k}(x_k)^T$ can be computed easily instead of updating B_{k+1} by utilizing some methods (e.g., BFGS). The new algorithm differs from other traditional filter methods [7, 27, 39] for nonlinear complementarity problems; subproblem (4) is consistent throughout, so we do not need any extra restoration procedure which means higher numerical expenditure. Besides, we used the gradient-projection technique which makes sure that the optimality condition of constrained optimization problem (2) is equivalent to the fact that the projected gradient is zero, so the multidimensional filter techniques based on unconstrained optimization problem is suitable for problem (2). Finally, in this context, we provide a reasonable and effective way to balance the projected gradient $\overline{g}_{\mu_k}(x_k)$ and the smooth parameter μ_k and then ensure that $\liminf_{k \longrightarrow +\infty} \|\overline{g}_{\mu_k}(x_k)\| + \mu_k = 0$. The results of numerical experiments show its efficiency.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest in this work.

Acknowledgments

The authors thank the anonymous reviewers for the careful reading of the manuscript and their valuable comments. This work was supported by the National Defence Pre-Research Foundation of China.

References

[1] A. Fischer, "Solution of monotone complementarity problems with locally Lipschitzian functions," *Mathematical Programming*, vol. 76, no. 3, pp. 513–532, 1997.

[2] A. Fischer, "New constrained optimization reformulation of complementarity problems," *Journal of Optimization Theory and Applications*, vol. 97, no. 1, pp. 105–117, 1998.

[3] C. Geiger and C. Kanzow, "On the resolution of monotone complementarity problems," *Computational Optimization and Applications*, vol. 5, no. 2, pp. 155–173, 1996.

[4] M. V. Solodov, "Stationary points of bound constrained minimization reformulations of complementarity problems," *Journal of Optimization Theory and Applications*, vol. 94, no. 2, pp. 449–467, 1997.

[5] N. Yamashita, "Properties of restricted NCP functions for nonlinear complementarity problems," *Journal of Optimization Theory and Applications*, vol. 98, no. 3, pp. 701–717, 1998.

[6] O. L. Mangasarian and M. V. Solodov, "Nonlinear complementarity as unconstrained and constrained minimization," *Mathematical Programming*, vol. 62, no. 1-3, pp. 277–297, 1993.

[7] H. Wang and D. Pu, "A kind of nonmonotone filter method for nonlinear complementarity problem," *Journal of Applied Mathematics and Computing*, vol. 36, no. 1-2, pp. 27–40, 2011.

[8] M. Huang and D. Pu, "A trust-region SQP method without a penalty or a filter for nonlinear programming," *Journal of Computational and Applied Mathematics*, vol. 281, pp. 107–119, 2015.

[9] K. Su, X. Lu, and W. Liu, "An improved filter method for nonlinear complementarity problem," *Mathematical Problems in Engineering*, vol. 2013, Article ID 450829, 7 pages, 2013.

[10] X. Fan, T. Xu, and F. Gao, "Solving nonlinear complementarity problem by a smoothing homotopy method," *Taiwanese Journal of Mathematics*, vol. 19, no. 1, pp. 51–63, 2015.

[11] B. Fan, C. Ma, A. Wu, and C. Wu, "A Levenberg-Marquardt method for nonlinear complementarity problems based on nonmonotone trust region and line search techniques," *Mediterranean Journal of Mathematics*, vol. 15, pp. 118–136, 2018.

[12] J. Liu and S. Li, "Unconstrained optimization reformulation for stochastic nonlinear complementarity problems," *Applicable Analysis*, pp. 1–22, 2019.

[13] H. Zheng and L. Liu, "The sign-based methods for solving a class of nonlinear complementarity problems," *Journal of Optimization Theory and Applications*, vol. 180, no. 2, pp. 480–499, 2019.

[14] H. Zheng, "Improved convergence theorems of modulus-based matrix splitting iteration method for nonlinear complementarity problems of H-matrices," *Calcolo*, vol. 54, no. 4, pp. 1481–1490, 2017.

[15] S.-L. Xie, H.-R. Xu, and J.-P. Zeng, "Two-step modulus-based matrix splitting iteration method for a class of nonlinear complementarity problems," *Linear Algebra and Its Applications*, vol. 494, pp. 1–10, 2016.

[16] H. Zheng and L. Liu, "A two-step modulus-based matrix splitting iteration method for solving nonlinear complementarity problems of H_+-matrices," *Computational and Applied Mathematics*, vol. 37, no. 4, pp. 5410–5423, 2018.

[17] C. Ma and N. Huang, "Modified modulus-based matrix splitting algorithms for a class of weakly nondifferentiable nonlinear complementarity problems," *Applied Numerical Mathematics*, vol. 108, pp. 116–124, 2016.

[18] R. Li and J.-F. Yin, "Accelerated modulus-based matrix splitting iteration methods for a restricted class of nonlinear complementarity problems," *Numerical Algorithms*, vol. 75, no. 2, pp. 339–358, 2017.

[19] R. Fletcher and S. Leyffer, "Nonlinear programming without a penalty function," *Mathematical Programming*, vol. 91, no. 2, pp. 239–269, 2002.

[20] C. Jiang, A. Zada, M. T. Senel, and T. Li, "Synchronization of bidirectional N-coupled fractional-order chaotic systems with ring connection based on antisymmetric structure," *Advances in Difference Equations*, vol. 2019, no. 1, pp. 456–471, 2019.

[21] A. Yang, L. Li, Z. Wang, and R. Guo, "Tracking control of a class of chaotic systems," *Symmetry*, vol. 11, no. 4, pp. 568–579, 2019.

[22] R. Xu and F. Zhang, "ε-Nash mean-field games for general linear-quadratic systems with applications," *Automatica*, vol. 114, pp. 108835–108840, 2020.

[23] A. Wächter and L. T. Biegler, "On the implementation of an interior-point filter linesearch algorithm for large scale nonlinear programming," *Mathematical Programming*, vol. 106, no. 1, pp. 25–57, 2006.

[24] R. Fletcher, S. Leyffer, and P. L. Toint, "On the global convergence of a filter—SQP algorithm," *SIAM Journal on Optimization*, vol. 13, no. 1, pp. 44–59, 2002.

[25] S. Ulbrich, "On the superlinear local convergence of a filter-SQP methods," *Mathematical Programming*, vol. 100, pp. 217–245, 2004.

[26] J. Long and S. Zeng, "A projection-filter method for solving nonlinear complementarity problems," *Applied Mathematics and Computation*, vol. 216, no. 1, pp. 300–307, 2010.

[27] P.-y. Nie, "A filter method for solving nonlinear complementarity problems," *Applied Mathematics and Computation*, vol. 167, no. 1, pp. 677–694, 2005.

[28] R. Fletcher, N. I. M. Gould, S. Leyffer, P. L. Toint, and A. Wächter, "Global convergence of a trust-region SQP-filter algorithm for general nonlinear programming," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 635–659, 2002.

[29] P. S. Ferreira, E. W. Karas, M. Sachine, and F. N. C. Sobral, "Global convergence of a derivative-free inexact restoration filter algorithm for nonlinear programming," *Optimization*, vol. 66, no. 2, pp. 271–292, 2017.

[30] N. I. M. Gould, C. Sainvitu, and P. L. Toint, "A filter-trust-region method for unconstrained optimization," *SIAM Journal on Optimization*, vol. 16, no. 2, pp. 341–357, 2005.

[31] N. I. M. Gould, S. Leyffer, and P. L. Toint, "A multidimensional filter algorithm for nonlinear equations and nonlinear least-squares," *SIAM Journal on Optimization*, vol. 15, no. 1, pp. 17–38, 2004.

[32] C. Sainvitu and P. L. Toint, "A filter-trust-region method for simple-bound constrained optimization," *Optimization Methods and Software*, vol. 22, no. 5, pp. 835–848, 2007.

[33] C. Kanzow and H. Pieper, "Jacobian smoothing methods for nonlinear complementarity problems," *SIAM Journal on Optimization*, vol. 9, no. 2, pp. 342–373, 1999.

[34] J. Zhu and B. Hao, "A new smoothing method for solving nonlinear complementarity problems," *Open Mathematics*, vol. 17, no. 1, pp. 104–119, 2019.

[35] A. R. Conn, N. I. M. Gould, and P. L. Toint, *Trust Region Method*, MPS and SIAM, Philadelphia, PA, USA, 2000.

[36] F. Facchinei and C. Kanzow, "A nonsmooth inexact Newton method for the solution of large-scale nonlinear complementarity problems," *Mathematical Programming*, vol. 76, no. 3, pp. 493–512, 1997.

[37] L. Lukšan, "Inexact trust region method for large sparse systems of nonlinear equations," *Journal of Optimization Theory and Applications*, vol. 81, pp. 569–590, 1994.

[38] J. D. Ding and H. Y. Yin, "A new homotopy method for nonlinear complementarity problems," *Numerical Mathematics Theory Methods and Applications*, vol. 16, pp. 155–163, 2007.

[39] K. Su and H.-p. Cai, "A modified SQP-filter method for nonlinear complementarity problem," *Applied Mathematical Modelling*, vol. 33, no. 6, pp. 2890–2896, 2009.
[40] T. Ni and P. Wang, "A smoothing-type algorithm for solving nonlinear complementarity problems with a non-monotone line search," *Applied Mathematics and Computation*, vol. 216, no. 7, pp. 2207–2214, 2010.
[41] C. Huang and S. Wang, "A power penalty approach to a nonlinear complementarity problem," *Operations Research Letters*, vol. 38, no. 1, pp. 72–76, 2010.
[42] H. Yu and D. Pu, "Smoothing Levenberg-Marquardt method for general nonlinear complementarity problems under local error bound," *Applied Mathematical Modelling*, vol. 35, no. 3, pp. 1337–1348, 2011.
[43] J.-l. Zhang and X. Zhang, "A smoothing Levenberg-Marquardt method for NCP," *Applied Mathematics and Computation*, vol. 178, no. 2, pp. 212–228, 2006.
[44] H. Zheng, W. Li, and S. Vong, "An iteration method for nonlinear complementarity problems," *Journal of Computational and Applied Mathematics*, vol. 372, Article ID 112681, 2020.