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Research Article

Property and Representation of *n*-Order Pythagorean Matrix

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Here we study the character and expression of n-order Pythagorean matrix using number theory. Theories of Pythagorean matrix are obtained. Using related algebra skills, we prove that the set which constitutes all n-order Pythagorean matrices is a finitely generated group of matrix multiplication and gives a generated tuple of this finitely generated group ($n \leq 10$) simultaneously.

1. Introduction and Theme

If integers $a, b,$ and c satisfy $a^2 + b^2 = c^2$, then we call $\{a, b, c\}$ a Pythagorean array; if Pythagorean array is written in vector form, then we call it a Pythagorean vector [1]. A Pythagorean vector is called primitive [2] if and only if $a, b,$ and c are coprime.

It is well known that every Pythagorean vector is either of the form $((k(m^2 - n^2))2kmn(k(m^2 + n^2)))$ or of the form $(2kmn(k(m^2 - n^2))(k(m^2 + n^2)))$ with $k, m, n \in Z$. Frisch and Vaserstein [3] pointed that there exists a parametrization of Pythagorean vectors by a single triple of integer-valued polynomials.

Estimates for the number of Pythagorean vectors with a given constraint are studied in [4–6]. Benito and Varona [4] found asymptotic estimates for the number of Pythagorean vectors with legs less than n. Omland [5] obtained the number of Pythagorean vectors with a given inradius. Okagbue et al. [6] gave statistical and algebraic properties of primitive Pythagorean vectors from the first 331 primitive Pythagorean vectors.

For any fixed primitive Pythagorean vector (a, b, c) such that $a^2 + b^2 = c^2$, Jesmanowicz' [7] studied the Diophantine equation $a^x + b^y = c^z$ and conjectured the equation has a unique solution. The authors of [8–11] obtained some conclusions on Jesmanowicz's conjecture.

The authors of [12–14] constructed the following three interesting matrices and obtained the following theorem.

Theorem 1. *If* $F_1 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$, $F_2 = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$, $F_3 = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{pmatrix}$, *$(a, b, c)$ is a 3-dimensional Pythagorean vector, and the vector satisfies $a^2 + b^2 = c^2$, then $(a, b, c)F_1$, $(a, b, c)F_2$, and $(a, b, c)F_3$ are still 3-dimensional Pythagorean vectors.*

Start with $(3, 4, 5)$ or $(4, 3, 5)$ and multiply F_1, F_2, or F_3 by it in any order any number of times. This yields another primitive Pythagorean vector (x, y, z), that is, a triple of positive integers without a common factor satisfying $x^2 + y^2 - z^2 = 0$. Furthermore, every primitive Pythagorean vector can be obtained uniquely this way. In other words, all primitive Pythagorean vectors can be given a tree-order structure with each edge representing a multiplication by F_j. Cha et al. [15] studied such trees that are applicable to any integral quadratic form.

Generally, 3-order integral square matrix A satisfies the following condition:

(i) $\alpha = (a, b, c)$ is a 3-dimensional Pythagorean vector, then $\beta = (a, b, c)A$ still is a Pythagorean vector
(ii) $|A|^2 = 1$, then square matrix A is a 3-order Pythagorean matrix [16]

Let T_3 be a set which is constituted by all 3-order Pythagorean matrices, namely, $T_3 = \{F \mid F \in Z^{3\times 3}, F$ is a three-order Pythagorean matrix}. Hence, we can calculate that the determinant values of F_1, F_2, and F_3 are 1; in other words, F_1, F_2, and F_3 are Pythagorean matrices, namely, $F_1 \in T_3$, $F_2 \in T_3$, and $F_3 \in T_3$.

Niu [17] researched algebraic properties of the set T_3 and proposed the following theorem.

Theorem 2. *T_3 constitutes a group about the matrix multiplication.*

In this paper, we further study algebraic properties and number-theoretic properties of the set T_3. Is T_3 a finitely generated group? If T_3 is a finitely generated group, then what are the generators of the finitely generated group? We prove our main theorem (Theorem 12). The theorem shows that T_3 is a finitely generated group, and the generators of the finitely generated group T_3 are given.

Furthermore, we also attempt to extend the Pythagorean vector and 3-order Pythagorean matrices to higher-order case and research algebraic properties and number-theoretic properties of the set formed from all n-order Pythagorean matrices ($n > 3$). Then, we get Theorem20.

This paper is organized as follows. The goal of Section 2 is to give some lemmas needed to prove the main conclusion of this paper. After we give some algebraic properties and number-theoretic properties of the set T_3 in Section 3, we prove our main theorem (Theorem 12) in Sections 4 and 5. Section 6 is devoted to the study of properties on n-order Pythagorean matrices ($4 < n$). Building on this, we prove our another main theorem (Theorem 20) in Section 7. Finally, in Section 8, we briefly discuss future work and prospects.

2. Some Preparations

Definition 1. If $a^2 + b^2 = c^2$ and $a, b,$ and c are coprime numbers, then we call $\{a, b, c\}$ a 3-dimensional primitive Pythagorean array and we call the correspondent vector a 3-order primitive Pythagorean vector [2].

Lemma 1. *If $\alpha = (a, b, c)$ is a 3-order primitive Pythagorean vector, then there exist an odd integer and even integer between a and b, where c must be odd.*

Lemma 2. *The necessary and sufficient conditions of 3-order integral square matrix $A \in T_3$ are as follows:*

(i) If $\alpha = (a, b, c)$ is a 3-order Pythagorean vector, then $\beta = (a, b, c)A$ is still a 3-order primitive Pythagorean vector

(ii) $|A|^2 = 1$

Lemma 3 (see [17]). *Given $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, then the necessary and sufficient condition of 3-order integral square matrix $A \in T_3$ is $ABA' = B$.*

Lemma 4 (see [17]). *The necessary and sufficient condition of matrix $A \in T_3$ is $A' \in T_3$.*

Hence, we can get Lemma5 as follows from Lemma3.

Lemma 5. *Given $A = (a_{ij}) \in Z^{3\times 3}$, then the necessary and sufficient condition of matrix $A \in T_3$ is the following equations exist integer solutions a_{ij}:*

$$a_{11}^2 + a_{21}^2 - a_{31}^2 = 1, \quad (1)$$

$$a_{12}^2 + a_{22}^2 - a_{32}^2 = 1, \quad (2)$$

$$-a_{13}^2 - a_{23}^2 + a_{33}^2 = 1, \quad (3)$$

$$a_{11}a_{12} + a_{21}a_{22} = a_{31}a_{32}, \quad (4)$$

$$a_{11}a_{13} + a_{21}a_{23} = a_{31}a_{33}, \quad (5)$$

$$a_{12}a_{13} + a_{22}a_{23} = a_{32}a_{33}. \quad (6)$$

3. Property of T_3

Let G be a set of 3-order integer square matrix A and satisfy the following two conditions:

(i) $A \in T_3$

(ii) If any a, b is even in 3-dimensional primitive Pythagorean vector, then $(a', b', c') = (a, b, c)A$ is still a 3-dimensional primitive Pythagorean [2] vector and b' is even.

Thus, $G \subset T_3$.

For clearer expression, we use some signs. Let $G_t = \{A \mid A = (a_{ij}) \in Z^{3\times 3}, \text{ and } A \in G, \ \max|a_{ij}| = t\}$. D_i ($i = 1, 2, \ldots, 8$) are the following diagonal matrices: $D_1 = \mathrm{diag}[1, 1, 1]$, $D_2 = \mathrm{diag}[-1, 1, 1]$, $D_3 = \mathrm{diag}[1, -1, 1]$, $D_4 = \mathrm{diag}[1, 1, -1]$, $D_5 = \mathrm{diag}[-1, -1, 1]$, $D_6 = \mathrm{diag}[-1, 1, -1]$, $D_7 = \mathrm{diag}[1, -1, -1]$, $D_8 = \mathrm{diag}[-1, -1, -1]$, and $D_9 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $F_4 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ -2 & -2 & -3 \end{pmatrix}$, $L(A_1, A_2, \ldots, A_n)$ express a finite group [18] about matrix multiplication, and $Z^{n\times n}$ express a set with all integer elements of n-order square matrix.

Theorem 3. *Let $A = (a_{ij}) \in Z^{3\times 3}$ and $A \in G$, then $|a_{33}| = \max_{i,j}|a_{ij}|$.*

Proof. Because $G \subset T_3$ and $A \in G$, $A \in T_3$. From Lemma 5, the above equalities (1)~(6) are tenable.

From equality (3), we can get the following inequalities:

$$\begin{aligned} |a_{33}| &\geq |a_{13}|, \\ |a_{33}| &\geq |a_{23}|, \\ |a_{33}| &\geq 1, \end{aligned} \quad (7)$$

and if $A \in T_3$, then $A' \in T_3$; therefore,

$$|a_{33}| \geq |a_{31}|,$$
$$|a_{33}| \geq |a_{32}|. \quad (8)$$

When a_{11}, a_{12}, a_{21}, and a_{22} are not equal to zero, we can obtain the following inequalities from equality (1):

$$|a_{31}| \geq |a_{11}|,$$
$$|a_{31}| \geq |a_{21}|. \quad (9)$$

From equality (2), we can obtain the following inequalities:

$$|a_{32}| \geq |a_{12}|,$$
$$|a_{32}| \geq |a_{22}|. \quad (10)$$

From (7)–(10), we know that equality $|a_{33}| = \max_{i,j}|a_{ij}|$ is tenable.

When at least one of the numbers in a_{11} and a_{21} is equal to zero but all of the two numbers in a_{12} and a_{22} are not equal to zero, we can infer from equality (2) that inequality (10) is true.

When at least one of the numbers in a_{11} and a_{21} is equal to zero, we can get equality $\max_i|a_{i1}| = 1$, where $|a_{33}| \geq 1$, so $|a_{33}| \geq \max_i|a_{i1}|$; from inequalities (7), (8), (10), and $|a_{33}| \geq \max_i|a_{i1}|$, we know that $|a_{33}| = \max_{i,j}|a_{ij}|$ is tenable.

When at least one of the numbers in a_{12} and a_{22} is equal to zero but all of the two numbers in a_{11} and a_{21} are not equal to zero, we can infer from equality (1) that inequality (9) is true.

When at least one of the numbers in a_{12} and a_{22} is equal to zero, we can get equality $\max_i|a_{i2}| = 1$, where $|a_{33}| \geq 1$, so $|a_{33}| \geq \max_i|a_{i2}|$; from the inequalities (7)–(9), and $|a_{33}| \geq \max_i|a_{i2}|$, we know that $|a_{33}| = \max_{i,j}|a_{ij}|$ is tenable.

When at least one of the numbers in a_{12} and a_{22} is equal to zero and at least one of the numbers in a_{11} and a_{21} is equal to zero, we get $\max_i|a_{i1}| = 1$, where $|a_{33}| \geq 1$, so $|a_{33}| \geq \max_i|a_{i1}|$; moreover, we also get $\max_i|a_{i2}| = 1$, where $|a_{33}| \geq 1$, so $|a_{33}| \geq \max_i|a_{i2}|$. From inequalities (7), $|a_{33}| \geq \max_i|a_{i1}|$ and $|a_{33}| \geq \max_i|a_{i2}|$, we know that $|a_{33}| = \max_{i,j}|a_{ij}|$ is tenable. □

Theorem 4. *If $A = (a_{ij}) \in Z^{3\times3}$ and $A \in G$, then $a_{ii} \equiv 1$ (mod 2) $(i = 1, 2, 3)$ and $a_{ii} \equiv 0$ (mod 2) $(i \neq j;\ i = 1, 2, 3;\ j = 1, 2, 3)$.*

Proof. Because $A \in G$, the above equalities (1)~(3) are tenable.

From equality $-a_{13}^2 - a_{23}^2 + a_{33}^2 = 1$, we get $a_{33} \equiv 1$ (mod 2), $a_{13} \equiv 0$ (mod 2), and $a_{23} \equiv 0$ (mod 2). From $A \in T_3$, we know $A' \in T_3$. So, $a_{31} \equiv 0$ (mod 2) and $a_{32} \equiv 0$ (mod 2) are tenable. Given $\alpha = (a, b, c)$ is an arbitrary primitive Pythagorean vector with even integer b, by Lemma 1, we know a and c are odd integers. Since $A \in G$, $(a', b', c') = (a, b, c)A$ is still a 3-order primitive Pythagorean vector with a even integer b'. By Lemma 1, we know a' and c' are odd integers. Since b is an even integer, a is an odd integer, $a_{31} \equiv 0$ (mod 2), and $a' = aa_{11} + ba_{21} + ca_{31}$, we get $a_{11} \equiv 1$ (mod 2). From $a_{11} \equiv 1$ (mod 2), $a_{31} \equiv 0$ (mod 2), and equality (1), we know $a_{21} \equiv 0$ (mod 2). Since a is an odd integer, b is an even integer, $a_{32} \equiv 0$ (mod 2), and $b' = aa_{12} + ba_{22} + ca_{32}$, we get $a_{12} \equiv 0$ (mod 2). From $a_{12} \equiv 0$ (mod 2), $a_{32} \equiv 0$ (mod 2). and equality (2), we obtain the equality $a_{22} \equiv 1$ (mod 2). From above, we know $a_{ii} \equiv 1$ (mod 2) $(i = 1, 2, 3)$ and $a_{ij} \equiv 0$ (mod 2) $(i \neq j; i = 1, 2, 3;\ j = 1, 2, 3)$ are tenable. □

Inference 1. (1) G constitutes a group on matrix multiplication; (2) G is a subgroup of T_3.

Proof

(1) Since any A or $B \in G$, A or $B \in T_3$, so $A * B \in T_3$. Given $\alpha = (a, b, c)$ is an arbitrary 3-order primitive Pythagorean vector with even integer b because $A \in G$, so $(a, b, c)A$ is also a 3-order primitive Pythagorean vector with second element which is an even integer, and $B \in G$, so $(a, b, c)AB$ is a 3-order primitive Pythagorean vector with second element which is an even integer; thus, $A * B \in G$, that is, G is a closed operator of matrix multiplication. Matrix multiplication is clear to meet the combination of law, and 3-order unitary matrix is a unit element in G. Now, we prove that G is closed for inverse matrix. Arbitrary $A \in G$, where $A = (a_{ij}) \in Z^{3\times3}$; we know $a_{ii} \equiv 1$ (mod 2) $(i = 1, 2, 3)$ and $a_{ij} \equiv 0$ (mod 2) $(i \neq j;\ i = 1, 2, 3;\ j = 1, 2, 3)$ are tenable by Theorem 4. Given $\alpha = (a, b, c)$ is an arbitrary 3-order primitive Pythagorean vector with even integer b, let $(a', b', c') = (a, b, c)A^{-1}$; since $A \in T_3$, T_3 constitutes a group on matrix multiplication, so $A^{-1} \in T_3$; accordingly, (a', b', c') is a primitive Pythagorean vector. One is an odd integer between a' and b' and another is an even integer by Lemma 1. Since (a, b, c) is an arbitrary 3-order primitive Pythagorean vector with even integer b, we know that a is an odd integer by Lemma 1. We obtain $(a, b, c) = (a', b', c')A$ from the equality $(a', b', c') = (a, b, c)A^{-1}$. We can conclude that $a = a'a_{11} + b'a_{21} + c'a_{31}$ because a is an odd integer and both a_{21} and a_{31} are even integer, so a' is an odd integer while b' must be an even integer. Accordingly, $(a', b', c') = (a, b, c)A^{-1}$ is a 3-order primitive Pythagorean vector with second element which is an even integer; hence, $A^{-1} \in G$.

From above, G constitutes a group on matrix multiplication. And that is what we wanted to prove.

(2) We can conclude that G is a subgroup of T_3 by $G \subset T_3$ and G constitutes a group on matrix multiplication. □

4. Expression of G_1 and G_3

Theorem 5. $G_1 = \{D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8\}$.

Proof. From Theorems 3 and 4, if $A \in G$ and $\max_{i,j}|a_{ij}| = 1$, then A must be one of D_i $(i = 1, 2, \ldots, 8)$. $D_i \in G_1$, $i = 1, 2, \ldots, 8$, is easy to verify, so Theorem 5 is tenable. □

Theorem 6. *If $A = (a_{ij}) \in Z^{3\times3}$, $A \in G$, and $\max_{i,j}|a_{ij}| = 3$, then $|a_{33}| = 3$, $|a_{31}| = |a_{32}| = |a_{13}| = |a_{23}| = |a_{12}| = |a_{21}| = 2$, and $|a_{11}| = |a_{22}| = 1$.*

Proof. From Theorem 4 we know, if $A \in G$ and $\max_{i,j}|a_{ij}| = 3$, then $|a_{33}| = \max_{i,j}|a_{ij}| = 3$. By $A \in G$, the former equations (1)~(3) are tenable. By equation (3), equalities $|a_{13}| = |a_{23}| = 2$ are true. From $A \in T_3$, we get $A' \in T_3$; hence, $|a_{31}| = |a_{32}| = 2$ are true. From Theorem 3, we know that $|a_{11}|$ and $|a_{22}|$ only possibly are 1 or 3, while from $|a_{13}| = |a_{23}| = |a_{31}| = |a_{32}| = 2$ and from equalities (1) and (2), we can conclude that $|a_{11}| = |a_{22}| = 1$ are true. Then, equalities $|a_{12}| = |a_{21}| = 2$ are also true. From above, Theorem 6 is tenable, and that is what we wanted to prove. □

Theorem 7. $G_3 = \{A \mid A = D_i F_1 D_j, D_i \in G_1, D_j \in G_1\}$.

Proof. Both arbitrary D_i $(i = 1, 2, \ldots, 8)$ and D_j $(j = 1, 2, \ldots, 8)$ also $\in G$ by Theorem 5. It is easy to prove that $F_1 \in G$. We obtain from Inference 1 (1) that $D_i F_1 D_j \in G$. Also, it is easy to verify that the maximum absolute value of elements of matrix $D_i F_1 D_j$ is equal to 3. So, we get that $D_i F_1 D_j \in G_3$ for arbitrary D_i $(i = 1, 2, \ldots, 8)$ and D_j $(j = 1, 2, \ldots, 8)$.

On the other hand, if $A \in G_3$, then $|a_{33}| = 3$, $|a_{11}| = |a_{22}| = 1$, and $|a_{31}| = |a_{32}| = |a_{13}| = |a_{23}| = |a_{12}| = |a_{21}| = 2$.

For A, there must be D_i and D_j, these three matrices produce a new matrix $C = D_i A D_j$, let $C = (c_{ij})$, among six elements of column 1 and column 2 in matrix C, there are two elements at most less than zero, and these less than zero elements are not in the same row. Since D_i, D_j, and F_1 belong to G, $C = D_i A D_j \in G$; thus, $C \in G_3$. Hence $|c_{33}| = 3$, $|c_{31}| = |c_{32}| = |c_{13}| = |c_{23}| = |c_{12}| = |c_{21}| = 2$, and $|c_{11}| = |c_{22}| = 1$. We obtain $c_{11}c_{12} + c_{21}c_{22} = c_{31}c_{32}$ by $C \in G$ and Lemma 5, so $c_{ij} > 0$ $(i = 1, 2, 3;\ j = 1, 2)$.

For C, there must be D_k; they produce a new matrix $H = CD_k$. Let $H = (h_{ij})$. Column vectors 1 and 2 of H are exactly the same as column vectors 1 and 2 of C, and one element at most in column vector 3 of H is less than zero. So, we get $|h_{33}| = 3$, $|h_{31}| = |h_{32}| = |h_{13}| = |h_{23}| = |h_{12}| = |h_{21}| = 2$, $|h_{11}| = |h_{22}| = 1$, and $h_{11}h_{13} + h_{21}h_{23} = h_{31}h_{33}$. Note that $h_{i1} = c_{i1} > 0$ $(i = 1, 2, 3)$, so $h_{i3} > 0$ $(i = 1, 2, 3)$; hence, $H = F_1$.

So, there exist D_i, D_j, and D_k, they generate a new matrix $F_1 = D_i A D_j D_k$. Let $P_1 = D_i^{-1}$ and $P_2 = (D_j D_k)^{-1}$. It is easy to verify that both P_1 and P_2 belong to G_1, and $A = P_1 F_1 P_2$.

From above, $G_3 = \{A \mid A = D_i F_1 D_j, D_i \in G_1, D_j \in G_1\}$ is obtained.

The following theorem is obtained by direct verification. □

Theorem 8. $D_1 = F_1 * F_1^{-1}$, $D_2 = F_1 * F_2^{-1}$, $D_3 = F_1 * F_3^{-1}$, $D_4 = F_1 * F_4^{-1}$, $D_5 = F_2 * F_3^{-1}$, $D_6 = F_2 * F_4^{-1}$, $D_7 = F_3 * F_4^{-1}$, *and* $D_8 = F_2 * F_3^{-1} * F_1 * F_4^{-1}$.

We can get Theorem 9 by Theorems 7 and 8.

Theorem 9. *(1)* $G_3 \subset L(F_1, F_2, F_3, F_4)$*; (2)* $G_1 \subset L(F_1, F_2, F_3, F_4)$.

5. Representation of G and T_3

Theorem 10. *Arbitrary* $A = (a_{ij}) \in G$ *if the maximum absolute value of elements of matrix* A *is equal to* y *and* $y > 3$*; let* $H_i = AF_i$ $(i = 1, 2, 3, 4)$*, then there must exist a matrix* H_i*, its maximum absolute values of elements are less than* y.

Proof. Since $F_i \in G$ $(i = 1, 2, 3, 4)$ and $A \in G$, $H_i = AF_i \in G$ $(i = 1, 2, 3, 4)$. So, we get that the maximum absolute value of elements of matrix A is $y = |a_{33}|$ by Theorem 3, while the maximum absolute value of elements of matrix H_i is $|2a_{31} + 2a_{32} + 3a_{33}|$, matrix H_2 is $|-2a_{31} + 2a_{32} + 3a_{33}|$, matrix H_3 is $|2a_{31} - 2a_{32} + 3a_{33}|$, and H_4 is $|2a_{31} + 2a_{32} - 3a_{33}|$.

Cases of $a_{31} \geq 0$, $a_{32} \geq 0$, and $a_{33} > 0$ are considered firstly.

Since $A \in G$, we get $A' \in G$; thus, equality $-a_{31}^2 - a_{32}^2 + a_{33}^2 = 1$ is tenable consequently. Also, because $|a_{33}| > 3$, $a_{31} \neq 0$ and $a_{32} \neq 0$; hence, it follows that $a_{31} > 0$ and $a_{32} > 0$; thus, $a_{33} > a_{31}$ and $a_{33} > a_{32}$. Finally, the inequality $2a_{31} + 2a_{32} - 3a_{33} < a_{33}$ is true as a consequence.

We obtain $a_{33} - a_{31} = (1 + a_{32}^2)/(a_{33} + a_{31})$ from $-a_{31}^2 - a_{32}^2 + a_{33}^2 = 1$, while $(1 + a_{32}^2)/(a_{33} + a_{31}) < (a_{32}a_{31} + a_{32}a_{33})/(a_{33} + a_{31}) = a_{32}$, so $a_{33} - a_{31} < a_{32}$ or $a_{33} < a_{32} + a_{31}$; thus, $-a_{33} < 2a_{31} + 2a_{32} - 3a_{33}$ is true.

By inequalities $2a_{31} + 2a_{32} - 3a_{33} < a_{33}$ and $-a_{33} < 2a_{31} + 2a_{32} - 3a_{33}$, we get $|2a_{31} + 2a_{32} - 3a_{33}| < y$; in other words, the maximum absolute value of elements of matrix H_4 is less than y, and that is Theorem 10 which we wanted to prove.

In the same way, we can prove that, under the cases $a_{31} \geq 0$, $a_{32} \leq 0$, and $a_{33} > 0$, Theorem 10 is still valid. □

Theorem 11. $G = L(F_1, F_2, F_3, F_4)$*, namely,* G *is a finitely generated group.*

Proof. It can be seen that $L(F_1, F_2, F_3, F_4) \subset G$.

Given arbitrary $A = (a_{ij}) \in G$; if $|a_{33}| \leq 3$, then $|a_{33}| = 1$ or $|a_{33}| = 3$ by Theorem 4. When $|a_{33}| = 1$, there must be $A \in G_1$; hence, it follows that $A \in L(F_1, F_2, F_3, F_4)$ by Theorem 9. When $|a_{33}| = 3$, there must be $A \in G_3$; hence, it follows that $A \in L(F_1, F_2, F_3, F_4)$ by Theorem 9.

If $|a_{33}| > 3$, from Theorem 10, we know that there exist $P_1, P_2, \ldots, P_k$, these matrices satisfy that $P_i \in \{F_1, F_2, F_3, F_4\}$ $(i = 1, \ldots, k)$ and they produce a new matrix $A * P_1 * P_2 * \cdots * P_k$ by multiplication of matrices, and the maximum absolute value of elements of the new matrix is less than or equal to 3. Indeed, the new matrix $A * P_1 * P_2 * \cdots * P_k \in G$, so $A * P_1 * P_2 * \cdots * P_k \in L(F_1, F_2, F_3, F_4)$; hence, $A \in L(F_1, F_2, F_3, F_4)$ and then $G \subset L(F_1, F_2, F_3, F_4)$.

From above, we get $G = L(F_1, F_2, F_3, F_4)$. □

Inference 2. (1) $G = L(F_1, D_2, D_3, D_4)$; (2) $G = L(F_1, D_2, D_4, D_5)$; (3) $G = L(F_i, D_2, D_3, D_4)$ $(i = 2, 3, 4)$.

Proof. We only prove that equality (1) is tenable, and others may prove similarly.

It is obvious that $L(F_1, D_2, D_3, D_4) \subset G$. Arbitrary $A = (a_{ij}) \in G$; we get $A \in L(F_1, F_2, F_3, F_4)$ by Theorem 11 and $D_2 = F_1 * F_2^{-1}$, $D_3 = F_1 * F_3^{-1}$, and $D_4 = F_1 * F_4^{-1}$ by Theorem 8; hence, $F_2 = D_2^{-1} * F_1$, $F_3 = D_3^{-1} * F_1$, and $F_4 = D_4^{-1} * F_1$, so $A \in L(F_1, D_2, D_3, D_4)$.

From above, $G = L(F_1, D_2, D_3, D_4)$ is tenable. □

Definition 2. If H is a finitely generated group and $X_1, X_2, \ldots, X_n$ satisfy $H = L(X_1, X_2, \ldots, X_n)$, then we call

$X_1, X_2, \ldots, X_n$ is a generated tuple of H. If $X_1, X_2, \ldots, X_n$ is a generated tuple of H with least element, then we call $X_1, X_2, \ldots, X_n$ is a minimum generated tuple of H and we call n is cardinality of H, expressed as $n = d(H)$.

Theorem 12. *$T_3 = L(F_1, D_2, D_4, D_9)$. In other words, T_3 is a finitely generated group; furthermore, $d(T_3) = 4$.*

Proof. It is easy to verify that $D_9 \in T_3$; hence, $L(F_1, D_2, D_4, D_9) \subset T_3$.

Owing to $D_3 = D_9 D_2 D_9$, $G = L(F_1, D_2, D_3, D_4) \subset L(F_1, D_2, D_4, D_9)$. Now, let arbitrary $A = (a_{ij}) \in T_3$ and $(x, y, z) = (3, 4, 5)A$ such that one in x, y is an even and another is an odd.

(1) First, consider x is odd and y is even.

Equality $y = 3a_{12} + 4a_{22} + 5a_{32}$ is obtained from $(x, y, z) = (3, 4, 5)A$. Because y is an even integer, both a_{12} and a_{32} are either even integer or odd integer.

Given (a, b, c) is an arbitrary three-order primitive Pythagorean vector with an even integer b; note $(a', b', c') = (a, b, c)$, so that (a', b', c') is also a three-order primitive Pythagorean vector and a, c are odd integer. Thanks to $b' = aa_{12} + ba_{22} + ca_{32}$, where b is an even integer and a_{12} and a_{32} are either odd integers or even integers simultaneously. From the previous reason, we know that b' is an even integer. So, $A \in G$; hence, $A \in (F_1, D_2, D_3, D_4)$. In addition, $L(F_1, D_2, D_3, D_4) \subset L(F_1, D_2, D_4, D_9)$; therefore, $A \in (F_1, D_2, D_4, D_9)$.

(2) Now, consider x is even and y is odd.

Let $(x', y', z') = (x, y, z)D_9 = (3, 4, 5)AD_9$, then x' is an odd integer and y' is an even integer. From $A \in T_3$ and $D_9 \in T_3$, we have $AD_9 \in T_3$. Thus, $AD_9 \in L(F_1, D_2, D_4, D_9)$; consequently, $A \in L(F_1, D_2, D_4, D_9)$.

From above, we get $T_3 \subset L(F_1, D_2, D_4, D_9)$; on this account, $T_3 \subset L(F_1, D_2, D_4, D_9)$ and $d(T_3) = 4$ are tenable. □

Inference 3. $T_3 = L(F_1, D_2, D_4, D_9) = L(F_1, D_3, D_4, D_9) = L(F_1, F_2, F_4, D_9) = L(F_1, F_3, F_4, D_9) = L(F_2, F_3, F_4, D_9)$.

6. Property of n-Order Pythagorean Matrix ($n \geq 4$)

Definition 3. If integers $a_1, a_2, \ldots, a_n$ satisfy $\sum_{k=1}^{n-1} a_k^2 = a_n^2$, then $\{a_1, a_2, \ldots, a_n\}$ is designated as an n-order Pythagorean array, while vector-style expression $(a_1, a_2, \ldots, a_n)$ is named as an n-order Pythagorean vector.

Definition 4. If an n-order square matrix A meets the following two conditions, then we name A *as* n-order Pythagorean matrix.

(1) If $\alpha = (a_1, a_2, \ldots, a_n)$ is an arbitrary n-order Pythagorean vector and $\beta = (a_1, a_2, \ldots, a_n)A$ is still a Pythagorean vector

(2) If $|A|^2 = 1$

Definition 5. Among an n-order Pythagorean vector $(a_1, a_2, \ldots, a_n)$, $a_1, a_2, \ldots, a_n$ are coprime numbers; then $(a_1, a_2, \ldots, a_n)$ is named as an n-order primitive Pythagorean vector.

Let $T_n = \{F \mid F \in Z^{n\times n}, F \text{ is a } n-\text{order Pythagorean matrix}\}$ and $B = \operatorname{diag}[1, 1, \ldots, 1, -1]$ in this chapter. Then, we have the following theorem.

Theorem 13. *A is an n-order integer matrix, and the necessary and sufficient condition of $A \in T_n$ is $ABA' = B$.*

Proof Sufficiency condition.

If A satisfies $ABA' = B$, then we can easily check that $|A|^2 = 1$. Let $\alpha = (a_1, a_2, \ldots, a_n)$ be an arbitrary n-order Pythagorean vector. In the expression of $\beta = (a_1, a_2, \ldots, a_n)A$, we can simplify it. Let $\beta = (a_{11}, a_{12}, \ldots, a_{1n})$. Clearly, $a_{11}, a_{12}, \ldots, a_{1n}$ are all integers. Since $\sum_{i=1}^{n-1} a_{1i}^2 - a_{1n}^2 = \beta B \beta' = \alpha ABA'\alpha' = \alpha B \alpha' = \sum_{i=1}^{n-1} a_i^2 - a_n^2 = 0$, β is an n-order Pythagorean vector. Thereby, $A \in T_n$.

Necessary condition.

If $A = (a_{ij}) \in T_n$ and $\alpha = (c_1, c_2, \ldots, c_n)$ is an n-order Pythagorean vector, then $\beta = (c_1, c_2, \ldots, c_n)A$ is still an n-order Pythagorean vector, that is to say $\beta B \beta' = \alpha ABA'\alpha' = 0$.

Hence, we can get the following equality:

$$\begin{aligned}
&\left[\sum_{i=1}^{n-1} a_{1i}^2 - a_{1n}^2\right] c_1^2 + \left[\sum_{i=1}^{n-1} a_{2i}^2 - a_{2n}^2\right] c_2^2 + \cdots + \left[\sum_{i=1}^{n-1} a_{ni}^2 - a_{nn}^2\right] c_n^2 + 2\left[\sum_{i=1}^{n-1} a_{1i}a_{2i} - a_{1n}a_{2n}\right] c_1 c_2 \\
&+ 2\left[\sum_{i=1}^{n-1} a_{1i}a_{3i} - a_{1n}a_{3n}\right] c_1 c_3 + \cdots + 2\left[\sum_{i=1}^{n-1} a_{1i}a_{ni} - a_{1n}a_{nn}\right] c_1 c_n + 2\left[\sum_{i=1}^{n-1} a_{2i}a_{3i} - a_{2n}a_{3n}\right] c_2 c_3 \\
&+ \cdots + 2\left[\sum_{i=1}^{n-1} a_{2i}a_{ni} - a_{2n}a_{nn}\right] c_2 c_n + \cdots + 2\left[\sum_{i=1}^{n-1} a_{n-1,i}a_{ni} - a_{n-1,n}a_{nn}\right] c_{n-1} c_n = 0.
\end{aligned} \tag{11}$$

Let $\alpha = (1, 0, \ldots, 0, 1)$, then substitute it in equality (11), and we can get the following equality:

$$\left[\sum_{i=1}^{n-1} a_{1i}^2 - a_{1n}^2\right] + \left[\sum_{i=1}^{n-1} a_{ni}^2 - a_{nn}^2\right] + 2\left[\sum_{i=1}^{n-1} a_{1i}a_{ni} - a_{1n}a_{nn}\right] = 0. \tag{12}$$

Let $\alpha = (1, 0, \ldots, 0, -1)$, then substitute it in equality (11), and we can get the following equality:

$$\left[\sum_{i=1}^{n-1} a_{1i}^2 - a_{1n}^2\right] + \left[\sum_{i=1}^{n-1} a_{ni}^2 - a_{nn}^2\right] - 2\left[\sum_{i=1}^{n-1} a_{1i}a_{ni} - a_{1n}a_{nn}\right] = 0. \tag{13}$$

Let $\alpha = (0, 1, 0, \ldots, 0, 1)$ and $\alpha = (0, 1, 0, \ldots, 0, -1)$, then substitute it in equality (11) Then, we can get

$$\left[\sum_{i=1}^{n-1} a_{2i}^2 - a_{2n}^2\right] + \left[\sum_{i=1}^{n-1} a_{ni}^2 - a_{nn}^2\right] + 2\left[\sum_{i=1}^{n-1} a_{2i}a_{ni} - a_{2n}a_{nn}\right] = 0, \tag{14}$$

$$\left[\sum_{i=1}^{n-1} a_{2i}^2 - a_{2n}^2\right] + \left[\sum_{i=1}^{n-1} a_{ni}^2 - a_{nn}^2\right] - 2\left[\sum_{i=1}^{n-1} a_{2i}a_{ni} - a_{2n}a_{nn}\right] = 0.$$

$$\vdots \tag{15}$$

Let $\alpha = (0, \ldots, 0, 1, 1)$, then substitute it in equality (11), and we obtain the following equation:

$$\left[\sum_{i=1}^{n-1} a_{n-1i}^2 - a_{n-1n}^2\right] + \left[\sum_{i=1}^{n-1} a_{ni}^2 - a_{nn}^2\right] + 2\left[\sum_{i=1}^{n-1} a_{n-1,i}a_{ni} - a_{n-1,n}a_{nn}\right] = 0. \tag{16}$$

Let $\alpha = (0, \ldots, 0, 1, -1)$, then substitute it in equality (11), andwe obtain the following equation:

$$\left[\sum_{i=1}^{n-1} a_{n-1i}^2 - a_{n-1n}^2\right] + \left[\sum_{i=1}^{n-1} a_{ni}^2 - a_{nn}^2\right] - 2\left[\sum_{i=1}^{n-1} a_{n-1,i}a_{ni} - a_{n-1,n}a_{nn}\right] = 0. \tag{17}$$

From equations (12) and (13), we obtain

$$\begin{gathered}\left[\sum_{i=1}^{n-1} a_{1i}^2 - a_{1n}^2\right] = -\left[\sum_{i=1}^{n-1} a_{ni}^2 - a_{nn}^2\right],\\ \sum_{i=1}^{n-1} a_{1i}a_{ni} - a_{1n}a_{nn} = 0.\end{gathered} \tag{18}$$

From equations (14) and (15), we get

$$\begin{gathered}\left[\sum_{i=1}^{n-1} a_{2i}^2 - a_{2n}^2\right] = -\left[\sum_{i=1}^{n-1} a_{ni}^2 - a_{nn}^2\right],\\ \sum_{i=1}^{n-1} a_{2i}a_{ni} - a_{2n}a_{nn} = 0,\end{gathered} \tag{19}$$

$$\vdots$$

From equations (16) and (17), we get

$$\begin{gathered}\left[\sum_{i=1}^{n-1} a_{n-1i}^2 - a_{n-1n}^2\right] = -\left[\sum_{i=1}^{n-1} a_{ni}^2 - a_{nn}^2\right],\\ \left[\sum_{i=1}^{n-1} a_{n-1,i}a_{ni} - a_{n-1,n}a_{nn}\right] = 0.\end{gathered} \tag{20}$$

Also, from equalities (18), (19), $\ldots$, (20), we obtain the following equations:

$$\sum_{i=1}^{n-1} a_{1i}^2 - a_{1n}^2 = \sum_{i=1}^{n-1} a_{2i}^2 - a_{2n}^2 = \cdots = \sum_{i=1}^{n-1} a_{n-1,i}^2 - a_{n-1,n}^2 = -\sum_{i=1}^{n-1} a_{ni}^2 - a_{nn}^2, \tag{21}$$

$$\begin{aligned}&\sum_{i=1}^{n-1} a_{1i}a_{ni} - a_{1n}a_{nn} = \sum_{i=1}^{n-1} a_{2i}a_{ni} - a_{2n}a_{nn} = \cdots = \sum_{i=1}^{n-1} a_{n-1,i}a_{ni}\\ &\quad - a_{n-1,n}a_{nn} = 0.\end{aligned} \tag{22}$$

Substituting equalities (21) and (22) into equality (11), we can get the following equality:

$$\begin{aligned}&2\left[\sum_{i=1}^{n-1} a_{1i}a_{2i} - a_{1n}a_{2n}\right]c_1c_2 + \cdots + 2\left[\sum_{i=1}^{n-1} a_{1i}a_{n-1,i} - a_{1n}a_{n-1,n}\right]\\ &\quad \cdot c_1c_{n-1} + 2\left[\sum_{i=1}^{n-1} a_{2i}a_{3i} - a_{2n}a_{3n}\right]c_2c_3 + \cdots\\ &\quad + 2\left[\sum_{i=1}^{n-1} a_{2i}a_{n-1,i} - a_{2n}a_{n-1,n}\right]c_2c_n + \cdots\\ &\quad + 2\left[\sum_{i=1}^{n-1} a_{n-2,i}a_{n-1,i} - a_{n-2,n}a_{n-1,n}\right]c_{n-2}c_{n-1} = 0.\end{aligned} \tag{23}$$

Hence, we can get equation (24) by randomicity of $\alpha = (c_1, c_2, \ldots, c_n)$ and equality (23):

$$\begin{aligned}&\sum_{i=1}^{n-1} a_{1i}a_{2i} - a_{1n}a_{2n} = \cdots = \sum_{i=1}^{n-1} a_{1i}a_{n-1,i} - a_{1n}a_{n-1,n} = \sum_{i=1}^{n-1} a_{2i}a_{3i} - a_{2n}a_{3n}\\ &= \cdots = \sum_{i=1}^{n-1} a_{2i}a_{n-1,i} - a_{2n}a_{n-1,n} = \cdots = \sum_{i=1}^{n-1} a_{n-2,i}a_{n-1,i} - a_{n-2,n}a_{n-1,n} = 0.\end{aligned} \tag{24}$$

For given $\sum_{i=1}^{n-1} a_{1i}^2 - a_{1n}^2 = k$, we know equality (25) is valid by equalities (21), (22), and (24):

$$ABA' = kB. \quad (25)$$

Thanks to $|ABA'| = |A|^2|B| = |B| = |kB| = k^n|B|$, where k is an integer, $k = 1$ or $k = -1$ when n is an even integer, but $k = 1$ when n is an odd integer. Also, because matrices B and kB are congruence relationship from (6.15), $k = -1$ is inappropriate while $k = 1$ is tenable; as a result, $ABA' = B$.

It i s easy to get Theorem 14 by Theorem 13. □

Theorem 14. *Given $A = (a_{ij}) \in Z^{n\times n}$, the necessary and sufficient condition of $A \in T_n$ is that a_{ij} is integer solution of the following equations:*

$$\sum_{i=1}^{n-1} a_{ik}^2 - a_{nk}^2 = 1, \quad k = 1, 2, \ldots, n-1, \quad (26)$$

$$-\sum_{i=1}^{n-1} a_{in}^2 + a_{nn}^2 = 1, \quad (27)$$

$$\sum_{i=1}^{n-1} a_{ip}a_{iq} = a_{np}a_{nq}, \quad 1 \le p < q \le n; p, q \in Z. \quad (28)$$

Lemma 6. *Given A is an n-order integer square matrix, the necessary and sufficient condition of $A \in T_n$ is A meets the following two cases:*

(1) If $\alpha = (a_1, a_2, \ldots, a_n)$ is an arbitrary n-order primitive Pythagorean vector, then $\beta = \alpha A$ is still an n-order primitive Pythagorean vector

(2) $|A|^2 = 1$

Lemma 7. *If $A \in T_n$, then $A^{-1} \in T_n$.*

Proof. Because $A \in T_n$, $ABA' = B$; therefore, $A^{-1}B(A^{-1})' = B$. Consequently, $A^{-1} \in T_n$. □

Lemma 8. *Given $A \in T_n$ and $C \in T_n$, then $AC \in T_n$.*

Proof. Because $A \in T_n$ and $C \in T_n$, $(AC)B(AC)' = ACBC'A' = ABA' = B$; as a result, $AC \in T_n$. □

Lemma 9. *The necessary and sufficient condition of $A \in T_n$ is $A' \in T_n$.*

Proof. If we want to prove Lemma 9 is proper, we need to prove only that if $A \in T_n$, then $A' \in T_n$.

Given $A \in T_n$, $ABA' = B$; hence, $A' = B^{-1}A^{-1}B$. Because both B and B^{-1} belong to T_n, we can get $A^{-1} \in T_n$ by Lemma 7 and we can conclude that $A' \in T_n$ from Lemma 8.

We can easily get the following theorem by using the above lemmas. □

Theorem 15. *T_n compose a group about matrix multiplication.*

Theorem 16. *If $A = (a_{ij}) \in Z^{n\times n}$ and $A \in T_n$, then $|a_{nn}| = \max_{i,j}|a_{ij}|$.*

Proof. Since $A \in T_n$, equations (6.16), (6.17), and (6.18) are tenable by Theorem 14.

From equality (27), we know $|a_{in}| \le |a_{nn}|$ $(i = 1, 2, \ldots, n-1)$ and $1 \le |a_{nn}|$ are true.

From $A \in T_n$, we get $A' \in T_n$, so

$$|a_{ni}| \le |a_{nn}|, \quad i = 1, 2, \ldots, n-1. \quad (29)$$

For arbitrary j $(j = 1, 2, \ldots, n-1)$, if $\max(|a_{1j}|, |a_{2j}|, \ldots, |a_{n-1,j}|) \le |a_{nj}|$, then we get $|a_{ij}| \le |a_{nn}|$ $(i = 1, 2, \ldots, n-1)$ by (29); if $\max(|a_{1j}|, |a_{2j}|, \ldots, |a_{n-1,j}|) > |a_{nj}|$, then we can conclude that $\max(|a_{1j}|, |a_{2j}|, \ldots, |a_{n-1,j}|) \le 1$ by (26). Because $1 \le |a_{nn}|$, $|a_{ij}| \le |a_{nn}|$ $(i = 1, 2, \ldots, n-1)$. Hence, we get

$$|a_{ij}| \le |a_{nn}|, \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n). \quad (30)$$

That is to say $|a_{nn}| = \max_{i,j}|a_{ij}|$ is tenable. □

7. Expression of *n*-Order Pythagorean Matrix ($4 \le n \le 10$)

Given $T_n^p = \{A \mid A = (a_{ij}) \in T_n \text{ and } \max_{ij} = |a_{ij}| = p\}$, $F_4^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$ and $F_n^2 = \begin{pmatrix} E_{n-4} & 0 \\ 0 & F_4^2 \end{pmatrix}$ when $5 \le n \le 10$. Clearly $F_n^2 \in T_n^2$. Now, we designate W_i $(i = 1, 2, \ldots, 2n-1)$ as the following n-order square matrix, $W_1 = \text{diag}\,[1, 1, \ldots, 1]$, $W_2 = \text{diag}[-1, 1, \ldots, 1]$, $W_3 = \text{diag}\ [1, 1, \ldots, 1, -1]$, $W_4 = E_n(1, 2)$, $W_5 = E_n(1, 3), \ldots$, $W_{n+1} = E_n(1, n-1)$, $W_{n+2} = \text{diag}[1, -1, 1, \ldots, 1]$, $W_{n+3} = \text{diag}\ [1, 1, -1, 1, \ldots, 1], \ldots$, and $W_{2n-1} = \text{diag}[1, \ldots, 1, -1, 1]$. Here, $E_n(i, j)$ is an n-order square matrix produced by unitary matrix exchange row i and j. $L(W_1, W_2, \ldots, W_{2n-1})$ represents the finitely generated group about matrix multiplication produced by $W_1, W_2, \ldots, W_{2n-1}$; $L(W_1, W_2, W_3, W_{n+2}, W_{n+3}, \ldots, W_{2n-1})$ represents the finitely generated group about matrix multiplication produced by $W_1, W_2, W_3, W_{n+2}, W_{n+3}, \ldots, W_{2n-1}$; $L(W_4, W_5, \ldots, W_{n+1})$ represents the finitely generated group about matrix multiplication produced by $W_4, W_5, \ldots, W_{n+1}$; $L(W_1, W_2, \ldots, W_{n+1})$ represents the finitely generated group about matrix multiplication produced by $W_1, W_2, \ldots, W_{n+1}$. It is easy to know that $L(W_1, W_2, \ldots, W_{2n-1}) = L(W_2, W_3, W_4 \ldots, W_{n+1})$.

Theorem 17. *$T_n^1 = L(W_1, W_2, \ldots, W_{n+1})$.*

Proof. It is easy to validate that $W_1, W_2, \ldots, W_{n+1}$ belong to T_n^1. So, we can conclude that $L(W_1, W_2, \ldots, W_{n+1}) \subset T_n^1$ by Theorem 15.

Considering $A = (a_{ij}) \in Z^{n\times n}$ and $A \in T_n^1$, equation $|a_{nn}| = 1$ becomes true by Theorem 16. We can infer that $a_{in} = 0$ $(i = 1, 2, \ldots, n-1)$ from equation (27) and $|a_{nn}| = 1$, and we can conclude that $A' \in T_n^1$ by $A \in T_n^1$, thereby $a_{ni} = 0$

$(i = 1, 2, \ldots, n-1)$ are tenable. Thus, we can obtain the following conclusion by equation (26) and $a_{ni} = 0$ $(i = 1, 2, \ldots, n-1)$.

There exists only one 1 among $|a_{11}|, |a_{21}|, \ldots, |a_{n-1,1}|$.
There exists only one 1 among $|a_{12}|, |a_{22}|, \ldots, |a_{n-1,2}|$.
$\vdots$
There exists only one 1 among $|a_{1,n-1}|, |a_{2,n-1}|, \ldots, |a_{n-1,n-1}|$.
We can get the other conclusion by $A' \in T_n^1$.
There exists only one 1 among $|a_{11}|, |a_{12}|, \ldots, |a_{1,n-1}|$.
There exists only one 1 among $|a_{21}|, |a_{22}|, \ldots, |a_{2,n-1}|$.
$\vdots$
There exists only one 1 among $|a_{n-1,1}|, |a_{n-1,2}|, \ldots, |a_{n-1,n-1}|$.

Accordingly, there exist $X_1, X_2, \ldots, X_{n-2}$ $(X_1, X_2, \ldots, X_{n-2} \in \{W_1, W_4, W_5, \ldots, W_{n+1}\})$, and they cause $C \triangleq AX_1X_2 \cdots X_{n-2} \triangleq (c_{ij})$ and satisfy that $c_{ij} = 0$ $(i \neq j)$ and $|c_{ii}| = 1$ $(i = 1, 2, \ldots, n)$. For C, there exist $X_{n-1}, X_n, \ldots, X_{2n-1}$ $(X_{n-1}, X_n, \ldots, X_{2n-1} \in \{W_1, W_2, W_3, W_{n+2}, W_{n+3}, \ldots, W_{2n-1}\})$, and they make $CX_{n-1}X_n \cdots X_{2n-1} = E_n$. So, $AX_1X_2 \cdots X_{n-2}X_{n-1}X_n \cdots X_{2n-1} = E_n$. Consequently, $A = X_{2n-1}^{-1} \cdots X_2^{-1}X_1^{-1}$. Hence, $A \in L(W_1, W_2, \ldots, W_{n+1})$ is true. From above, $T_n^1 = L(W_1, W_2, \ldots, W_{n+1})$ is tenable.

All n which are mentioned hereinafter satisfy the condition of $4 \le n \le 10$, and no longer explained. □

Theorem 18. *$T_n^2 = \{A \mid A = XF_n^2Y$, both X and $Y \in L(W_1, W_2, \ldots, W_{2n-1})\}$.*

Proof. Suppose $A = XF_n^2Y$ and $X, Y \in L(W_1, W_2, \ldots, W_{2n-1})$, then we get $A \in T_n$ because $F_n^2 \in T_n^2$ and $W_i \in T_n$. If we express $A = (a_{ij})$, then it is easy to check $\max_{i,j}|a_{ij}| = 2$. Therefore, $A \in T_n^2$, namely, $\{A \mid A = XF_n^2Y$, both X and $Y \in L(W_1, W_2, \ldots, W_{2n-1})\} \subset T_n^2$.

If $A \in T_n^2$, then A can left or right multiply by matrix in $L(W_1, W_2, \ldots, W_{2n-1})$; hence, matrix $P_1 \triangleq (p_{ij})_{n \times n}$ is obtained, and it makes equations $p_{22} = 2$, $|p_{n,n-1}| = |p_{n,n-2}| = |p_{n,n-3}| = 1 = |p_{n-1,n}| = |p_{n-2,n}| = |p_{n-3,n}|$, and $p_{nj} = p_{jn} = 0$ $(j < n-3)$ tenable.

If P_1 is partitioned into $P_1 = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$, in which P_{11} is $(n-4)$-th-order square matrix and P_{22} is 4-th-order square matrix, then there exists only one number which absolute value is equal to 1 in the former $(n-4)$ column (include no. $(n-4)$ column) of P_1 from equation (26). In a similar way, there exists only one number which absolute value is equal to 1 in the former $(n-4)$ row (include no. $(n-4)$ row) of P_1 from equation (26). Use equation (28) and reductio ad absurdum, we can get $P_{12} = 0$ and $P_{21} = 0$, that is to say $P_1 = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}$.

For matrix P_2, it can be obtained by matrix P_1 which left or right multiply by matrix in $L(W_4, W_5, \ldots, W_{n+1})$. It causes $P_2 = \begin{pmatrix} R_2 & 0 \\ 0 & Q \end{pmatrix}$, in which R_2 is a diagonal matrix and the absolute value of diagonal elements is equal to 1.

For matrix P_3, it can be obtained by matrix P_2 which left or right multiply by matrix in $L(W_1, W_2, \ldots, W_{2n-1})$. It causes $P_3 = \begin{pmatrix} E_{n-4} & 0 \\ 0 & Q \end{pmatrix}$.

For matrix P_4, it can be obtained by matrix P_3 which left or right multiply by matrix in $L(W_1, W_2, \ldots, W_{2n-1})$. It causes $P_4 = \begin{pmatrix} E_{n-4} & 0 & 0 \\ 0 & U & \beta \\ 0 & \alpha & 2 \end{pmatrix}$, in which $\alpha = (1, 1, 1)$ and $\beta = (1, 1, 1)^T$. We know that only two elements' absolute values are equal to 1 in every row (column) of U by equation (26), and only two elements in every row (column) of U are equal to 1 by equation (28). For matrix P_4, it can left or right multiply by matrix of $L(W_4, W_5, \ldots, W_{n+1})$. The result is matrix F_n^2.

From above, matrix F_n^2 can be obtained by A which left or right multiply by matrix in $L(W_1, W_2, \ldots, W_{2n-1})$. In other words, there exist $X_1, Y_1 \in L(W_1, W_2, \ldots, W_{2n-1})$; they cause $X_1AY_1 = F_n^2$. The other form is $A = (X_1)^{-1}F_n^2(Y_1)^{-1}$. Let $X = (X_1)^{-1}$ and $Y = (Y_1)^{-1}$, then $X, Y \in L(W_1, W_2, \ldots, W_{2n-1})$; so, $A = XF_n^2Y$. Therefore, $T_n^2 \subset \{A \mid A = XF_n^2Y$, X and $Y \in L(W_1, W_2, \ldots, W_{2n-1})\}$; it follows that $T_n^2 = \{A \mid A = XF_n^2Y$, X and $Y \in L(W_1, W_2, \ldots, W_{2n-1})\}$. □

Inference 4. $T_n^2 \subset L(F_n^2, W_2, W_3, \ldots, W_{n+1})$.

Theorem 19. *Arbitrary $A = (a_{ij}) \in T_n$, if maximum absolute value of A's element is y; furthermore, $y > 2$, and then there exist $Q_i \in L(W_2, W_3, W_4, \ldots, W_{n+1})$ $(i = 1, 2, \ldots, n)$ which make maximum absolute value of elements of matrix $H = AQ_1Q_2 \cdots Q_nF_n^2$ be less than y.*

Proof. Clearly, there exist $Q_1 \in L(W_2, W_3, W_4, W_5, \ldots, W_{n+1})$ which make the former $(n-1)$ elements of last row in matrix AQ_1 be nonnegative, and the last element of last row in matrix AQ_1 is equal to $-y$; there exist $Q_2, Q_3, \ldots, Q_n \in L(W_2, W_3, W_4, W_5, \ldots, W_{n+1})$ which make $Q \triangleq AQ_1Q_2 \cdots Q_n \triangleq (q_{ij})_{n \times n}$, of which $q_{nn} = -y$, $q_{ni} \ge 0$ $(i = 1, \ldots, n-1)$ and $q_{n1} \le q_{n2} \le \cdots \le q_{n,n-1}$. Let $H = QF_n^2 = (h_{ij})_{n \times n}$, and then we get $h_{nn} = q_{n,n-3} \cdot 1 + q_{n,n-2} \cdot 1 + q_{n,n-1} \cdot 1 - 2y$. Obviously, $q_{n,n-3} + q_{n,n-2} + q_{n,n-1} < 3y$.

Now we must prove that $q_{n,n-3} + q_{n,n-2} + q_{n,n-1} > y$ is tenable when $4 \le n \le 10$.

When $q_{n,n-3} = 0$, $q_{n,n-3} + q_{n,n-2} + q_{n,n-1} = \sum_{i=1}^{n-1} q_{ni} > y$ is true. Now we must prove that $q_{n,n-3} + q_{n,n-2} + q_{n,n-1} > y$ is still true when $4 \le n \le 10$ and $q_{n,n-3} > 0$. Otherwise, from $q_{n,n-3} + q_{n,n-2} + q_{n,n-1} \le y$ we can get $(q_{n,n-3} + q_{n,n-2} + q_{n,n-1})^2 \le y^2$. From $1 + \sum_{i=1}^{n-1} q_{ni}^2 = q_{nn}^2 = y^2$, we can get $1 + \sum_{i=1}^{n-4} q_{ni}^2 + q_{n,n-3}^2 + q_{n,n-2}^2 + q_{n,n-1}^2 - (\sum_{i=n-3}^{n-1} q_{ni})^2 = y^2 - (\sum_{i=n-3}^{n-1} q_{ni})^2 \ge 0$. Hence, $1 + \sum_{i=1}^{n-4} q_{ni}^2 \ge 2(q_{n,n-3}q_{n,n-2} + q_{n,n-3}q_{n,n-1} + q_{n,n-2}q_{n,n-1}) \ge 6q_{n,n-3}^2$. If $q_{n,n-3} > 1$, then $n - 4 \ge 6$; from previous agreement $n \le 10$, we know that $n - 4 = 6$; then from $1 + \sum_{i=1}^{n-4} q_{ni}^2 \ge 6q_{n,n-3}^2$ we get

$q_{n1} = q_{n2} = \cdots = q_{n,n-1}$, so $9q_{n1}^2 = q_{nn}^2 - 1$. But, it is out of question.

If $q_{n,n-3} = 1$ and $q_{n,n-1} \geq 2$, then we get $1 + (n-4) \geq 1 + \sum_{i=1}^{n-4} q_{ni}^2 \geq 2(2\cdot 1 + 1\cdot 1 + 1\cdot 1) = 8$, so $n \geq 11$; it contradicts with previous agreement $n \leq 10$; if $q_{n,n-3} = 1$ and $q_{n,n-1} = 1$, then $|q_{ni}| \leq 1$ is true of arbitrary $i \leq n-1$; from $1 + \sum_{i=1}^{n-4} q_{ni}^2 \geq 6q_{n,n-3}^2$ we know that $n-3 \geq 6$, namely, $n \geq 9$ and $q_{n,n-4} = 1$; hence, matrix H can be expressed as the form of $H = \begin{pmatrix} H_{11} & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & H_{16} \\ H_{21} & 1 & 1 & 1 & 1 & -y \end{pmatrix}$. From equation (27), we know that only two elements' absolute value of the former $n-1$ rows of $n-i\,(i = 1,2,3,4)$ columns in matrix H is equal to 1, and the rest elements are equal to zero. And this conclusion is incompatible with equation (6.18).

From above, $y < q_{n,n-3} + q_{n,n-2} + q_{n,n-1} < 3y$ is true; in other words, $|q_{n,n-3} + q_{n,n-2} + q_{n,n-1} - 2y| < |y|$ is true, namely, $|h_{nn}| < |y|$ is tenable. Hence, Theorem 19 is established. □

Theorem 20. $T_n = L(F_n^2, W_2, W_3, W_4, \ldots, W_{n+1})$. *In other words, T_n is a finitely generated group, and $F_n^2, W_2, W_3, W_4, \ldots, W_{n+1}$ is a generated tuple of T_n.*

Proof. Clearly $L(F_n^2, W_2, W_3, W_4, \ldots, W_{n+1}) \subset T_n$. Arbitrary $A \in T_n$, and it can be written as $A = (a_{ij})_{n\times n}$. If $|a_{nn}| > 2$, then $\exists\, Q_1, Q_2, \ldots, Q_n \in L(W_2, W_3, W_4, \ldots, W_{n+1})$, which make $H_1 = AQ_1Q_2 \ldots Q_n F_n^2 \triangleq (h_{ij}^1)_{n\times n}$ and $|h_{nn}^1| < |a_{nn}|$ by Theorem 19. In other words, $\exists X_1 \in L(F_n^2, W_2, W_3, W_4, \ldots, W_{n+1})$, which make $H_1 = AX_1 \triangleq (h_{ij}^1)_{n\times n}$, of which $|h_{nn}^1| < |a_{nn}|$; if $|h_{nn}^1| > 2$, then we can apply the theorem time after time; hence, we get $\exists\, X_1, X_2, \ldots, X_k \in L(F_n^2, W_2, W_3, W_4, \ldots, W_{n+1})$, which make $H_{m+1} = H_m X_{m+1}\,(m = 1, 2, \ldots, k-1) \triangleq (h_{ij}^m)_{n\times n}$ and $|h_{nn}^k| < |h_{nn}^{k-1}| < \cdots < |h_{nn}^1| < |a_{nn}|$. Because lower bound of $|h_{nn}^k|$ is 1 or 2, which is to say there exist k, which make $|h_{nn}^k| = 2$ or $|h_{nn}^1| = 1$. If $|h_{nn}^k| = 2$, then $H_k \in T_n^2$; $T_n^2 \subset L(F_n^2, W_2, W_3, W_4, \ldots, W_{n+1})$ is true by Theorem 18, so $A \in L(F_n^2, W_2, W_3, W_4, \ldots, W_{n+1})$. If $|h_{nn}^1| = 1$, then $H_k \in L(W_1, W_2, \ldots, W_{2n-1})$ is true by Theorem 17, so $H_k \in L(F_n^2, W_2, W_3, W_4, \ldots, W_{n+1})$.

From above, $T_n \subset L(F_n^2, W_2, W_3, W_4, \ldots, W_{n+1})$ is true, accordingly $T_n = L(F_n^2, W_2, W_3, W_4, \ldots, W_{n+1})$. This is what we want to prove.

We suppose that T_n $(n \geq 11)$ still is a finitely generated group, but the presentation of T_n need to be further studied. □

8. Future Work and Prospects

Let $W_n = \{(a_1, a_2, \ldots, a_n) \mid (a_1, a_2, \ldots, a_n)$ is n-order primitive Pythgorean vector$\}$.

Start with $(3, 4, 5)$ or $(4, 3, 5)$ and multiply F_1, F_2, or F_3 by it in any order any number of times, and all 3-dimensional primitive Pythagorean vectors can be formed trees which Cha et al. [15] call Berggren trees.

Since $F_1 \in T_3$, $F_2 \in T_3$, $F_3 \in T_3$, $D_i \in T_3$ $(1 \leq i \leq 9)$, and $T_3 = L(F_1, D_2, D_4, D_9)$, we get that every 3-order primitive Pythagorean vector can be obtained from multiplying F_1, D_2, D_4, or D_9 by $(3, 4, 5)$ in any order any number of times. Can all 3-dimensional primitive Pythagorean vectors be formed a Berggren tree starting with a primitive Pythagorean vector?

Using the definition and properties of T_3, we can obtain the another representation of W_3; that is, we have that $W_3 = \{(a, b, c) \mid (a, b, c) = (3, 4, 5) * F, \forall F \in T_3\}$. Does W_n $(n \geq 4)$ have a similar representation?

In this paper, we have given the generators of the finitely generated group T_n $(n \leq 10)$. Is T_n $(n > 10)$ a finitely generated group? If T_n $(n > 10)$ is a finitely generated group, what are the generators of T_n $(n > 10)$?

These appear to be interesting questions, which we hope to take up in the near future.

Data Availability

All data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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