Research Article

On a Discrete Markov-Modulated Risk Model with Random Premium Income and Delayed Claims

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In this paper, a discrete Markov-modulated risk model with delayed claims, random premium income, and a constant dividend barrier is proposed. It is assumed that the random premium income and individual claims are affected by a Markov chain with finite state space. The model proposed is an extension of the discrete semi-Markov risk model with random premium income and delayed claims. Explicit expressions for the total expected discounted dividends until ruin are obtained by the method of generating function and the theory of difference equations. Finally, the effect of related parameters on the total expected discounted dividends are shown in several numerical examples.

1. Introduction

The study of dividend strategies for the insurance risk model was first proposed by De Finetti [1], and he found that the optimal strategy must be a barrier strategy in the studied discrete time model. Since then, many scholars have tried to work out the dividend problems under more general and more realistic model assumptions, for example, Claramunt et al. [2], Zhou [3], Landriault [4], Gerber et al. [5], Chen et al. [6], Chen and Yuen [7], and Peng et al. [8, 9].

In practice, insurance claims may be delayed for various reasons. Yuen and Guo [10] considered a reasonable claim structure that includes two types of independent claims, namely, main claims and by-claims, but the occurrence of by-claims may be delayed. For example, for a catastrophe such as an earthquake or a rainstorm, it is very likely that there exist other insurance claims after the immediate ones. From then on, the related issues about a risk process with such time-correlated claims have been studied by many authors. Li and Wu [11] calculated the total expected discounted dividends up to the time of ruin in a discrete time risk model with delayed claims and a constant dividend barrier. Other risk models involving delayed claims were studied by Xiao and Guo [12], Bao and Liu [13], Zhou et al.

[14], Yuen et al. [15], Liu and Zhang [16], Deng et al. [17], and the references therein.

As an extension of the classical risk model, the risk model with random premium income has attracted a great deal of attention during the last few years. Boikov [18] generalized the classical risk model to the case where the premium process is determined by a compound Poisson process. Bao [19] extended the classical risk model to the case where the premium income process is a Poisson process and studied the expected discounted penalty at ruin. Assuming that the aggregate premium process is modeled as a compound Poisson process, Zou et al. [20] studied the expected discounted penalty function and optimal dividend strategy for a risk model with interclaim-dependent claim sizes. For the risk models with random premium income, one can also see [21–24] for more details.

Markov-modulated risk model where the surplus process is affected by an environmental Markov chain is a quite important model which attributed to the structural changes in conditions of economic, politic, or social, etc. Zhu and Yang [25] considered some ruin problems in a Markov regime-switching risk model where dividends were paid out according to a certain threshold strategy depending on the underlying Markovian environment process. Chen et al. [26]

studied the survival probability for a discrete semi-Markov risk model, which assumes individual claims are influenced by a Markov chain with finite state space. Yuen et al. [27] investigated the expected penalty functions for a discrete semi-Markov risk model with randomized dividends. Other related works can be found in Lu and Li [28], Diko and Usábel [29], Chen et al. [30], Huo et al. [31], etc.

In the discrete time risk model with random premium income, the authors often assumed that the random premium income follows a binomial process with a certain parameter, which means that the premium received in each period is a sequence of independent variables with the same Bernoulli distribution. In this paper, we propose a discrete time risk model where the random premium process and the individual claims are all influenced by a Markov chain. The model proposed in this paper is an extension of the discrete semi-Markov risk model with paying dividends and delayed claims. As mentioned in Chen et al. [26], the discrete semi-Markov risk model includes several existing risk models such as the compound binomial model and the compound Markov binomial model as special cases. So the discrete model studied in this paper is quite general. The explicit expressions for the expected discounted dividends until ruin are obtained for the model.

The outline of this paper is as follows. In Section 2, we introduce the model of this paper, including various parameters and notations. In Section 3, explicit expressions for the total expected discounted dividends until ruin are obtained by the method of generating function and the theory of difference equations. In Section 4, numerical examples are provided to illustrate the impact of the related parameters on the total expected discounted dividends.

2. Problem Formulation

We first recall the discrete time risk model with delayed claims that are studied by Yuen and Guo [10]. Denote the discrete time period by $t = 0, 1, 2, \dots$, and assume that premiums are received with a constant premium rate of 1 in each time period. The probability of a main claim occurring in any time period is p ($0 < p < 1$), and the probability of no main claim is $q = 1 - p$. Each main claim causes a by-claim, the probability that the main claim and by-claim occurs in the same period is θ , or the occurrence of by-claim is delayed to next time period with probability $1 - \theta$. Suppose that $\{\xi_k, k = 1, 2, \dots\}$ be a sequence of Bernoulli random variables with $\mathbb{P}(\xi_k = 1) = p$ and $\mathbb{P}(\xi_k = 0) = 1 - p$, describing whether or not a main claim occurs in the k -th time period,

and $\{\eta_k, k = 1, 2, \dots\}$ be another sequence of Bernoulli random variables with $\mathbb{P}(\eta_k = 1) = \theta$ and $\mathbb{P}(\eta_k = 0) = 1 - \theta$, describing whether or not the k -th by-claim occurs simultaneously with its corresponding main claim. Let $\{X_k, k = 1, 2, \dots\}$ be the independent and identically distributed (i.i.d.) main claims amounts and $\{Y_k, k = 1, 2, \dots\}$ be the independent and identically distributed (i.i.d.) by-claims amounts. Then the total amount of the main claims and by-claims until the end of t -th time period are given by

$$\begin{aligned} S_t^X &= \sum_{k=1}^t \xi_k X_k, \quad t = 1, 2, \dots, \\ S_1^Y &= \xi_1 \eta_1 Y_1, \\ S_t^Y &= \sum_{k=1}^t \xi_k \eta_k Y_k + \sum_{k=2}^t \xi_{k-1} (1 - \eta_{k-1}) Y_{k-1}, \quad t = 2, 3, \dots \end{aligned} \quad (1)$$

Hence, the surplus process for an insurance company is described as follows:

$$S(t) = u + t - S_t^X - S_t^Y, \quad (2)$$

where u is the initial surplus of the insurance company.

Now we introduce the risk model with delayed claims, a constant dividend barrier, and random premium income. We assume that premium income and dividends occur at the beginning of the period, and each claim occurs at the end of the period. If the surplus is above a constant barrier b , then the excess is paid out as a dividend. In addition, if the initial surplus $u > b$, $u - b$ is paid out as a dividend immediately. Denote by Z_k the amount of premium received in the k -th time period. Then the surplus process at the end of t -th time period of the risk model with delayed claims, a constant dividend barrier, and random premium income can be expressed as

$$U(t) = u + \sum_{k=1}^t Z_k - S_t^X - S_t^Y - \sum_{k=0}^t d_k, \quad (3)$$

where d_k is the amount of dividend paid out in the k -th period with $d_0 = \max\{u - b, 0\}$ and $\sum_{k=1}^0 Z_k = 0$.

In this paper, we assume that $\{J_k, k = 0, 1, \dots\}$ is a Markov chain with the state space $M = \{1, \dots, r\}$, and its transition matrix is written as $\mathbf{P} = (p_{ij})_{i,j \in M}$, where $p_{ij} = \mathbb{P}(J_{k+1} = j | J_k = i, J_t, t \leq k - 1)$. Furthermore, the distributions of premium income, main claims, and by-claims that depend on the Markov chain are defined as follows:

$$\begin{aligned} q_{ij}(l) &= \mathbb{P}(Z_t = l | J_t = j, J_{t-1} = i, J_k, Z_k, k \leq t - 1), \quad l = 0, 1, \\ g_{ij}(n) &= \mathbb{P}(X_t = n | J_t = j, J_{t-1} = i, J_k, X_k, k \leq t - 1), \quad n = 1, 2, \dots, \\ f_{ij}(m) &= \mathbb{P}(Y_t = m | J_t = j, J_{t-1} = i, J_k, Y_k, k \leq t - 1), \quad m = 1, 2, \dots \end{aligned} \quad (4)$$

For simplicity, this article only considers the case of $M = \{1, 2\}$. In addition, we agree that $\sum_{i=1}^2 \sum_{j=1}^2 q_{ij}(1) \neq 0$;

otherwise, there must be no premium income in any period. Define $T_{u,b} = \inf\{k \geq 1: U(k) < 0\}$ to be the ruin time of

model (3) and the one-period discount factor is $0 < v \leq 1$. Then the total expected discounted dividends until ruin given the initial surplus u and initial state i are defined as follows:

$$V_i(u, b) = E \left[\sum_{k=0}^{T_{u,b}} v^k d_k \mid U(0) = u, J_0 = i \right], \quad i = 1, 2. \quad (5)$$

The rest of this paper is devoted to obtain the explicit expressions of $V_i(u, b)$ for calculation purposes.

3. The Total Expected Discounted Dividends $V_i(u, b)$

In order to obtain the explicit expressions of the total expected discounted dividends, we need to consider the scenario that the by-claim induced by the main claim is delayed

to the next period (see Yuen and Guo [10]). According to this scenario, we define a complementary surplus process as follows:

$$U_1(t) = u + \sum_{k=1}^t Z_k - S_t^X - S_t^Y - \sum_{k=0}^t d_k - YI_{(t \geq 1)}, \quad (6)$$

where I_A is the indicator function of event A , Y is a random variable following the same probability function with $\{Y_k, k = 1, 2, \dots\}$ and is independent of all other claim amounts random variables X_i and Y_j for all i and j . The total expected discounted dividends of the complementary surplus process until ruin are denoted by $V_{1,i}(u, b)$.

Based on the condition of premium income, claims, and dividends, the following equations can be obtained by using the law of total probability for the first period.

For $u = 0, 1, \dots, b$,

$$\begin{aligned} V_i(u, b) = & vq \sum_{j=1}^2 p_{ij} q_{ij}(0) V_j(u, b) + vp\theta \sum_{j=1}^2 \sum_{m+n \leq u} p_{ij} q_{ij}(0) g_{ij}(m) f_{ij}(n) V_j(u - m - n, b) \\ & + vq \sum_{j=1}^2 p_{ij} q_{ij}(1) V_j(u + 1, b) + vp\theta \sum_{j=1}^2 \sum_{m+n \leq u+1} p_{ij} q_{ij}(1) g_{ij}(m) f_{ij}(n) V_j(u + 1 - m - n, b) \\ & + vp(1 - \theta) \sum_{j=1}^2 \sum_{m=1}^u p_{ij} q_{ij}(0) g_{ij}(m) V_{1,j}(u - m, b) \\ & + vp(1 - \theta) \sum_{j=1}^2 \sum_{m=1}^{u+1} p_{ij} q_{ij}(1) g_{ij}(m) V_{1,j}(u + 1 - m, b), \quad i = 1, 2, \end{aligned} \quad (7)$$

$$V_{1,i}(0, b) = vq \sum_{j=1}^2 p_{ij} q_{ij}(1) f_{ij}(1) V_j(0, b), \quad i = 1, 2, \quad (8)$$

and for $u = 1, \dots, b$,

$$\begin{aligned} V_{1,i}(u, b) = & vq \sum_{j=1}^2 \sum_{l=1}^u p_{ij} q_{ij}(0) f_{ij}(l) V_j(u - l, b) + vq \sum_{j=1}^2 \sum_{l=1}^{u+1} p_{ij} q_{ij}(1) f_{ij}(l) V_j(u + 1 - l, b) \\ & + vp\theta \sum_{j=1}^2 \sum_{m+n+l \leq u} p_{ij} q_{ij}(0) g_{ij}(m) f_{ij}(n) f_{ij}(l) V_j(u - m - n - l, b) \\ & + vp\theta \sum_{j=1}^2 \sum_{m+n+l \leq u+1} p_{ij} q_{ij}(1) g_{ij}(m) f_{ij}(n) f_{ij}(l) V_j(u + 1 - m - n - l, b) \\ & + vp(1 - \theta) \sum_{j=1}^2 \sum_{m+l \leq u} p_{ij} q_{ij}(0) g_{ij}(m) f_{ij}(l) V_{1,j}(u - m - l, b) \\ & + vp(1 - \theta) \sum_{j=1}^2 \sum_{m+l \leq u+1} p_{ij} q_{ij}(1) g_{ij}(m) f_{ij}(l) V_{1,j}(u + 1 - m - l, b), \quad i = 1, 2. \end{aligned} \quad (9)$$

According to the barrier dividend strategy, for $u = b + 1, \dots$, we have

$$\begin{aligned} V_i(u, b) &= u - b + V_i(b, b), \quad i = 1, 2, \\ V_{1,i}(u, b) &= u - b + V_{1,i}(b, b), \quad i = 1, 2. \end{aligned} \quad (10)$$

From (7) and (9), $V_{1,i}(u, b)$ can be rewritten as

$$V_{1,i}(u, b) = \sum_{l=1}^u V_i(u-l, b) \sum_{j=1}^2 p_{ij} f_{ij}(l), \quad i = 1, 2. \quad (11)$$

This result can also be obtained from model (6) as

$$\begin{aligned} V_{1,i}(u, b) &= E[V_i(u-Y, b)] \\ &= \sum_{l=1}^u V_i(u-l, b) \sum_{j=1}^2 p_{ij} f_{ij}(l), \quad i = 1, 2. \end{aligned} \quad (12)$$

Substituting (8) into (7), we get

$$\begin{aligned} V_i(0, b) &= vq \sum_{j=1}^2 p_{ij} q_{ij}(0) V_j(0, b) \\ &\quad + vq \sum_{j=1}^2 p_{ij} q_{ij}(1) V_j(1, b) \\ &\quad + v^2 pq(1-\theta) \sum_{j=1}^2 p_{ij} q_{ij}(1) g_{ij}(1) \\ &\quad \cdot \sum_{k=1}^2 p_{jk} q_{jk}(1) f_{jk} \\ &\quad \cdot (1) V_k(0, b), \quad i = 1, 2. \end{aligned} \quad (13)$$

Similarly, substituting (11) into (7), we have

$$\begin{aligned} V_i(u, b) &= vq \sum_{j=1}^2 p_{ij} q_{ij}(0) V_j(u, b) \\ &\quad + vq \sum_{j=1}^2 p_{ij} q_{ij}(1) V_j(u+1, b) \\ &\quad + vp \sum_{j=1}^2 \sum_{m+n \leq u} p_{ij} h_{ij}^X(m) V_j(u-m-n, b) \\ &\quad \cdot \left[\theta f_{ij}(n) + (1-\theta) \sum_{k=1}^2 p_{jk} f_{jk}(n) \right], \\ u &= 1, 2, \dots, b, \quad i = 1, 2, \end{aligned} \quad (14)$$

where $h_{ij}^X(m) = q_{ij}(0)g_{ij}(m) + q_{ij}(1)g_{ij}(m+1)$ with $g_{ij}(0) = 0, m = 0, 1, 2, \dots$

To obtain the explicit expressions of $V_i(u, b)$, we define new functions $W_i(u)$ as follows:

$$\begin{aligned} W_i(0) &= vq \sum_{j=1}^2 p_{ij} q_{ij}(0) W_j(0) + vq \sum_{j=1}^2 p_{ij} q_{ij}(1) W_j(1) \\ &\quad + v^2 pq(1-\theta) \sum_{j=1}^2 p_{ij} q_{ij}(1) g_{ij}(1) \sum_{k=1}^2 p_{jk} q_{jk}(1) f_{jk} \\ &\quad \cdot (1) W_k(0), \quad i = 1, 2. \end{aligned} \quad (15)$$

$$\begin{aligned} W_i(u) &= vq \sum_{j=1}^2 p_{ij} q_{ij}(0) W_j(u) + vq \sum_{j=1}^2 p_{ij} q_{ij}(1) W_j(u+1) \\ &\quad + vp \sum_{j=1}^2 \sum_{m+n \leq u} p_{ij} h_{ij}^X(m) W_j(u-m-n) \\ &\quad \cdot \left[\theta f_{ij}(n) + (1-\theta) \sum_{k=1}^2 p_{jk} f_{jk}(n) \right], \\ u &\in \mathbb{N}_+, \quad i = 1, 2. \end{aligned} \quad (16)$$

We use $\tilde{h}_{ij}^X(s)$, $\tilde{f}_{ij}(s)$, and $\tilde{W}_i(s)$ to represent the generating functions of $h_{ij}^X(k)$, $f_{ij}(k)$, and $W_i(k)$, respectively. Multiplying both sides of (15) and (16) by s^{u+1} , next summing over u from 0 to ∞ gives

$$\begin{aligned} \frac{s}{v} \tilde{W}_i(s) &= sq \sum_{j=1}^2 p_{ij} q_{ij}(0) \tilde{W}_j(s) + q \sum_{j=1}^2 p_{ij} q_{ij}(1) (\tilde{W}_j(s) - W_j(0)) \\ &\quad + sp \sum_{j=1}^2 p_{ij} \tilde{W}_j(s) \tilde{h}_{ij}^X(s) \left[\theta \tilde{f}_{ij}(s) + (1-\theta) \sum_{k=1}^2 p_{jk} \tilde{f}_{jk}(s) \right] \\ &\quad + svpq(1-\theta) \sum_{j=1}^2 p_{ij} q_{ij}(1) g_{ij}(1) \sum_{k=1}^2 p_{jk} q_{jk}(1) f_{jk}(1) W_k(0), \quad i = 1, 2. \end{aligned} \quad (17)$$

For the convenience of the following derivations, we define

$$\begin{aligned}\tilde{e}_1(s) &= q \sum_{j=1}^2 p_{1j} q_{1j}(1) W_j(0) - svpq(1-\theta) \sum_{j=1}^2 p_{1j} q_{1j}(1) g_{1j}(1) \sum_{k=1}^2 p_{jk} q_{jk}(1) f_{jk}(1) W_k(0), \\ \tilde{e}_2(s) &= q \sum_{j=1}^2 p_{2j} q_{2j}(1) W_j(0) - svpq(1-\theta) \sum_{j=1}^2 p_{2j} q_{2j}(1) g_{2j}(1) \sum_{k=1}^2 p_{jk} q_{jk}(1) f_{jk}(1) W_k(0), \\ \tilde{e}_{ij}(s) &= qp_{ij} q_{ij}(1) + s \left[qp_{ij} q_{ij}(0) + pp_{ij} \tilde{h}_{ij}^X(s) \left(\theta \tilde{f}_{ij}(s) + (1-\theta) \sum_{k=1}^2 p_{jk} \tilde{f}_{jk}(s) \right) \right], \quad i, j = 1, 2.\end{aligned}\tag{18}$$

Now, (17) can be written as

$$\begin{cases} \left(\tilde{e}_{11}(s) - \frac{s}{v} \right) \tilde{W}_1(s) + \tilde{e}_{12}(s) \tilde{W}_2(s) = \tilde{e}_1(s), \\ \tilde{e}_{21}(s) \tilde{W}_1(s) + \left(\tilde{e}_{22}(s) - \frac{s}{v} \right) \tilde{W}_2(s) = \tilde{e}_2(s). \end{cases}\tag{19}$$

To facilitate the following derivation, denote

Simplifying the above equation yields

$$\begin{aligned}a_i(0) &= qp_{ii} q_{ii}(1), \\ a_i(1) &= qp_{ii} q_{ii}(0) + pp_{ii} h_{ii}^X(0) \left[\theta f_{ii}(0) + (1-\theta) \sum_{k=1}^2 p_{ik} f_{ik}(0) \right] - \frac{1}{v}, \\ a_i(k) &= pp_{ii} \sum_{n=0}^{k-1} h_{ii}^X(k-1-n) \left[\theta f_{ii}(n) + (1-\theta) \sum_{k=1}^2 p_{ik} f_{ik}(n) \right], \quad k = 2, 3, \dots, i = 1, 2, \\ b_{ij} &= v p q (1-\theta) \sum_{k=1}^2 p_{ik} q_{ik}(1) g_{ik}(1) p_{kj} q_{kj}(1) f_{kj}(1), \quad i, j = 1, 2, \\ f_k &= \sum_{n=0}^k [a_1(n) a_2(k-n) - e_{12}(n) e_{21}(k-n)], \\ g_k^{(1)} &= \sum_{n=0}^k W_1(n) f_{k-n}, \\ h_k^{(1)} &= e_1(0) a_2(k) + e_1(1) a_2(k-1) - e_2(0) e_{12}(k) - e_2(1) e_{12}(k-1), \quad k \in \mathbb{N},\end{aligned}\tag{21}$$

where $e_i(k)$ and $e_{ij}(k)$ are the coefficients of s^k on functions $\tilde{e}_i(s)$ and $\tilde{e}_{ij}(s)$. In addition, $a_i(-1) = 0, e_{ij}(-1) = 0, i, j = 1, 2$.

Comparing the coefficients of s^k in both sides of (20), we get $g_k^{(1)} = h_k^{(1)}, k \in \mathbb{N}$. So we have

$$\sum_{n=0}^k W_1(n) f_{k-n} = h_k^{(1)}, \quad k \in \mathbb{N}.\tag{22}$$

In a similar way, we obtain that

$$\sum_{n=0}^k W_2(n) f_{k-n} = h_k^{(2)}, \quad k \in \mathbb{N},\tag{23}$$

where $h_k^{(2)} = -e_1(0) e_{21}(k) - e_1(1) e_{21}(k-1) + e_2(0) a_1(k) + e_2(1) a_1(k-1)$.

From (22) and (23), we know that f_k play a decisive role in the value of $\{W_i(k), k \in \mathbb{N}, i = 1, 2\}$. So we discuss the values of f_k in the following lemma.

Lemma 1. *If $f_0 = 0$, then $f_1 \neq 0$.*

Proof. Note that

$$\begin{aligned} f_0 &= e_{11}(0)e_{22}(0) - e_{12}(0)e_{21}(0), \\ f_1 &= e_{11}(0)\left[e_{22}(1) - \frac{1}{\nu}\right] + e_{22}(0)\left[e_{11}(1) - \frac{1}{\nu}\right] \\ &\quad - e_{21}(0)e_{12}(1) - e_{12}(0)e_{21}(1). \end{aligned} \quad (24)$$

If $f_0 = 0$, we can derive $f_1 \neq 0$ by using reduction to absurdity. Assuming $f_1 = 0$ yields

$$\begin{aligned} e_{11}(0) &= 0, \\ e_{22}(0) &= 0, \\ e_{21}(0)e_{12}(1) &= 0, \\ e_{21}(1)e_{12}(0) &= 0, \end{aligned} \quad (25)$$

where $e_{11}(0) = qp_{11}q_{11}(1)$ means $q_{11}(1) = 0$, and $e_{22}(0) = qp_{22}q_{22}(1)$ means $q_{22}(1) = 0$.

Since $f_0 = 0$, we get $e_{21}(0)e_{12}(0) = 0$. Note that $\sum_{i=1}^2 \sum_{j=1}^2 q_{ij}(1) \neq 0$, there are only two situations:

$$\begin{aligned} \xi_k^{(1)} &= e_{11}(0)a_2(k) - b_{11}a_2(k-1) - e_{21}(0)e_{12}(k) + b_{21}e_{12}(k-1), \\ \eta_k^{(1)} &= e_{12}(0)a_2(k) - b_{12}a_2(k-1) - e_{22}(0)e_{12}(k) + b_{22}e_{12}(k-1), \\ \xi_k^{(2)} &= e_{21}(0)a_1(k) - b_{21}a_1(k-1) - e_{11}(0)e_{21}(k) + b_{11}e_{21}(k-1), \\ \eta_k^{(2)} &= e_{22}(0)a_1(k) - b_{22}a_1(k-1) - e_{12}(0)e_{21}(k) + b_{12}e_{21}(k-1). \end{aligned} \quad (28)$$

Note that $h_k^{(i)}$ can be rewritten as

$$h_k^{(i)} = \xi_k^{(i)}W_1(0) + \eta_k^{(i)}W_2(0), \quad i = 1, 2, k \in \mathbb{N}, \quad (29)$$

which are determined by $W_1(0)$ and $W_2(0)$, and we can see from (27) that $\{W_1(k), W_2(k)\}_{k \in \mathbb{N}_+}$ is also determined by the initial values $W_1(0)$ and $W_2(0)$. Therefore, apart from a multiplicative constant, the solution of (15) and (16) is unique. For $j = 1, 2$, we set $\{C_{1j}(k), C_{2j}(k)\}$ to be the linearly independent particular solutions of (15) and (16) with initial conditions $C_{ij}(0) = I_{(i=j)}$. Then the general solution of (15) and (16) is given by

$$W_i(u) = \sum_{j=1}^2 W_j(0)C_{ij}(u), \quad u \in \mathbb{N}, i = 1, 2. \quad (30)$$

The same procedure may be easily adapted to obtain the solutions of (13) and (14), that is,

(i) If $e_{12}(0) = 0$ and $e_{21}(0) \neq 0$, we have

$$\begin{aligned} e_{12}(1) &= qp_{12}q_{12}(0) + pp_{12}h_{12}^X(0) \\ &\quad \cdot \left[\theta f_{12}(0) + (1 - \theta) \sum_{k=1}^2 p_{12}f_{12}(0) \right] = 0, \end{aligned} \quad (26)$$

which means $q_{12}(0) = 0$, and $e_{12}(0) = 0$ yields $q_{12}(1) = 0$. The result $\sum_{l=0}^1 q_{12}(l) \neq 1$ is obviously not in agreement with reality, so there is no such situation.

(ii) If $e_{21}(0) = 0$ and $e_{12}(0) \neq 0$, we can derive in a similar way that $\sum_{l=0}^1 q_{21}(l) \neq 1$, which is also not realistic. The proof is complete.

In order to obtain the explicit expressions of $V_i(u, b)$, we shall distinguish two cases.

(1) If $f_0 \neq 0$, equations (22) and (23) yield

$$W_i(k) = \frac{1}{f_0} \left[h_k^{(i)} - \sum_{n=0}^{k-1} W_i(n)f_{k-n} \right], \quad i = 1, 2, k \in \mathbb{N}_+. \quad (27)$$

Denote

$$V_i(u, b) = \sum_{j=1}^2 V_j(0, b)C_{ij}(u), \quad i = 1, 2, u = 0, 1, \dots, b+1. \quad (31)$$

Let

$$\begin{aligned} \vec{V}(u, b) &= (V_1(u, b), V_2(u, b))^T, \\ \mathbf{C}(u) &= (C_{ij}(u))_{i,j=1,2}, \\ \vec{1} &= (1, 1)^T. \end{aligned} \quad (32)$$

Rewriting (31) into matrix form as follows:

$$\vec{V}(u, b) = \mathbf{C}(u)\vec{V}(0, b), \quad i = 1, 2, u = 0, 1, \dots, b+1. \quad (33)$$

Combining (33) with the boundary condition $\vec{V}(b+1, b) = \vec{1} + \vec{V}(b, b)$, we get

$$\vec{V}(0, b) = [C(b+1) - C(b)]^{-1} \vec{1}, \quad (34)$$

which in turn yields that

$$\vec{V}(u, b) = C(u)[C(b+1) - C(b)]^{-1} \vec{1}, \quad u = 0, 1, \dots, b. \quad (35)$$

(2) If $f_0 = 0$, Lemma 1 guarantees that $f_1 \neq 0$. Based on (22) and (23), we get $W_i(0) = (1/f_1)h_1^{(i)}$, $i = 1, 2$, and

$$W_i(k) = \frac{1}{f_1} \left[h_{k+1}^{(i)} - \sum_{n=0}^{k-1} W_i(n) f_{k+1-n} \right], \quad i = 1, 2, k \in \mathbb{N}_+. \quad (36)$$

Define

$$\begin{aligned} J_1 &= \begin{pmatrix} \xi_1^{(1)} - f_1 & \eta_1^{(1)} \\ \xi_1^{(2)} & \eta_1^{(2)} - f_1 \end{pmatrix}, \\ \vec{W}(0) &= (W_1(0), W_2(0))^T, \\ \vec{0} &= (0, 0)^T. \end{aligned} \quad (37)$$

We obtain that

$$J_1 \vec{W}(0) = \vec{0}. \quad (38)$$

Let

$$\begin{aligned} D_0 &= \begin{pmatrix} e_{22}(0) & -e_{12}(0) \\ -e_{21}(0) & e_{11}(0) \end{pmatrix}, \\ E &= \begin{pmatrix} \frac{1}{v} - e_{11}(1) & -e_{12}(1) \\ -e_{21}(1) & \frac{1}{v} - e_{22}(1) \end{pmatrix}. \end{aligned} \quad (39)$$

Because of $J_1 = D_0 E$, $D_0 \neq 0$, $|D_0| = f_0 = 0$ and $|E| > 0$, we have $\text{rank}(J_1) = \text{rank}(D_0) = 1$. Equation (38) shows that only one of $W_1(0)$ and $W_2(0)$ is free variable. Assuming that $W_1(0)$ is the free variable without loss of generality, then we know that $\{W_1(k), W_2(k)\}_{k \in \mathbb{N}_+}$ is determined by the initial value $W_1(0)$. Set $\{W_1^{(1)}(u), W_2^{(1)}(u), u \in \mathbb{N}\}$ to be the linearly independent particular solution of (15) and (16) with initial conditions $W_1^{(1)}(0) = 1$, and then the general solution of (15) and (16) is of the form:

$$(W_1(u), W_2(u)) = W_1(0)(W_1^{(1)}(u), W_2^{(1)}(u)), \quad u \in \mathbb{N}. \quad (40)$$

Therefore, the solution of (13) and (14) is given by

$$(V_1(u, b), V_2(u, b)) = V_1(0, b)(W_1^{(1)}(u), W_2^{(1)}(u)), \quad u = 0, 1, \dots, b+1. \quad (41)$$

Combining the above equation with $V_1(b+1, b) = 1 + V_1(b, b)$ gives

$$V_1(0, b) = [W_1^{(1)}(b+1) - W_1^{(1)}(b)]^{-1}. \quad (42)$$

So we have

$$\begin{aligned} (V_1(u, b), V_2(u, b)) &= [W_1^{(1)}(b+1) - W_1^{(1)}(b)]^{-1} \\ &\quad \cdot (W_1^{(1)}(u), W_2^{(1)}(u)), \\ &u = 0, 1, \dots, b. \end{aligned} \quad (43)$$

To sum up, the main result of this paper is as follows. $\square$

Theorem 1. For any given dividend barrier $b \in \mathbb{N}_+$ and initial surplus $u = 0, 1, \dots, b$, the total expected discounted dividends until ruin satisfies the following explicit expression:

$$\vec{V}(u, b) = \begin{cases} C(u)[C(b+1) - C(b)]^{-1} \vec{1}, & f_0 \neq 0, \\ [W_1^{(1)}(b+1) - W_1^{(1)}(b)]^{-1} (W_1^{(1)}(u), W_2^{(1)}(u)), & f_0 = 0, \end{cases} \quad (44)$$

where $C(u) = (C_{ij}(u))_{i,j=1,2}$ with $C_{ij}(0) = I_{(i=j)}$ and $C_{ij}(k) = (1/f_0)[\xi_k^{(i)} I_{\{j=1\}} + \eta_k^{(i)} I_{\{j=2\}} - \sum_{n=0}^{k-1} C_{ij}(n) f_{k+1-n}]$, 2
 $k \in \mathbb{N}_+$, $i, j = 1, 2$; $W_1^{(1)}(0) = 1, W_2^{(1)}(0) = 0, W_i^{(1)}(k) = (1/f_1)[\xi_{k+1}^{(i)} - \sum_{n=0}^{k-1} W_i^{(1)}(n) f_{k+1-n}]$, $k \in \mathbb{N}_+$, $i = 1, 2$.

4. Numerical Simulation

In this section, we use several examples to illustrate the conclusions of $f_0 \neq 0$ in Section 3 and explain the impact of u, b, v, p , and θ on the total expected discounted dividends. In the following numerical examples, the transition

TABLE 1: The distribution of premium income.

l	$q_{11}(l)$	$q_{12}(l)$	$q_{21}(l)$	$q_{22}(l)$
0	1/10	2/9	1/6	0
1	9/10	7/9	5/6	1

TABLE 2: The distribution of main claim.

n	$g_{11}(n)$	$g_{12}(n)$	$g_{21}(n)$	$g_{22}(n)$
1	13/16	3/4	5/6	1
2	1/8	1/4	0	0
3	1/16	0	1/6	0
≥ 4	0	0	0	0

TABLE 3: Values of $V_i(u, b)$ when $\nu = 0.93$.

	i	$b = 0$	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$	$b = 6$	$b = 7$	$b = 8$
$u = 0$	1	2.7495	2.6384	2.4991	2.2867	2.0104	1.7367	1.4871	1.2678	1.0786
	2	3.0406	2.9996	2.8248	2.5835	2.2715	1.9625	1.6805	1.4327	1.2188
$u = 1$	1	3.7495	3.5290	3.3566	3.0723	2.7009	2.3331	1.9977	1.7031	1.4488
	2	4.0406	4.0274	3.7846	3.4606	3.0428	2.6290	2.2513	1.9193	1.6328
$u = 2$	1	4.7495	4.5290	4.3473	3.9821	3.5001	3.0231	2.5884	2.2066	1.8771
	2	5.0406	5.0274	4.7007	4.2965	3.7781	3.2645	2.7956	2.3835	2.0277
$u = 3$	1	5.7495	5.5290	5.3473	4.9066	4.3112	3.7225	3.1867	2.7164	2.3107
	2	6.0406	6.0274	5.7007	5.2055	4.5783	3.9566	3.3887	2.8892	2.4580
$u = 4$	1	6.7495	6.5290	6.3473	5.9066	5.1856	4.4744	3.8288	3.2631	2.7756
	2	7.0406	7.0274	6.7007	6.2055	5.4603	4.7208	4.0440	3.4483	2.9337
$u = 5$	1	7.7495	7.5290	7.3473	6.9066	6.1856	5.3283	4.5553	3.8807	3.3003
	2	8.0406	8.0274	7.7007	7.2055	6.4603	5.5906	4.7916	4.0867	3.4773

TABLE 4: Values of $V_i(u, b)$ when $\nu = 0.95$.

	i	$b = 0$	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$	$b = 6$	$b = 7$	$b = 8$
$u = 0$	1	3.0651	3.1731	3.2602	3.2280	3.0292	2.7687	2.4935	2.2277	1.9819
	2	3.3921	3.6103	3.6819	3.6419	3.4173	3.1235	2.8130	2.5132	2.2360
$u = 1$	1	4.0651	4.1265	4.2627	4.2236	3.9637	3.6227	3.2625	2.9148	2.5932
	2	4.3921	4.7233	4.8039	4.7500	4.4569	4.0736	3.6688	3.2778	2.9162
$u = 2$	1	5.0651	5.1265	5.3592	5.3184	4.9918	4.5622	4.1086	3.6706	3.2656
	2	5.3921	5.7233	5.7831	5.7131	5.3603	4.8994	4.4126	3.9424	3.5075
$u = 3$	1	6.0651	6.1265	6.3592	6.3342	5.9470	5.4349	4.8941	4.3722	3.8898
	2	6.3921	6.7233	6.7831	6.6872	6.2730	5.7339	5.1644	4.6142	4.1052
$u = 4$	1	7.0651	7.1265	7.3592	7.3342	6.8909	6.2966	5.6691	5.0641	4.5051
	2	7.3921	7.7233	7.7831	7.6872	7.2082	6.5892	5.9353	5.3032	4.7183
$u = 5$	1	8.0651	8.1265	8.3592	8.3342	7.8909	7.2077	6.4869	5.7934	5.1534
	2	8.3921	8.7233	8.7831	8.6872	8.2082	7.5049	6.7616	6.0422	5.3762

probability matrix of Markov chain is set to be $\begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$, the distribution of premium income is shown in Table 1, the distribution of main claim is shown in Table 2, and we assume that the distribution of by-claim is the same as main claim.

Example 1. In this example, we set $p = 0.25$ and $\theta = 0.3$. The results of $V_i(u, b)$ for $\nu = 0.93, 0.95, 0.97$ are listed in Tables 3–5, respectively. We can see that the expected discounted dividends are gradually increasing with the increase of the initial surplus u or one-period discount factor ν . Given the initial surplus u , set the optimal dividend boundary b^* be the boundary value which

maximizes the expected discounted dividends $V_i(u, b)$. From Table 3, it is easy to see that $b^* = 0$ when $\nu = 0.93$. Tables 4 and 5 show that $b^* = 2$ when $\nu = 0.95$ and $b^* = 4$ when $\nu = 0.97$. The results in this example imply that it would be better to pay dividends earlier when the discount factor ν is smaller.

Example 2. In this example, we take $\nu = 0.95$, $b = 8$, and $\theta = 0.3$ and investigate the effect of p on the expected discounted dividends $V_i(u, b)$. It can be seen from Table 6 that the expected discounted dividends are a decreasing function of p , and the result is in accordance with the fact.

TABLE 5: Values of $V_i(u, b)$ when $v = 0.97$.

	i	$b = 0$	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$	$b = 6$	$b = 7$	$b = 8$
$u = 0$	1	3.4458	3.9036	4.4567	4.9317	5.0810	5.0235	4.8360	4.5801	4.2966
	2	3.8163	4.4453	5.0289	5.5564	5.7232	5.6582	5.4469	5.1587	4.8394
$u = 1$	1	4.4458	4.9370	5.6744	6.2860	6.4775	6.4044	6.1654	5.8391	5.4777
	2	4.8163	5.6693	6.3918	7.0582	7.2693	7.1866	6.9183	6.5522	6.1466
$u = 2$	1	5.4458	5.9370	6.9269	7.6922	7.9298	7.8409	7.5483	7.1489	6.7064
	2	5.8163	6.6693	7.4579	8.2241	8.4681	8.3714	8.0589	7.6324	7.1600
$u = 3$	1	6.4458	6.9370	7.9269	8.8545	9.1369	9.0361	8.6991	8.2386	7.7287
	2	6.8163	7.6693	8.4579	9.2962	9.5667	9.4566	9.1034	8.6217	8.0881
$u = 4$	1	7.4458	7.9370	8.9269	9.8545	10.1930	10.0848	9.7090	9.1950	8.6258
	2	7.8163	8.6693	9.4579	10.2962	10.5815	10.4572	10.0664	9.5338	8.9438
$u = 5$	1	8.4458	8.9370	9.9269	10.8545	11.1930	11.0854	10.6735	10.1082	9.4821
	2	8.8163	9.6693	10.4579	11.2962	11.5815	11.4388	11.0106	10.4282	9.7831

TABLE 6: Values of $V_i(u, b)$ when $v = 0.95, b = 8$, and $\theta = 0.3$.

	i	$p = 0.05$	$p = 0.10$	$p = 0.15$	$p = 0.20$	$p = 0.25$	$p = 0.30$	$p = 0.40$	$p = 0.50$
$u = 0$	1	8.8392	6.6890	4.8048	3.2247	1.9819	1.0907	0.2202	0.0250
	2	9.1448	7.0774	5.1970	3.5635	2.2360	1.2551	0.2627	0.0307
$u = 1$	1	9.6611	7.5912	5.6875	4.0016	2.5932	1.5145	0.3523	0.0479
	2	9.9655	7.9947	6.1201	4.4017	2.9162	1.7408	0.4222	0.0595
$u = 2$	1	10.4853	8.4901	6.5824	4.8166	3.2656	2.0094	0.5327	0.0863
	2	10.6805	8.7312	6.8512	5.0866	3.5075	2.1997	0.6087	0.1030
$u = 3$	1	11.2369	9.2628	7.3470	5.5332	3.8898	2.5039	0.7512	0.1456
	2	11.4095	9.4604	7.5651	5.7595	4.1052	2.6877	0.8413	0.1712
$u = 4$	1	11.9809	9.9978	8.0648	6.2146	4.5051	3.0202	1.0216	0.2383
	2	12.1553	10.1900	8.2728	6.4317	4.7183	3.2124	1.1328	0.2781
$u = 5$	1	12.7585	10.7568	8.8011	6.9170	5.1534	3.5870	1.3640	0.3842
	2	12.9415	10.9558	9.0143	7.1396	5.3762	3.7960	1.5035	0.4462

TABLE 7: Values of $V_i(u, b)$ when $v = 0.95, b = 8$, and $p = 0.25$.

	i	$\theta = 0$	$\theta = 0.2$	$\theta = 0.4$	$\theta = 0.6$	$\theta = 0.8$	$\theta = 1$
$u = 0$	1	2.0822	2.0141	1.9509	1.8920	1.8370	1.7855
	2	2.3755	2.2807	2.1930	2.1117	2.0361	1.9656
$u = 1$	1	2.6327	2.6059	2.5810	2.5578	2.5361	2.5158
	2	2.9512	2.9274	2.9055	2.8854	2.8669	2.8499
$u = 2$	1	3.2963	3.2755	3.2559	3.2374	3.2197	3.2029
	2	3.5201	3.5114	3.5038	3.4971	3.4914	3.4864
$u = 3$	1	3.9092	3.8961	3.8836	3.8715	3.8599	3.8487
	2	4.1132	4.1077	4.1029	4.0989	4.0955	4.0927
$u = 4$	1	4.5186	4.5095	4.5008	4.4924	4.4844	4.4767
	2	4.7226	4.7196	4.7172	4.7150	4.7133	4.7119
$u = 5$	1	5.1637	5.1567	5.1500	5.1436	5.1374	5.1313
	2	5.3776	5.3766	5.3758	5.3753	5.3749	5.3747

Example 3. In this example, we discuss the impact of θ on the expected discounted dividends $V_i(u, b)$. Let $v = 0.95$, $b = 8$, and $p = 0.25$. The values of $V_i(u, b)$ are shown in Table 7. It is easy to see that the expected discounted dividends $V_i(u, b)$ gradually decrease with the increase of θ , and the effect of θ on $V_i(u, b)$ is decreasing with respect to the initial surplus u .

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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