Research Article

Nonlinear Buckling Analysis of Thin-Walled Box Beams considering Shear Lag

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In this work, a general geometric nonlinear model of straight thin-walled box beams (STBBs) under combined eccentric and axial loads is established. In order to accurately reflect the behavior of STBB, the additional shear lag warping is added to enrich the displacement field. It is necessary to define the section shape function to describe the local section deformation. Therefore, extension, bending, torsion, distortion, and shear lag effects are expressed by the generalized coordinate method. Based on the stability of transverse unconstrained box beam theory, meaningful higher-order solutions can be obtained by defining a set of coupled deformation modes. The equilibrium equation is discretized by the Galerkin method, and the Newton–Raphson incremental method is used to derive and solve the nonlinear governing equations. On this basis, the analytical expression of stiffness matrix is established. For solving the stability problem, the effectiveness of the proposed method is verified by comparing the calculation results of shell element (Ansys) with other theories. Numerical examples even show that the proposed method can not only get the influence of shear lag but also obtain the variation of lateral buckling of the beam model.

1. Introduction

As a kind of important load-bearing structural member, performance and design are a very significant aspect of thin-walled elements because they minimize the weight and cost of the structure. The classical Vlasov beam theory, St. Venant torsion theory, and the thin-walled beam theory put forward several calculation formulas including transverse deformation and displacement, according to the research of different open and closed cross sections [1]. On the account of the kinematics hypothesis that the shear strain was ignored in the central line of the cross section, classical formulas about distortion theory were proposed on the basis of Vlasov theory. Correia et al. discussed the buckling behavior and strength of glass fiber reinforced polymer (GFRP) under uniform compression and conducted experimental, numerical, and analytical studies, focusing on local buckling [2]. Based on the theory of small strain and medium rotations, Ascione et al. established the kinematics model of composite thin-wall beam. The effects of geometrical

imperfections and shear deformation on the members were fully considered, and the prebuckling behavior, the global buckling, and local buckling analyses were carried out [3–6].

In general, the deflection of box beam under self-weight and load may be quite different when the shear lag effect is ignored in the traditional beam theory. Reissner was the first one to use the variational energy method to analyze the biaxial symmetric rectangular box girder and the quadratic parabola as the displacement function of shear lag buckling [7]. Later, many scholars adopted quadratic, cubic, or even quadrilateral parabolas as the displacement function of shear lag on the basis of Reissner's theory [8–12]. Based on these methods, Luo and Wu selected three independent generalized displacements of top, bottom, and side cantilever plates, respectively, to represent different shear lag warping displacements between box girder plates with trapezoidal sections [8]. According to the energy method, Lin and Zhao described the nonuniform longitudinal displacement of flanges by using the higher-order polynomial of the infinite term and further quantified the shear lag effect of

thin-walled bending members such as T-beams and box beams [13]. In order to solve the problem of plastic deformation of steel box girder in the limit condition, Lin and Zhao proposed a method for modeling the behavior of inelastic shear lag [14]. Lertsima et al. used shell element modeling for the entire simply supported box girder and studied the stress concentration problem at the flange under the influence of shear lag by using the three-dimensional finite element method [15]. In the same way, Sa-nguanmanasak treated the influence of finite element mesh on shear lag in detail by using the method of multi-degree-of-freedom extrapolation and derived the empirical formula to calculate the stress concentration coefficient including the shear lag effect [16]. Then, Zhang put forward generalized formulas, which included the geometric properties of sections and generalized internal forces concerning shear lag moments and bending moments [17]. Chen et al. obtained a closed solution of shear lag based on the variational method of energy and carried out harmonic analysis on the closed beam [18]. In addition, Li improved the PM-DP analysis method, which can be used to correct the defects of shear lag warping deformation of the top cantilever flange [19].

The good structural performance of thin-walled box girder has always been the focus of research by many scholars. Considering the coupling of bending, torsion, shear deformation, and warping shear stress, Wang proposed a new type of nonshearing locking thin-wall section space beam element [20]. In view of distortion, a new one-dimensional theory was proposed by Law and Kim, in which the static and dynamic studies on thin-walled closed beams with general cross sections were carried out [21]. The research of Xu and Ye showed that the single box beam had no obvious pure distortion, and the influence of distortion shear stress on shear deformation was analyzed [22]. Razaqpur and Li developed a new finite element model of curved thin-walled multi-compartment box girder by combining the generalized Vlasov theory with the FEA technique, which could simulate the extension, bending, torsion, torsional warping, distortion, distortional warping, and shear lag effects of thin-walled box beam [23]. In order to calculate the section deformation, Choi and Jang adopted the beam frame method of two-dimensional Euler beam for simulation and developed the FEA model of quadrilateral thin-walled beam with variable section using the high-order beam theory [24]. It was approximated using the Ritz–Galerkin method's independent descriptions of stress and displacement fields. According to the previous work, Tan further analyzed nonlinear lateral buckling of thin-walled box beam's webs with the unequal thickness by considering the coupling of the bending, torsion, and distortion [25].

The deformation of box girder section under eccentric load and axial force is also studied. Genoese and Pietraszkiewicz established the linear kinematic equations for a thin-walled beam of heterogeneous isotropic material [26]. Opoka and Pietraszkiewicz showed the complete boundary value problem (BVP) for geometrical nonlinearity of thin shells, which was the stress action and bending of the reference surface of shells [27]. Coupling both material anisotropy and geometric nonlinearity, Vo presented a general geometric nonlinear model of arbitrary spatial composite thin-walled

beams under different loading [28]. Miranda and Gutiérrez developed a new generalized beam theory (GBT), which modified the kinematics formulations of shearing deformation GBT put forward by Silvestre and Camotim in the early period [29]. Considering the effects of axial warping effect and material anisotropy on the surface shear strain in the shell, Zhang proposed an improved structural mechanics model of composite beams with single closed thin-walled section [30]. Meanwhile, Sahraei extended the shear deformation theory and FEA method with computational efficiency to the calculation of the transverse torsional buckling capacity of single symmetric I-beams under general load [31]. Moreover, Cambronero-Barrientos identified a beam element with three nodes, each of which has five degrees of freedom, which exceeded six degrees of freedom of traditional three-dimensional beam [32]. And the influence of homogeneous and nonhomogeneous torsion, distortion, and shear lag had been considered in the normal stress distribution at the same time. Then, Meftah investigated the elastic lateral buckling distortional deformation of thin-walled box girder under the combined action of bending and axial loads [33].

In this paper, shear lag effect is considered as a separate mechanical property in the displacement deformation. So, the displacements of straight thin-walled box beams (STBBs) are divided into the sum of the partial displacements of six deformation fields. The main purpose of this work is to consider the influence of shear lag on nonlinear buckling performance of box girder under the combined action of an eccentric load and axial force.

2. Theoretical Model for Buckling Analysis

In this section, the main symbols and conventions of symbols are given, then the displacement field are described, and the kinematic equations are given on the basis of references [12, 21]. Therefore, the stiffness matrix is finally obtained.

It is assumed that the displacement field is the sum of the modes of each displacement deformation. Here, since the shear forces Q_z and Q_y are unknown, it is essential to modify it through the orthogonal warping shear function $\omega_{\zeta z}$ and $\omega_{\zeta y}$ when the shear lag effect is analyzed.

2.1. Basic Assumptions and Considerations. The present assumptions about the structural model are as follows:

- (1) The material of the model is linear elastic and homogeneous. The influence of the thickness of each wall on warping is ignored.
- (2) The strain on the contour of each plane is zero, that is, the plane of STBB cannot expand in the tangential direction.
- (3) The member buckles under the nonextensional mode, so the curvature and central longitudinal strain of the xz plane remain zero during the instability process. This means that members are supposed to buckle under combined axial and bending loads.

2.2. Deformation Modes. Figure 1 shows the global Cartesian coordinate system (x, y, z) , where the x -axis is the same as the longitudinal direction of STBB. (s, n, x) is the local coordinate system. For each wall, s passes through the whole cross section along the center line of each plate, and n is the coordinate of the local normal line.

The generalized displacement theory of STBB is based on the assumption of deformation modes (in Figures 2 and 3) corresponding to the extension, bending, torsion, shear lag, and distortion. These modes can be separated into a group of coupled axial and transverse displacement fields, each of which can be decomposed into the functions of the plane and axial coordinates. In the following, it makes sense to note that the displacement modes including global and local modes are described in detail.

Extensional mode, which only produces constant displacement u in the direction of length, is the simplest kind of displacement.

Two bending modes generate a displacement and a rotation, respectively. The rigid solid displacements (v and w) are in the plane of cross section and the rotations (θ_y and θ_z) are about the y - and z -axes.

Shear lag deflections considering shear forces, the generalized displacement of shear lag deformations (ζ_z and ζ_y), and shear lag warping displacement functions ($\omega_{\zeta z}$ and $\omega_{\zeta y}$) are introduced, which are proportional to shear forces Q_z and Q_y , respectively. Therefore, the additional displacements caused by the shear lag are $u_{\zeta z} = -\omega_{\zeta z}\zeta_z$ and $u_{\zeta y} = -\omega_{\zeta y}\zeta_y$. In the analysis of shear lag effect of thin-walled box beam, reference [12] interprets the quadratic parabola theory as a reasonable model of warping displacement function mode, the specific form of which will be detailed in the following part.

Torsion mode (no transverse deformation occurs in the cross section) generates homogeneous and nonhomogeneous torsional warping ω_a and ω_{at} , which are related to the normal stress $\sigma_x = E\omega_a d^2\theta_x/dx^2$ and the shear stress $\tau_{sx} = G\partial\omega_{at}/\partial s \cdot d^3\theta_x/dx^3$, respectively. Therefore, the displacement field contains a rigid solid rotation θ_x in regard to the shear center (x_0, y_0) of the cross section and two longitudinal displacements—the first- and third-order derivatives of torsion rotation are multiplied by the two warping functions, respectively. Nonhomogeneous torsional warping ω_{at} is caused by the shear stresses. For the calculation of longitudinal displacement, the influence of nonhomogeneous torsion on normal stresses can be ignored.

Distortion mode is the transverse deformation of the cross section. The difference between homogeneous and nonhomogeneous distortion is not discussed in traditional theories. For distortion mode, transverse displacements are $(\eta\chi, \psi_s\chi)$ in the plane of the cross section. And in terms of free warping or nonfree warping, homogeneous distortion and nonhomogeneous distortion can be obtained. Considering that the thickness of the beam girder compared with other dimensions is very small, the compression of the box wall along the direction of the wall thickness was neglected currently. Then, the longitudinal displacement at any point of the beam can be given by the homogeneous and nonhomogeneous distortion warping ω_d and ω_{dt} multiplied, respectively, by the first- and third-order derivatives of distortion rotation χ .

In the analysis of deformation modes, the transverse and longitudinal displacements are introduced as

$$v(y, z, x) = v_0 - (z - z_0)\theta_x(x) + \eta(s, n)\chi(x), \quad (1a)$$

$$w(y, z, x) = w_0 - (y - y_0)\theta_x(x) + \psi_s(s)\chi(x), \quad (1b)$$

$$\begin{aligned} u(y, z, x) = & u_0 - z\theta_y - y\theta_z - \omega_{\zeta z}\zeta_z(x) - \omega_{\zeta y}\zeta_y(x) \\ & + \omega_a \frac{d\theta_x(x)}{dx} + \omega_{at} \frac{d^3\theta_x(x)}{dx^3} + \omega_d \frac{d\chi(x)}{dx} \\ & + \omega_{dt} \frac{d^3\chi(x)}{dx^3}. \end{aligned} \quad (1c)$$

2.3. Structural Mechanisms. In order to illustrate the general method of obtaining deformation modes of STBB under combined loads, the geometric dimension of STBB is defined as follows: b is the width; h is the height; L denotes the entire beam length; and t_i denotes the thickness of each wall which is much smaller than other dimensions.

It is well known that the shear lag effect inevitably increases the deflection of the box girder. The shear lag deformation state is described by generalized displacement, that is, additional deflection caused by shear lag effect.

The shear lag warping displacement function is not a continuous function throughout the beam that directly affects the size and distribution of cross section stress. Therefore, many authors studied and proved these interpolation of functions [8–12]. The shear lag warping stress $\sigma_{\omega_{\zeta z}} = -E\omega_{\zeta z}\omega'_{\zeta z}(x)$ (without the influence of axial force) of the cross section of the thin-walled structure should meet the self-balancing condition, that is,

$$\int_A \sigma_{\omega_{\zeta z}} dA = 0, \quad (2)$$

$$\int_A \sigma_{\omega_{\zeta z}} z dA = 0. \quad (3)$$

When the thin-walled box beams are bent, the normal stress of the cross section must satisfy the axial equilibrium condition, which means $\int_A \omega_{\zeta z} dA = 0$. After many comparisons, it is reasonable to select the quadratic parabola as the shear lag warping displacement function. In this paper, the quadratic parabola expression of the selected shear lag warping displacement function is

$$\omega_{\zeta z}^i(s) = \begin{cases} -\left(\frac{h}{2} - z_0\right)\left(1 - \frac{4s^2}{b^2}\right) + d, & i = 1, \\ d, & i = 2, 4, \\ \left(\frac{h}{2} + z_0\right)\left(1 - \frac{4s^2}{b^2}\right)\left(\frac{h + 2z_0}{h - 2z_0}\right) + d, & i = 3, \end{cases} \quad (4)$$

$$d = \frac{2(h/2 - z_0)}{3A} \left[bt_1 - bt_3 \left(\frac{h + 2z_0}{h - 2z_0} \right)^2 \right], \quad (5)$$

FIGURE 1: Cartesian coordinate system and the local coordinate of each plane.

FIGURE 2: The transverse displacements.

where d is the additional uniform displacement function modified by the shear lag warping function and A is the cross section area; this method is valid and correct for this paper.

The microelement of generalized distortion displacement in STBB wall is shown in Figure 4. In accordance with

any point N_0 in the middle of plane, the longitudinal, normal, and circumferential displacements of the plane at any point N_1 can be obtained by ignoring the compression of the wall in the direction of the plane thickness. In the light of middle shear strain $\gamma_{xs} = (\partial\psi_s/\partial\chi)/\partial x + (-\partial\omega d\chi'/\partial s) = 0$, the

FIGURE 3: The longitudinal displacements $u(y, z, x)$, $v(y, z, x)$, and $w(y, z, x)$.

distortion warping can be expressed as $\omega_d(s) = \int_s^0 \psi_s(s) ds$. From the self-equilibrium condition of the warping stress, $\int_A \omega_d dA = 0$.

In the case that the box girder has no torsion but only distortion deformation, the interposition pure torsion force system $1/\Omega$ is added at both ends of the microsection dx of STBB and the circumferential displacement $\psi_s \delta\chi$ of the distorted microsection is taken as the virtual displacement. Then, the virtual work done by the unit pure torque force on the virtual displacement is zero: $\int [(1/\Omega)\psi_s(s)\delta\chi] ds = 0$, that is, $\int \psi_s(s) ds = 0$. Through the frame angular point displacement and the continuity of bending moment, the coefficients can be obtained in reference [21]:

$$\psi_s^i(s) = \xi_1^2 \left(\frac{h}{2} + \xi_1 z_0 \right) + \xi_2^2 \left(\frac{b}{2} - \xi_2 y_0 \right), \quad (6a)$$

$$\psi_n^i(s) = \left(-\frac{L_{\text{side}}}{2} + \frac{3}{L_{\text{side}}} s^2 - \frac{2}{L_{\text{side}}^2} s^3 \right) (-1)^{i+1} \alpha, \quad (6b)$$

$$\alpha = \begin{cases} \frac{2b}{b+h}, & i = 1, 3, \\ \frac{2h}{b+h}, & i = 2, 4, \end{cases}$$

where L_{side} is the length of each wall of the cross section and α is a coefficient to satisfy the static force equilibrium. The displacement of STBB is given by

FIGURE 4: The distortion displacement of microelement.

$$u^i(s, x, n) = u_0(x) + Z^i(s)w_0'(x) + Y^i(s)v_0'(x) - \omega_{\zeta_z}^i(s)\zeta_z'(x) - \omega_a^i(s)\theta'(x) - \omega_d^i(s)\chi'(x), \quad (7a)$$

$$v^i(s, x, n) = \xi_1^i v_0(x) - \xi_1^i w_0(x) - h^i(s)\theta(x) + \eta^i(s, n)\chi(x), \quad (7b)$$

$$w^i(s, x, n) = -\xi_1^i w_0(x) - \xi_2^i v_0(x) - s\theta(x) + \psi_n^i(s)\chi(x), \quad (7c)$$

in which $i = 1 \sim 4$ are the wall numbers. The functions $Z^i(s)$, $Y^i(s)$, $h^i(s)$, $\omega_a^i(s)$, and $\eta(s, n)$ in above equations are expressed as follows:

$$Z^i(s) = \xi_1 \left(\frac{h}{2} + \xi_1 z_0 \right) + \xi_2 s, \quad (8a)$$

$$Y^i(s) = \xi_2 \left(\frac{b}{2} - \xi_2 y_0 \right) + \xi_1 s, \quad (8b)$$

$$h^i(s) = \xi_1^2 \left(\frac{h}{2} + \xi_1 z_0 \right) + \xi_2^2 \left(\frac{b}{2} - \xi_2 y_0 \right), \quad (8c)$$

$$\omega_a^i(s) = \left[\xi_2^2 \left(\frac{b}{2} - \xi_2 y_0 \right) - \xi_1^2 \left(\frac{h}{2} + \xi_1 z_0 \right) \right] s, \quad (8d)$$

$$\eta^i(s, n) = \psi_s^i(s) - n \frac{d\psi_n^i(s)}{ds}, \quad (8e)$$

$$\xi_1 = \frac{1 + (-1)^{i+1}}{2} (i - 2), \quad (8f)$$

$$\xi_2 = \frac{1 + (-1)^i}{2} (3 - i). \quad (8g)$$

2.4. Expressions of Strains and Stresses. In order to explain the strain tensor of STBB, the normal and shear strains containing the linear and nonlinear strains are expressed as follows:

$$\epsilon_{xx}^i = \frac{\partial u^i(s, x, n)}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u^i(s, x, n)}{\partial x} \right)^2 + \left(\frac{\partial v^i(s, x, n)}{\partial x} \right)^2 + \left(\frac{\partial w^i(s, x, n)}{\partial x} \right)^2 \right], \quad (9a)$$

$$\gamma_{xs}^i = \frac{\partial u^i(s, x, n)}{\partial s} + \frac{\partial v^i(s, x, n)}{\partial x} + \frac{\partial u^i(s, x, n)}{\partial x} \frac{\partial u^i(s, x, n)}{\partial s} + \frac{\partial v^i(s, x, n)}{\partial x} \frac{\partial v^i(s, x, n)}{\partial s} + \frac{\partial w^i(s, x, n)}{\partial x} \frac{\partial w^i(s, x, n)}{\partial s}. \quad (9b)$$

In the case of normal strain, the expression according to the linear and nonlinear conditions can be denoted by $\epsilon i \cdot l$ and $\epsilon i \cdot nl$:

$$\epsilon_{xx}^i = \epsilon_l^i + \epsilon_{nl}^i, \quad (10a)$$

where

$$\epsilon_l^i = u_0' + Z^i(s)w_0''(x) + Y^i(s)v_0''(x) - \omega_a^i(s)\theta''(x) - \omega_d^i(s)\chi''(x) - \omega_{\zeta_z}^i(s)\zeta_z'(x), \quad (10b)$$

$$\begin{aligned} \epsilon_{nl}^i = & \frac{1}{2} \left[u_0'(x)^2 + v_0'(x)^2 + w_0'(x)^2 \right. \\ & + \omega_{\zeta_z}^i(s)^2 \zeta_z'(x)^2 + \left(\eta^i(s, n)^2 + \psi_n^i(s)^2 \right) \chi'(x)^2 \\ & + \left(h^i(s)^2 + s^2 + s^2 \right) \theta'(x)^2 + Y^i(s)^2 v_0''(x)^2 \\ & + Z^i(s)^2 w_0''(x)^2 + \omega_a^i(s)^2 \theta''(x)^2 + \omega_d^i(s)^2 \chi''(x)^2 \left. \right] \\ & - \left(\xi_1 h^i(s) - \xi_2 s \right) v_0'(x) \theta'(x) + \left(\xi_2 h^i(s) - \xi_1 s \right) w_0'(x) \theta'(x) \\ & + \left(\xi_2 \eta^i(s, n) + \xi_2 \psi_n^i(s) \right) v_0'(x) \chi'(x) - \omega_{\zeta_z}^i(s) u_0'(x) \zeta_z'(x) \\ & - \left(\xi_2 \eta^i(s, n) + \xi_1 \psi_n^i(s) \right) w_0'(x) \chi'(x) \\ & - \left(h^i(s) \eta^i(s, n) + s \psi_n^i(s) \right) \theta'(x) \chi'(x) \\ & + Y^i(s) u_0'(x) v_0''(x) - Y^i(s) \omega_{\zeta_z}^i(s) v_0''(x) \zeta_z'(x) \\ & - \omega_a^i(s) u_0'(x) \theta''(x) + \omega_a^i(s) \omega_{\zeta_z}^i(s) \theta''(x) \zeta_z'(x) \\ & + Z^i(s) u_0'(x) w_0''(x) - Z^i(s) \omega_{\zeta_z}^i(s) w_0''(x) \zeta_z'(x) \\ & - \omega_d^i(s) u_0'(x) \chi''(x) + \omega_d^i(s) \omega_{\zeta_z}^i(s) \chi''(x) \zeta_z'(x) \\ & - Y^i(s) \omega_a^i(s) v_0''(x) \theta''(x) + Z^i(s) Y^i(s) w_0''(x) v_0''(x) \\ & - Y^i(s) \omega_d^i(s) v_0''(x) \chi''(x) - Z^i(s) \omega_a^i(s) w_0''(x) \theta''(x) \\ & - Z^i(s) \omega_d^i(s) w_0''(x) \chi_0''(x) + \omega_d^i(s) \omega_d^i(s) \theta''(x) \chi''(x). \end{aligned} \quad (10c)$$

The shear strain γ_{xs}^i can be represented as

$$\begin{aligned}
\gamma_{xs}^i = & -\omega_{\zeta_z}'(s)\zeta_z'(x) - (\omega_a'(s) - h(s))\theta'(x) - (\omega_d'(s) - \eta(s, n))\chi'(x) - \psi_n(s)\theta(x)\chi'(x) \\
& - (h(s)\eta'(s, n) + s\psi_n'(s))\theta'(x)\chi(x) + s\theta(x)\theta'(x) + (\eta(s, n)\eta'(s, n) + \psi_n(s)\psi_n'(s))\chi(x)\chi'(x) \\
& + \omega_a(s)\omega_{\zeta_z}'(s)\theta''(x)\zeta_z'(x) + \omega_d(s)\omega_{\zeta_z}'(s)\chi''(x)\zeta_z'(x) + \omega_a(s)\omega_d'(s)\theta''(x)\chi'(x) \\
& + \omega_d(s)\omega_a'(s)\theta'(x)\chi'(x) + \omega_a(s)\omega_a'(s)\theta'(x)\theta''(x) + \omega_d(s)\omega_d'(s)\chi'(x)\chi''(x).
\end{aligned} \quad (10d)$$

The stresses are carried out by multiplying the normal and shear strain by E (Young's modulus constant) and G (shear modulus constant), respectively.

$$\begin{aligned}
\sigma_{xx}^i &= E\varepsilon_{xx}^i; \\
\tau_{xs}^i &= G\gamma_{xs}^i.
\end{aligned} \quad (11)$$

2.5. Variational Formulation. According to the energy variational principle, the total system potential energy is composed of external potential and internal strain energy

(W and U). Therefore, the first-order expression of the total potential energy of the system is

$$\delta(W - U) = 0, \quad (12)$$

$$\delta U = \sum_{i=1}^4 \int_L \int_{F_i} (\sigma_{xx}^i \delta \varepsilon_{xx}^i + \tau_{xs}^i \delta \gamma_{xs}^i) dF_i dx. \quad (13)$$

Substitute longitudinal and shear strain equations (10b)–(10d) into the elastic energy equation (13) to get the internal strain energy expression:

$$\begin{aligned}
\delta U = & \int_L N (\delta u_0' + u_0' \delta u_0' + v_0' \delta v_0' + w_0' \delta w_0') dx + \int_L M_z (\delta v'' + w_0' \delta \theta' + \theta' \delta w_0' + u_0' \delta v_0' + v_0' \delta u_0') dx \\
& + \int_L M_y (\delta w'' + v_0' \delta \theta' + \theta' \delta v_0' + u_0' \delta w_0' + w_0' \delta u_0') dx - \int_L B_{\omega_a} (\delta \theta_0' + u_0' \delta \theta'' + \theta'' \delta u_0') dx + \int_L M_R \theta' \delta \theta' dx \\
& - \int_L B_{\omega_d} (\delta \chi'' + u_0' \delta \chi'' + \chi'' \delta u_0') dx + \int_L M_{\psi} \chi' \delta \chi' dx - \int_L M_{\psi R} (\theta' \delta \chi' + \chi' \delta \theta') dx + \int_L M_{\psi y} (v_0' \delta \chi' + \chi' \delta v_0') dx \\
& - \int_L M_{\psi z} (w_0' \delta \chi' + \chi' \delta w_0') dx - \int_L \int_L M_{z\omega_{\zeta_z}} (\zeta_z' \delta v_0'' + v_0'' \delta \zeta_z') dx + \int_L M_{\omega_a \omega_{\zeta_z}} (\zeta_z' \delta \theta'' + \theta'' \delta \zeta_z') dx + \int_L M_{\omega_{\zeta_z}} \delta \zeta_z' dx \\
& + \int_L M_{\omega_d \omega_{\zeta_z}} (\zeta_z' \delta \chi'' + \chi'' \delta \zeta_z') dx + \int_L M_{Y^2} w_0'' \delta w_0'' dx + \int_L M_{Z^2} v_0'' \delta v_0'' dx + \int_L M_{\omega_a^2} \theta'' \delta \theta'' dx + \int_L M_{\omega_d^2} \chi'' \delta \chi'' dx \\
& + \int_L M_{ZY} (w_0'' \delta v_0'' + v_0'' \delta w_0'') dx - \int_L M_{y\omega_a} (w_0'' \delta \theta'' + \theta'' \delta w_0'') dx - \int_L M_{y\omega_d} (v_0'' \delta \theta'' + \theta'' \delta v_0'') dx \\
& - \int_L M_{y\omega_d} (w_0' \delta \chi'' + \chi'' \delta w_0'') dx - \int_L M_{z\omega_d} (v_0'' \delta \chi'' + \chi'' \delta v_0'') dx + \int_L M_{\omega_a \omega_d} (\chi'' \delta \theta'' + \theta'' \delta \chi'') dx \\
& - \int_L M_{\omega_{\zeta_z}} \delta \zeta_z dx - \int_L M_{sv} \delta \theta' dx + \int_L M_{sv\psi} \delta \chi' dx + \int_L M_1 (\theta \delta \theta' + \theta' \delta \theta) dx - \int_L M_2 (\theta \delta \chi' + \chi' \delta \theta) dx \\
& + \int_L M_3 (\chi \delta \chi' + \chi' \delta \chi) dx + \int_L M_4 (\theta' \delta \chi + \chi \delta \theta') dx + \int_L M_5 (\theta' \delta \theta'' + \theta'' \delta \theta') dx + \int_L M_6 (\chi' \delta \chi'' + \chi'' \delta \chi') dx \\
& + \int_L M_7 (\theta'' \delta \chi' + \chi' \delta \theta'') dx + \int_L M_8 (\theta' \delta \chi'' + \chi'' \delta \theta') dx + \int_L M_{\omega_a \omega_{\zeta_z}'} (\theta'' \delta \zeta_z + \zeta_z \delta \theta'') dx + \int_L M_{\omega_a \omega_{\zeta_z}'} (\zeta_z \delta \chi'' + \chi'' \delta \zeta_z) dx,
\end{aligned} \quad (14)$$

where N represents axial force, B_{ω} represents the bimoment, M_z and M_y represent the bending moments of the z and y directions, respectively, M_{sv} represents St. Venant torsion moment, M_{ψ} represents distortion moment, and B_{ω_a} and B_{ω_d} represent the bimoment and distortion bimoment; similar to M_z and M_y , $M_{\psi y}$ and $M_{\psi z}$ are distortion warping moments on behalf of the z and y directions; $M_{sv\psi}$ is in reference to St. Venant principle of distortion moment; M_R and $M_{\psi R}$ represent high-order stress synthetic; $M_{\omega_{\zeta_z}^2}$ is shear lag moment; $M_{\omega_a^2}$ is high-order bimoment; $M_{\omega_d^2}$ is the

high-order distortion bimoment; $M_{z\omega_{\zeta_z}}$ is shear lag bending moment; $M_{\omega_a \omega_{\zeta_z}}$ and $M_{\omega_d \omega_{\zeta_z}}$ are shear lag coupled with torsion and distortion moments, respectively; $M_{z\omega_a}$ and $M_{y\omega_a}$ are tangential and transverse bimoments; $M_{z\omega_d}$ and $M_{y\omega_d}$ are tangential and transverse distortion bimoments; $M_{\omega_a \omega_d}$ is the high-order coupling bimoment; $Q_{\omega_{\zeta_z}'}$ is the generalized shear force; M_1 to M_8 are high-order torques caused by torsion and distortion; $M_{\omega_a \omega_{\zeta_z}'}$ is the shear lag coupling torque; and $M_{\omega_d \omega_{\zeta_z}'}$ is the shear lag distortion coupling torque.

$$\begin{aligned}
N &= E \int_A \varepsilon_{xx} dA, \\
M_y &= E \int_A z \cdot \varepsilon_{xx} dA, \\
M_z &= E \int_A y \cdot \varepsilon_{xx} dA, \\
M_{\Psi_z} &= E \int_A (\xi_1 \Psi_n + \xi_2 \eta) \cdot \varepsilon_{xx} dA, \\
M_{\Psi_y} &= E \int_A (\xi_2 \Psi_n + \xi_1 \eta) \cdot \varepsilon_{xx} dA, \\
M_R &= E \int_A (h^2 + s^2) \cdot \varepsilon_{xx} dA, \\
M_\psi &= E \int_A (\psi_n^2 + \eta^2) \cdot \varepsilon_{xx} dA, \\
M_{\psi R} &= E \int_A (s\psi_n + h\eta) \cdot \varepsilon_{xx} dA, \\
B_{\omega_a} &= E \int_A \omega_a \cdot \varepsilon_{xx} dA, \\
B_{\omega_d} &= E \int_A \omega_d \cdot \varepsilon_{xx} dA, \\
M_{\omega_{\zeta_z}} &= E \int_A \omega_{\zeta_z} \cdot \varepsilon_{xx} dA, \\
M_{z\omega_{\zeta_z}} &= E \int_A z\omega_{\zeta_z} \cdot \varepsilon_{xx} dA, \\
M_{\omega_a\omega_{\zeta_z}} &= E \int_A \omega_a\omega_{\zeta_z} \cdot \varepsilon_{xx} dA, \\
M_{\omega_d\omega_{\zeta_z}} &= E \int_A \omega_d\omega_{\zeta_z} \cdot \varepsilon_{xx} dA, \\
M_{z^2} &= E \int_A z^2 \cdot \varepsilon_{xx} dA, \\
M_{y^2} &= E \int_A y^2 \cdot \varepsilon_{xx} dA, \\
M_{\omega_a^2} &= E \int_A \omega_a^2 \cdot \varepsilon_{xx} dA, \\
M_{\omega_d^2} &= E \int_A \omega_d^2 \cdot \varepsilon_{xx} dA, \\
M_{zy} &= E \int_A zy \cdot \varepsilon_{xx} dA, \\
M_{z\omega_a} &= E \int_A z\omega_a \cdot \varepsilon_{xx} dA, \\
M_{y\omega_a} &= E \int_A y\omega_a \cdot \varepsilon_{xx} dA, \\
M_{z\omega_d} &= E \int_A z\omega_d \cdot \varepsilon_{xx} dA, \\
M_{y\omega_d} &= E \int_A y\omega_d \cdot \varepsilon_{xx} dA, \\
M_{\omega_a\omega_d} &= E \int_A \omega_a\omega_d \cdot \varepsilon_{xx} dA, \\
Q_{\omega_{\zeta_z}'} &= G \int_A \omega_{\zeta_z}' \cdot \gamma_{xs} dA, \\
M_{sv} &= G \int_A (\omega_a' - h) \cdot \gamma_{xs} dA, \\
M_{sv\psi} &= G \int_A (\eta - \omega_d') \cdot \gamma_{xs} dA, \\
M_1 &= G \int_A s \cdot \gamma_{xs} dA, \\
M_2 &= G \int_A \psi_n \gamma_{xs} dA, \\
M_3 &= G \int_A (\eta\eta' + \psi_n\psi') \cdot \gamma_{xs} dA, \\
M_4 &= G \int_A (h\eta' + s\psi') \cdot \gamma_{xs} dA, \\
M_5 &= G \int_A \omega_a\omega_a' \cdot \gamma_{xs} dA, \\
M_6 &= G \int_A \omega_d\omega_d' \cdot \gamma_{xs} dA, \\
M_7 &= G \int_A \omega_a\omega_d' \cdot \gamma_{xs} dA, \\
M_8 &= G \int_A \omega_a'\omega_d \cdot \gamma_{xs} dA, \\
M_{\omega_a\omega_{\zeta_z}'} &= G \int_A \omega_a\omega_{\zeta_z}' \cdot \gamma_{xs} dA, \\
M_{\omega_d\omega_{\zeta_z}'} &= G \int_A \omega_d\omega_{\zeta_z}' \cdot \gamma_{xs} dA,
\end{aligned} \tag{15}$$

and detailed expressions of these forces and torques can be found in Appendix A.

The general formulas about of STBB nonlinear and the buckling analysis are obtained when it is subjected to external forces of combined axial load F_x and eccentric load F_z . In the initial transverse buckling, the external energy is given:

$$\delta W = \int_L (F_x \delta u_0(x) + F_z \delta w_r(s, x)) dx, \tag{16}$$

in which $W_r(s, r)$ stands for the vertical displacement. In consideration of the classical quadratic kinematics hypothesis, the change of external energy can be expressed as

$$\begin{aligned}
\delta W &= \int_L F_x \delta u_0(x) dx + \int_L F_z \delta (w_0(x) + \zeta_z(x)) dx \\
&\quad - (e_z + e_y) \int_L F_z (\theta(x) + \chi(x)) \delta (\theta(x) + \chi(x)) dx,
\end{aligned} \tag{17}$$

where e_y and e_z are eccentricities of the concentrated load F_z relative to the shear center.

According to equations (14) and (17), the virtual displacements and their derivatives can be used to express the change of the total potential energy. Due to the generalized variational principle, balance equations with reference to a first-order variation $\partial u(x)$, $\partial v(x)$, $\partial w(x)$, $\partial \theta(x)$, $\partial \chi(x)$, and $\partial \zeta(x)$ can be written as

$$N' + (Nu')' + M_y w'' + M_z v'' - B_{\omega_a} \theta'' - B_{\omega_d} \chi'' = F_x, \quad (18a)$$

$$Nv' + M_z' + M_y \theta' + M_{\psi_y} \chi' + (M_z u')' - (M_{z\omega_{\zeta_z}} \zeta_z')' + (M_{z^2} v'')' + (M_{ZY} w'')' - (M_{z\omega_a} \theta'')' - (M_{z\omega_d} \chi'')' = 0, \quad (18b)$$

$$(Nw')' + (M_z \theta')' + M_y'' - (M_{\psi_z} \chi')' + (M_y u'')' + (M_{Y^2} w'') + (M_{ZY} v'')'' - (M_{y\omega_a} \theta'')'' - (M_{y\omega_d} \chi'')'' = F_z, \quad (18c)$$

$$\begin{aligned} & -B_{\omega_a}'' + (M_y v')' + (M_R \theta') - (M_{\psi_R} \chi')' - (B_{\omega_a} u')'' + (M_{\omega_a \omega_{\zeta_z}} \zeta_z')'' + (M_{\omega_a^2} \theta'')'' - (M_{y\omega_a} w'')'' - (M_{z\omega_a} v'')'' \\ & + (M_{\omega_a \omega_d} \chi'')'' - M_{sv}' + (M_s \theta')' + M_1 \theta' - M_2 \chi' - (M_4 \chi')' + (M_5 \theta'')'' + (M_5 \theta'')' + (M_7 \chi'')'' + (M_8 \chi'')' \\ & + (M_{\omega_a \omega_{\zeta_z}}'')'' = -F_z(e_z + e_y)(\theta + \chi), \end{aligned} \quad (18d)$$

$$\begin{aligned} & -B_{\omega_d}'' + (M_{\psi} \chi'')'' + (M_{\psi_y} v') - (M_{\psi_z} w')' - (M_{\psi_R} \theta')' - (B_{\omega_d} u')'' + (M_{\omega_d \omega_{\zeta_z}} \zeta_z')'' + (M_{\omega_d^2} \chi'')'' - (M_{y\omega_d} w'')'' \\ & - (M_{z\omega_d} v'')'' + (M_{\omega_a \omega_d} \theta'')'' + M_{sv\psi}' + (M_3 \chi')' + M_3 \chi' - (M_2 \theta')' - M_4 \theta' + (M_6 \chi'')'' + (M_6 \chi'')' + (M_7 \theta'')' \\ & + (M_8 \theta'')'' + (M_{\omega_d \omega_{\zeta_z}}'')'' = -F_z(e_z + e_y)(\theta + \chi), \end{aligned} \quad (18e)$$

$$M_{\omega_{\zeta_z}}' + (M_{\omega_{\zeta_z}} u')' + (M_{y\omega_{\zeta_z}} w'')' + (M_{z\omega_{\zeta_z}} v'')' - (M_{\omega_d \omega_{\zeta_z}} \chi'')' + Q_{\omega_{\zeta_z}} - M_{\omega_d \omega_{\zeta_z}} \theta'' - M_{\omega_d \omega_{\zeta_z}} \chi'' = F_z. \quad (18f)$$

2.6. The Discrete Equilibrium Problem. For nonlinear analysis, the governing equations are simplified to generalized coordinates by the Ritz variational method. Based on the Newton–Raphson method, an incremental-iterative method is proposed to solve the nonlinear equilibrium equations. And the buckling load is first determined according to the singularity condition of the tangential stiffness matrix determinant. The discrete kinematic equation can be used to analyze the performance of simply supported structures. The displacement modes adopt the approximate functions that are consistent with the boundary conditions of STBB:

$$\frac{u(x)}{u_0} = \cos \frac{\pi x}{L}, \quad (19a)$$

$$\frac{v(x)}{v_0} = \frac{w(x)}{w_0} = \frac{\theta(x)}{\theta_0} = \frac{\chi(x)}{\chi_0} = \frac{\zeta_z(x)}{\zeta_{z0}} = \sin \frac{\pi x}{L}, \quad (19b)$$

where u_0 , v_0 , w_0 , θ_0 , χ_0 , and ζ_{z0} are related to displacement amplitudes. In addition, these functions can also represent exact solutions of nonlinear equilibrium equations. Through Galerkin's approximation method and integration, the algebraic coupling equilibrium equations of displacement modulus can be obtained:

$$P_1 u_0 + P_2 w_0 \zeta_{z0} + P_3 u_0 w_0^2 = \left(\frac{4L^2}{\pi^2} \right) F_x, \quad (20a)$$

$$P_4 v_0 + P_5 \chi_0 + P_6 w_0 \theta_0 + P_7 w_0 \chi_0 + P_8 v_0 w_0^2 = 0, \quad (20b)$$

$$P_9 w_0 + P_{10} \chi_0 + P_{11} w_0 \chi_0 + P_{12} w_0^3 - \left(\frac{16L}{\pi^3} \right) M_0 = 0, \quad (20c)$$

$$P_{13} \theta_0 + P_{14} \chi_0 + P_{15} v_0 w_0 + P_{16} w_0 \chi_0 + P_{17} w_0^2 + P_{18} w_0^2 \theta_0 + P_{19} w_0^2 \chi_0 + \frac{4(e_y + e_z)L}{\pi^2} M_0 = 0 \quad (20d)$$

$$P_{20} w + P_{21} v + P_{22} \theta_0 + P_{23} \chi_0 + P_{24} v_0 w_0 + P_{25} w_0 \theta_0 + P_{26} w_0 \chi_0 + P_{27} w_0^2 + P_{28} w_0^2 \theta_0 + P_{29} w_0^2 \chi_0 + \frac{4(e_y + e_z)L}{\pi^2} M_0 = 0, \quad (20e)$$

$$P_{30} \zeta_{z0} - \left(\frac{16L}{\pi^3} \right) M_0 = 0, \quad (20f)$$

in which $M_0 = F_z L/4$ represents the maximal bending moment when the eccentric load is applied at the middle length of the beam. The detailed expressions of P_i algebraic system is shown in Appendix B.

2.7. Tangential Stiffness Matrix for Lateral Buckling. The beam deflects under the action of axial and bending loads. If the beam is loaded to a certain extent, its cross section rotates around the axis of the beam, at which time lateral buckling may occur. This is transverse buckling, and the buckling load

is the critical load. Corresponding to the prebuckling state, the deflection is called the “fundamental state.”

On account of the Jacobian Matlab function, the tangent stiffness matrix $[K_t]$ is automatically defined by the equilibrium equations (20a)–(20f). In the case of nonlinear high-coupling high-order model and considering the prebuckling state, it can be reasonably assumed that each displacement $(u_0, v_0, w_0, \theta_0, \chi_0, \zeta_{z0})$ is approximately equal to $(0, 0, w_0, 0, 0, 0)$ by analyzing the geometric matrix. Here, the tangent stiffness matrix of the orthogonal displacement modes is

$$[K_t] = \begin{bmatrix} P_1 + P_3 w^2 & 0 & 0 & 0 & 0 & P_2 w \\ 0 & P_4 + P_8 w^2 & 0 & P_6 w & P_5 + P_7 w & 0 \\ 0 & 0 & P_9 + P_{12} w^2 & 0 & P_{10} + P_{11} w & 0 \\ 0 & P_{15} w & P_{17} w & \begin{bmatrix} P_{13} + P_{18} w^2 \\ +\frac{4L}{\pi^2}(e_z + e_y)M_0 \end{bmatrix} & \begin{bmatrix} P_{14} + P_{16} w + P_{19} w^2 \\ +\frac{4L}{\pi^2}(e_z + e_y)M_0 \end{bmatrix} & 0 \\ 0 & P_{21} + P_{24} w & P_{20} + P_{27} w & \begin{bmatrix} P_{22} + P_{25} w + P_{28} w^2 \\ +\frac{4L}{\pi^2}(e_z + e_y)M_0 \end{bmatrix} & \begin{bmatrix} P_{23} + P_{26} w + P_{29} w^2 \\ +\frac{4L}{\pi^2}(e_z + e_y)M_0 \end{bmatrix} & 0 \\ 0 & 0 & 0 & 0 & 0 & P_{30} \end{bmatrix}. \quad (21)$$

Finally, after eliminating the determinant of the tangent stiffness matrix, we get polynomial equations containing F_x and F_z . In this paper, the roots of the equation yield to transverse buckling loads in different buckling modes (extension, bending, torsion, distortion, and shear lag modes).

3. Application and Numerical Examples

The goal of the section is to verify the current theoretical model of STBB buckling behavior so as to demonstrate the accuracy and practical application of the formulas. At the same time, the results are compared with those calculated by FEA (Ansys software), Meftah, and Tan.

The analysis procedure for the nonlinear buckling analysis of STBB is the same as the general nonlinear analysis process. In the Ansys software, the STBB is idealized by the 4-node 181 isoparametric quadrangle shell elements (Figure 5). First, a “perfect” model is established, and the sparse matrix solver is used to conduct eigenvalue buckling analysis, which can predict the critical buckling force and provide a basis for the nonlinear buckling analysis. The grid can generate 3888 nodes and 3840 elements. Meanwhile, the grid is verified to discretize these patterns accurately. In the

following analysis, perturbation is introduced in the geometry by adding geometric imperfections using the UPGEOM command. The geometric nonlinear analysis of structures with geometric perturbation is conducted using the arclength method, where the postbuckling behavior can be obtained. The size of the perturbations is usually a small percentage of a relative structure, and the relevant scaling factor used in this case is 0.001. Finally, the bifurcation point of the buckling load displacement curve is found out, which is the critical buckling load.

In Ref. [33], Meftah studied the lateral buckling of thin-walled box girder under the combined action of bending and axial loads. And in the stress calculation, the nonlinear higher-order terms were omitted. Similarly, in Ref. [25], Tan further analyzed nonlinear lateral buckling of STBB's webs with the unequal thickness, but the main difference was that the geometrically nonlinear high-order coupled terms ($w\theta^2, w\chi^2, w^2\theta, \dots$) had been taken into account. Both of them did not consider the shear deformation and the shear lag effect.

3.1. Example 1: Variation of Length and Cross Section. Figure 6 shows a simply supported beam under combined eccentric and axial load. The critical load F_z imposed in the

FIGURE 5: Grid division of quadrilateral shell of STBB with unequal thickness webs.

FIGURE 6: The geometry of STBB under combined axial and bending loads.

TABLE 1: Cross section dimension of the models.

Category	I	II	III	IV	V
b (mm)	200	200	150	150	100
h (mm)	100	150	150	200	200

middle of beam length takes action on the intersection of the upper flange and the center line of the main web, and the axial load F_x is applied on one end of the upper flange. The geometrical and material properties are $F_x = 2.5$ kN, $E = 210$ GPa, $t_1 = t_3 = 4$ mm, $t_2 = 8$ mm, and $t_4 = 5$ mm. Poisson's ratio is 0.3. The length ranges from 1.0 to 8.0 m. In addition, in order to illustrate the influence of the length-width ratio of cross section on nonlinear lateral buckling of beams, five different groups of cross sections are selected in this paper. Table 1 shows the detailed cross section dimension of the models.

Figures 7–11 illustrate the comparison between the present and other three theories under different lengths and cross section ratios. The parameters of the beam can indicate the sufficient effectiveness and accuracy of the method in this paper. Tables 2–4 report percentage differences of the displacements (Δu_i , Δw_i , Δv_i) obtained by theoretical methods and those obtained by FEA. Where, $i = 1 \sim 3$ represents the present, Meftah, and Tan's methods, respectively.

As can be seen from Figure 7, the numerical values of the longitudinal displacement are almost between the values of Meftah and FEA. It is easy to see from the picture that Meftah's axial displacements are independent of the change of cross section. At the same length, the longitudinal displacements of the other three methods increase gradually with the decrease of cross section ratio (b/h). It should be noted that the relative errors of present method are the smallest while those of Meftah's method are the largest in Tables 2–4. It is shown that ignoring the shear lag effect will lead to an overestimation of percentage of the longitudinal

displacements at any length. With reference to Tan's beams, the values Δu_1 and Δu_3 are the closest, when the aspect ratio of cross section is 1. This is due to the geometrical nonlinear effect caused by bending, torsion, distortion and shear lag coupling, which leads to a decrease in the axial stiffness of STBB. It is confirmed that the axial displacements of the present theory are sufficiently close to those predicted by FEA. Especially in IV and V, the values of present method are closest to those of FEA when the length is under 7 and 8 m (the smallest error $\Delta u_1 = 6.6\%$). This shows that adding additional shear lag displacement is effective for the prediction of longitudinal displacement.

It can be seen from Figure 8 that the vertical displacement deformation trend is completely corresponding to the rigid body performance, which is true for all models. The results of the present theory are confirmed to be closer to those of FEA, which supports the validity of this theory. Comparing the present theory with the other two, the results are satisfactory and can be considered as a supplement to the shear lag. In the same case, the values of FEA are always greater than those of other methods, with the lowest values of Meftah. Because the three theoretical approaches do not take the vertical compressibility of webs into account under vertical load and more nonlinear factors are considered, the vertical displacement decreases. According to the relative errors of vertical displacements from Table 3, when the beam length is less than 2 m, the shear lag effect has little influence on the vertical displacement of STBB. When the beam tends to be slender, the percentage of vertical displacements is smaller (Δw_1 are almost all less than 10%). The present method can well capture the effect of shear deformation on the beam, so it has high prediction accuracy.

In Figure 9, we note that the transversal displacement distribution curve obtained by the present theory is in good agreement with the results obtained by the FEA, which indicates that the proposed beam theory can accurately

FIGURE 7: Axial displacement comparison.

predict the structural behavior of STBB under different cross sections. It shows that the postbuckling equilibrium paths are stable. In I ($b/h = 2/1$), the change of transversal displacement is tiny and the curve is relatively stable when the length is more than 3 m. Again, the same thing happens in III when the length is greater than 5 m. In the other three cases, the curve can be considered as increasing almost linearly. It can be seen that the five displacement modes without considering the shear lag effect are consistent with the prediction of the transverse displacement by present theory. In Table 4, the percentages of transverse

displacement of present method are closer than those of the other methods, indicating that the shear lag has a greater impact on the large ratio of the cross section. The results show that although six modes need to deal with more complex deformation relations to obtain the characteristics of transverse, the beam under different lengths and cross sections can be calculated arbitrarily and accurately.

Furthermore, as expected, Tables 2–4 point out that the relative displacement errors of Meftah are the largest. The shear lag displacement has a significant influence on the displacements of the long beam ($L > 3$ m), but a slight

FIGURE 8: Vertical displacement comparison.

influence on the short beam ($L \leq 3 \text{ m}$). The results highlight that the influence of shearing deformation on displacement is great.

In Figure 10, the present results confirm that the proposed methodology agrees well with that of the shell analysis. When the length-width ratio is greater than 1, Meftah's torsional rotations are always bigger than other values, but when the length-width ratio is less than 1, those of FEA become the maximum. Because of ignoring the effect of shear deformation according to Meftah's theory, torsional

stiffness is underestimated. When the length is more than 3 m, the torsional rotations start to become more sensitive. Interestingly, when the length between 4 m and 6 m, the torsional rotations grow slowly and the curve flattens out. The calculated values of Tan are always slightly larger than those of Meftah. And the variation of the two is more moderate with the length. It is easy to see from Figure 10 that the present torsional rotations are smaller than those of other methods, especially when the length-width ratio is less than 1. The torsion rotations of FEA are slightly different

FIGURE 9: Transversal displacement comparison.

from those of the present model, even in the distribution pattern. The reason is that the shear deformation effect is considered in present model (6 modes), and the shear stress terms not only include the generalized shear lag deformation, torsion, and distortion shear stress terms but also include the coupling torque of torsion and distortion.

The variation trends of distortion rotation of the three methods are in good agreement with each other in Figure 11. The present distortion curve is similar to that of Tan. The results of this theory (6 models) are satisfactory, especially when $b/h \leq 1$, and the curve value is always slightly smaller

than that of Tan. In the case of the same length, the distortional rotations of IV case ($b/h = 3/4$) are smaller than those of other cross section states. In fact, shear effect reduces the distortion value of the short beam relative to the nonshearable theory. The distortion analysis with shear lag effect shows a conservative behavior compared with a more comprehensive distortion analysis of nonlinear terms.

Figure 12 shows the shear lag warping displacement. It can be seen that the curve trend under the different cross sections is in good agreement within this length range. In general, the shear lag warping displacements gradually increase with the increase of

FIGURE 10: Torsional rotation comparison.

length. For $2 \leq L \leq 6$, the relative rate of change is very small ($\zeta_{z \min} = 3.9287E - 4m$, $\zeta_{z \max} = 7.8926E - 4m$). In the same case, the larger the length-width ratio is, the larger the shear lag warping displacement is. We can observe that the curve almost tends to a flat line in I ($b/h = 1/2$). In the case of long beams, the shear lag effect has a significant effect on the buckling behavior.

From Figures 7–12, it can be observed that the influence of shear deformation on buckling behavior is more significant when the length increases from 4 m. Due to the influence of defects, the mechanical properties of STBB by the

present method are obviously lower than those of FEA. When the cross section length-width ratio is $3/4$, the geometric nonlinearity of STBB is optimal. Therefore, the theory presented in this paper is valid for the thin-wall aspect ratio assumed in the next example.

3.2. Example 2: Variation of Height and Thickness. In this numerical example, the effects of beam length, web thickness, and load height on lateral resistance of STBB are studied. Figure 13 shows three different load positions.

FIGURE 11: Distortional rotation comparison.

Considering the appropriate combination of displacement modes, the higher-order model can also obtain the load of global buckling modes. In order to demonstrate the effectiveness of the proposed method, critical moments of linear and nonlinear under different lengths, web thicknesses, and load heights are calculated.

To evaluate the results, the relative errors between finite element analysis (M_{FEA}), Meftah (M_1), present nonlinear (M_2), present linear (M_3), Tan's nonlinear (M_4), and Tan's

linear (M_5) are given. The relative errors of moment are shown as follows:

$$\Delta_i = \left| \frac{M_i - M_{FEA}}{M_{FEA}} \right| \times 100\%. \quad (22)$$

Here, the material and geometrical properties are the same as the previous example. In example 1, it can be seen that when the aspect ratio of cross section is $b/h = 3/4$, the

TABLE 2: The percentage differences of the axial displacement (Δu_i).

L (m)	I			II			III			IV			V		
	Δu_1 (%)	Δu_2 (%)	Δu_3 (%)	Δu_1 (%)	Δu_2 (%)	Δu_3 (%)	Δu_1 (%)	Δu_2 (%)	Δu_3 (%)	Δu_1 (%)	Δu_2 (%)	Δu_3 (%)	Δu_1 (%)	Δu_2 (%)	Δu_3 (%)
1	18.5	29.9	26.3	12.3	21.0	19.7	17.1	22.5	19.7	14.3	15.2	15.7	13.1	23.2	16.8
2	12.8	22.0	16.9	12.3	20.1	14.4	18.9	32.7	19.4	15.7	17.5	17.3	11.2	14.4	11.9
3	13.5	19.7	15.9	14.9	18.0	16.7	19.9	27.0	21.8	18.0	21.9	19.0	9.7	24.3	18.4
4	21.9	28.9	25.8	19.9	26.2	23.0	20.3	34.0	21.7	20.1	25.6	21.7	7.5	29.1	19.0
5	13.5	26.6	19.7	20.9	32.1	26.8	25.0	31.0	28.9	20.5	30.3	27.6	5.2	23.6	17.6
6	15.8	29.8	25.3	15.3	26.7	21.3	14.1	28.9	25.0	21.1	28.9	22.0	8.5	17.6	14.6
7	16.2	23.1	17.3	13.4	27.0	13.5	9.6	28.6	12.6	13.3	29.7	19.0	9.8	24.4	22.3
8	18.9	26.3	19.7	14.9	25.2	17.1	8.5	18.5	9.0	6.6	26.4	16.2	6.6	20.0	9.4

TABLE 3: The percentage differences of the vertical displacement (Δw_i).

L (m)	I			II			III			IV			V		
	Δw_1 (%)	Δw_2 (%)	Δw_3 (%)	Δw_1 (%)	Δw_2 (%)	Δw_3 (%)	Δw_1 (%)	Δw_2 (%)	Δw_3 (%)	Δw_1 (%)	Δw_2 (%)	Δw_3 (%)	Δw_1 (%)	Δw_2 (%)	Δw_3 (%)
1	5.2	14.4	5.0	9.0	10.4	9.1	17.1	21.1	20.0	2.6	4.8	4.6	11.0	14.9	11.0
2	3.3	10.4	4.6	17.9	24.7	18.0	17.3	26.9	20.7	2.4	8.7	5.1	11.0	17.5	16.3
3	9.0	16.7	9.6	12.7	18.8	16.3	13.9	27.7	19.2	5.4	17.7	12.9	6.6	14.6	11.0
4	1.4	16.8	9.7	14.5	20.8	16.3	13.5	26.0	18.9	5.9	16.2	13.5	6.3	16.9	14.7
5	1.6	17.6	6.5	15.6	26.1	20.0	16.3	24.9	17.5	8.6	17.4	16.8	5.5	19.6	18.3
6	3.8	16.5	6.3	19.1	26.1	25.4	13.9	24.3	16.0	8.0	16.3	11.8	6.0	20.1	15.9
7	2.1	16.3	6.9	5.2	18.4	16.6	11.7	16.8	16.6	9.2	17.7	12.6	8.8	13.6	10.7
8	2.3	10.9	6.5	7.6	11.7	10.7	5.9	20.3	15.9	8.8	9.5	9.2	9.3	21.3	19.9

TABLE 4: The percentage differences of the transversal displacement (Δv_i).

L (m)	I			II			III			IV			V		
	Δv_1 (%)	Δv_2 (%)	Δv_3 (%)	Δv_1 (%)	Δv_2 (%)	Δv_3 (%)	Δv_1 (%)	Δv_2 (%)	Δv_3 (%)	Δv_1 (%)	Δv_2 (%)	Δv_3 (%)	Δv_1 (%)	Δv_2 (%)	Δv_3 (%)
1	18.1	24.5	21.1	11.2	25.1	23.4	3.0	8.1	7.2	18.1	22.1	21.8	10.6	25.9	13.3
2	11.0	22.6	19.2	13.0	17.4	15.1	7.5	23.0	17.1	2.8	13.1	4.1	11.0	25.2	17.3
3	11.8	21.8	13.8	13.0	31.9	30.0	8.7	23.1	17.5	4.2	11.4	13.9	12.2	22.6	16.8
4	13.6	19.8	18.2	12.7	15.6	17.8	6.1	24.1	10.5	9.2	16.5	14.7	2.7	24.0	17.3
5	10.3	21.1	20.4	10.2	19.6	21.4	9.0	21.6	12.9	9.5	23.3	16.3	6.6	22.3	18.9
6	10.7	19.1	17.8	14.0	18.7	21.9	2.3	18.2	5.5	4.6	20.9	8.2	6.5	24.2	17.8
7	5.6	25.7	15.6	9.2	17.4	14.1	2.0	19.7	7.3	2.5	18.6	4.8	3.8	21.1	17.2
8	2.9	20.0	11.2	2.8	4.0	9.9	1.8	11.7	2.5	3.6	10.7	8.0	9.6	20.6	17.3

comprehensive performance of STBB is the best. Therefore, the parameters are chosen as $h = 200$ mm, $b = 150$ mm, and $L = 3$ m, 4 m, and 5 m, respectively. The comparison of buckling moments relative to upper, center, and bottom flange and their difference is shown in Tables 5–7.

It can be seen from the three tables that all of the models have good consistency. The critical moments of the FEA model are almost bigger than those obtained by linear and nonlinear analysis. This is because the FEA model (generally speaking, a membrane shell with restricted movement) is stiffer than theoretical model. In order to verify the correlation of the buckling phenomenon, four rigid transverse webs with different thickness are carried out by the FEA method. The results of FEA analysis confirm the correlation of grid deformation.

From the three tables, it is obvious that the critical moments decrease with the increase of beam length. And at the same length, the critical moments increase with the increase of web thickness. The results show that the influence of length on the critical moments is always greater than that of web thickness. This fact is clear and useful to all theories. Furthermore, it can be obtained that the Meftah, Tan, and present nonlinear numerical calculations of the critical buckling moment are in good agreement with the FEA results. Tan's results and present linear results are more conservative than the nonlinear buckling moment calculated by FEA. It should be noted that the nonlinear solution of Meftah is the smallest, followed by the nonlinear solution of Tan's method. That is to say, the more the nonlinear coupling terms are considered, the higher the calculation accuracy is. It should be noted that Meftah and Tan's solutions

FIGURE 12: Shear lag warping displacement comparison.

FIGURE 13: Different heights applied by eccentric loads.

TABLE 5: Buckling moments and relative errors when the load height is $e_z = 0.1$ m.

L (m)	t_2 (mm)	t_4 (mm)	Buckling moments (kN·m)						Δ_1 (%)	Δ_2 (%)	Δ_3 (%)	Δ_4 (%)	Δ_5 (%)
			FEA	Meftah	Present nonlinear	Present linear	Tan nonlinear	Tan linear					
3	5	5	225667	211010.28	219611.56	209332.79	217552.50	199428.90	6.49	2.68	7.24	3.60	11.63
	8	5	276309	251003.81	259641.05	251027.69	252463.59	243544.89	9.16	6.03	9.15	8.63	11.86
	6.5	6.5	262569	235351.25	243006.22	234300.30	238937.94	227981.57	10.37	7.45	10.77	9.00	13.17
	8	8	316051	281436.29	291725.75	279336.82	286842.87	271458.02	10.95	7.70	6.71	9.24	14.11
4	5	5	180955	172317.23	178798.82	168978.27	173469.49	163869.54	4.77	1.19	6.62	4.14	9.44
	8	5	225538	200787.04	210657.63	198495.28	204972.31	192801.25	10.97	6.60	11.99	9.12	14.51
	6.5	6.5	208127	184138.17	193421.92	181271.11	186802.36	176519.44	11.53	7.07	12.90	10.25	15.19
	8	8	258906	226604.06	236073.46	224016.84	231506.06	218561.01	12.48	8.82	13.48	10.58	15.58
5	5	5	136689	128247.39	131239.14	128715.02	129486.48	126401.88	6.18	3.99	8.57	5.27	7.53
	8	5	161808	151369.70	155009.15	150871.19	152871.55	146421.74	6.45	4.20	6.76	5.52	9.51
	6.5	6.5	149180	139012.91	142197.01	138213.96	140702.31	134633.33	6.82	4.68	7.35	5.68	9.75
	8	8	184686	170382.46	175750.00	169385.73	173025.08	163285.86	7.74	4.84	8.28	6.31	11.59

TABLE 6: Buckling moments and relative errors when the load height is $e_z = 0.0$ m.

L (m)	t_2 (mm)	t_4 (mm)	Buckling moments (KN-m)						Δ_1 (%)	Δ_2 (%)	Δ_3 (%)	Δ_4 (%)	Δ_5 (%)
			FEA	Meftah	Present nonlinear	Present linear	Tan nonlinear	Tan linear					
3	5	5	225835	213002.21	223166.87	211266.27	220136.36	206738.29	5.68	1.18	6.45	2.52	8.46
	8	5	284822	259501.48	263471.40	252847.55	261261.62	248003.56	8.89	7.50	11.23	8.27	12.93
	6.5	6.5	279215	250135.71	255005.17	246604.78	253219.72	242447.34	10.41	8.67	11.68	9.31	13.17
	8	8	319554	285775.59	291819.43	282277.02	288867.14	276512.60	10.57	8.68	11.67	9.60	13.47
4	5	5	185347	173739.28	179231.77	170477.61	179247.22	166251.83	6.26	3.30	8.02	3.29	10.30
	8	5	232178	214779.69	222355.92	202710.09	217341.62	200551.73	7.49	4.23	12.69	6.39	13.62
	6.5	6.5	224293	204061.08	214146.40	194871.77	205642.26	191828.59	9.02	4.52	13.12	8.32	14.47
	8	8	295826	268658.00	281732.67	256199.83	269211.21	252482.85	9.18	4.76	13.40	9.00	14.62
5	5	5	138531	129328.49	133717.80	128925.13	130852.96	127093.14	6.64	3.47	6.93	5.54	8.26
	8	5	162876	148774.34	155911.16	147076.83	152361.35	144382.62	8.66	4.28	9.70	6.46	11.35
	6.5	6.5	159572	145679.79	152418.63	143205.59	147403.37	140670.70	8.71	4.48	10.26	7.63	11.84
	8	8	185976	168907.25	176252.37	165829.93	171598.74	162866.93	9.18	5.23	10.83	7.73	12.43

TABLE 7: Buckling moments and relative errors when the load height is $e_z = -0.1$ m.

L (m)	t_2 (mm)	t_4 (mm)	Buckling moments in (KN-m)						Δ_1 (%)	Δ_2 (%)	Δ_3 (%)	Δ_4 (%)	Δ_5 (%)
			FEA	Meftah	Present nonlinear	Present linear	Tan nonlinear	Tan linear					
3	5	5	227568	218984.20	224733.00	213629.25	220487.77	208314.22	3.77	1.25	6.13	3.11	8.46
	8	5	286538	271574.24	279651.51	267208.35	271402.49	261601.91	5.22	2.40	6.75	5.28	8.70
	6.5	6.5	280801	261425.78	273238.50	261238.57	264030.17	253839.75	6.90	2.69	6.97	5.97	9.60
	8	8	321619	286037.73	309914.91	285676.66	291390.65	278627.90	11.06	3.64	11.18	9.40	13.37
4	5	5	193686	176416.22	184194.28	175451.18	180291.54	171150.55	8.92	4.90	9.41	6.92	11.64
	8	5	252012	227827.91	239620.01	225511.64	230640.22	221674.48	9.60	4.92	10.52	8.48	12.04
	6.5	6.5	231983	210841.97	220204.92	206846.99	211561.88	201911.15	9.11	5.08	10.84	8.80	12.96
	8	8	297230	270095.40	282007.76	263162.24	270816.00	257204.08	9.13	5.12	11.46	8.89	13.47
5	5	5	138719	129404.94	136127.97	130180.71	134545.10	129216.81	6.71	1.87	6.16	3.01	6.85
	8	5	163506	151520.11	157885.73	149802.14	154473.85	144995.82	7.33	3.44	8.38	5.52	11.32
	6.5	6.5	160061	147786.97	152596.97	143508.83	150411.54	140978.89	7.67	4.66	10.34	6.03	11.92
	8	8	186072	170100.49	177063.44	166817.47	174786.55	163819.47	8.58	4.84	10.35	6.07	11.96

under the same conditions are lower than those of the present approach; this is due to the generalized eigenvectors of the classical models and the initial assumptions—essentially without taking the shear deformation into account, which is very important for the different thicknesses of the web model. The nonlinear critical moments obtained by present method is closest to those obtained by FEA. And the linear solution of present method is larger than the linear solution of Tan's method; it can be seen that considering the effect of shear lag effect can more comprehensively supplement the mechanical properties of STBB. The difference between linear and nonlinear relative error is very impressive. The linear and nonlinear relative errors of Tan and the present method can reach about 8% and 8.63% at the maximum, respectively. From the relative errors Δ_1 , Δ_2 , and Δ_4 , it can be seen that the numerical and analytical solutions are consistent and highly nonlinear. The relative error Δ_2 is almost the smallest value, while Δ_5 is the biggest one. In this case, by comparing the moments of STBB before and after buckling, it can be seen that only the nonlinear solution is more accurate.

By comparing the three tables, it can be seen that when the load is applied to the top plane of STBB, the bending moment is the smallest. And when the load is applied to the bottom plane, the beam resistance to lateral buckling is the greatest. Therefore, we can think that the lateral buckling strength depends on the load height parameter. In the postbuckling state, the shear lag effect term is added to maximize the resistance of STBB to lateral buckling. In general, as the web becomes thicker, the relative error becomes larger overall. Regardless of beam length and the location where the concentrated load was applied, M_{cr} values for nonuniform web thickness ($t_1 = 5$ mm and $t_2 = 8$ mm) are higher than those for the same web thickness ($t_1 = t_2 = 6.5$ mm). With the increase of the length, the difference of the critical moments (for both uniform and unequal webs) becomes more obvious, and the unequal web of beams show better mechanical properties. Buckling moment is mainly affected by the web resistance. It is clear that the phenomenon is due solely to coupling-induced geometrical nonlinear effect.

4. Conclusion

A new kinematics model is used to analyze the stability of straight thin-walled box beam (STBB). Based on the conventional motion variables, the shear lag modes are added to obtain that the beam section contains six modes of extension, bending, torsion, distortion, and shear lag under the action of shear deformation. The analytical formulas take the deformation, load height, and unequal web thickness into account under combined axial and vertical loads. In this theory, six sets of coupled equilibrium differential equations are derived to describe the longitudinal and transversal deformation through six displacement fields. In particular, when the straight thin-walled box beam is subjected to combined axial and vertical loads, these formulas are naturally applicable to capture bending, torsional, distortion, and shear lag responses.

Present theory puts forward a theoretical basis for studying the mechanical properties of STBB and the ways to be affected by adding shear lag effect. Unlike traditional Meftah and Tan's theories, present theory shows that the beam response is largely dependent on the shear lag. Various numerical results are compared with those of Meftah, Tan, and shell finite element (Ansys).

With regard to the different cross sections and lengths, the following conclusions are given:

- (i) When the cross section ratio is greater than 1, the geometric variation of cross section has little influence on the axial length.
- (ii) The shear lag displacement is conducive to the prediction of longitudinal displacement.
- (iii) For the long beam, the shear lag warping displacement can reflect the vertical displacement more accurately.
- (iv) Generalized shear lag deformation increases the predicted torsional stiffness of the beam.

- (v) The distortion under the coupling effect of the six modes (adding shear lag effect) is more conservative than that of the five modes.

- (vi) The shear lag warping displacement increases with the increase of cross section ratio (b/h).

For the different load positions and the thicknesses of webs, the following results are reached:

- (i) The nonlinear critical moments of present method are closest to those of FEA.
- (ii) The relative error Δ_2 is almost the smallest value, while Δ_5 is the biggest one.
- (iii) The lateral buckling strength depends on the load height parameter.
- (iv) M_{cr} values for nonuniform web thickness ($t_1 = 5$ mm and $t_2 = 8$ mm) are higher than those for the same web thickness ($t_1 = t_2 = 6.5$ mm).

The effect of shear lag can more comprehensively supplement the mechanical properties of STBB, and the nonlinear solution is more accurate. In the postbuckling state, the shear lag effect term is added to maximize the resistance of STBB to lateral buckling.

The calculation results are in good agreement with the results of finite element and other theoretical analysis. The above observation results show that the geometry has a great influence on the nonlinear buckling of STBB. For these reasons, further development of this research should focus on the relationship between the material properties of beams and the modes. The developed analytical method is sufficiently efficient and accurate.

Appendix

A. Detailed Expressions of These Forces and Torques

$$\begin{aligned}
 N &= E \left[\sum_{i=4} F_i u'_0(x) + \frac{1}{2} \sum_{i=4} F_i [u'_0(x)^2 + v'_0(x)^2 + w'_0(x)^2] + \frac{1}{2} \left(\sum_{i=1,3} A_2^i + \sum_{i=2,4} B_2^i + \sum_{i=1,3} A_5^i + \sum_{i=2,4} B_5^i \right) \theta(x)^2 - \sum_{i=1,3} A_6^i v''_0(x) \chi''_0(x) \right. \\
 &\quad + \frac{1}{2} \left(\sum_{i=1,3} A_{19}^i + \sum_{i=2,4} B_{19}^i + \sum_{i=1,3} A_{20}^i + \sum_{i=2,4} B_{20}^i \right) \chi'(x)^2 + \frac{1}{2} \sum_{i=1,3} A_{36}^i \zeta'_z(x)^2 - \left(\sum_{i=1,3} A_{21}^i + \sum_{i=2,4} B_{21}^i \right) \theta'(x) \chi'(x) - \sum_{i=2,4} B_6^i w''_0(x) \chi''_0(x) \\
 &\quad \left. - \sum_{i=1,3} A_{37}^i \zeta'_z(x) w''_0(x) + \frac{1}{2} \left(\sum_{i=1,3} A_2^i + \sum_{i=2,4} B_5^i \right) w''_0(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_5^i + \sum_{i=2,4} B_2^i \right) v''_0(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) [\theta''_0(x)^2 + \chi'_0(x)^2] \right] \\
 M_y &= E \left[\left(\sum_{i=1,3} A_2^i + \sum_{i=2,4} B_5^i \right) [w''_0(x) + u'_0(x) w''_0(x)] = \sum_{i=1,3} A_{37}^i [\zeta'_z(x) + u'_0(x) \zeta'_z(x)] + \left(\sum_{i=1,3} A_7^i - \sum_{i=2,4} B_7^i \right) v''_0(x) \theta''_0(x) \right. \\
 &\quad \left. + \left(\sum_{i=2,4} B_5^i - \sum_{i=1,3} A_2^i \right) v'_0(x) \theta'(x) + \left(\sum_{i=1,3} A_2^i - \sum_{i=2,4} B_{13}^i \right) v'_0(x) \chi'(x) - \sum_{i=1,3} A_{11}^i w'_0(x) \chi'(x) + \sum_{i=2,4} B_6^i [\chi''_0(x) + u'_0(x) \chi''_0(x)] \right]
 \end{aligned}$$

$$\begin{aligned}
M_z &= E \left[\begin{aligned} &\left(\sum_{i=1,3} A_5^i + \sum_{i=2,4} B_2^i \right) [v_0''(x) + u_0'(x)v_0''] - \frac{1}{4} \sum_{i=1,3} [A_{38}^i \zeta_z'(x) + \theta_0''(x)\zeta_z'(x)] + \left(\sum_{i=1,3} A_7^i - \sum_{i=2,4} B_7^i \right) w_0''(x)\theta_0''(x) \\ &+ \left(\sum_{i=2,4} B_2^i + \sum_{i=1,3} A_5^i \right) w_0'(x)\theta'(x) - \sum_{i=2,4} B_{11}^i v_0'(x)\chi'(x) - \left(\sum_{i=1,3} A_{11}^i + \sum_{i=2,4} B_2^i \right) w_0'(x)\chi'(x) - \sum_{i=2,4} A_6^i [\chi_0''(x) + u_0'(x)\chi_0''(x)] \end{aligned} \right], \\
M_{\psi_z} &= E \left[\begin{aligned} &\left(\sum_{i=2,4} B_2^i - \sum_{i=1,3} A_{13}^i \right) v_0''(x) - \sum_{i=1,3} A_{11}^i (w_0''(x) + u_0'(x)w_0''(x)) - \sum_{i=1,3} A_{14}^i (\chi_z''(x) + u_0'(x)\chi_z''(x)) \\ &+ \sum_{i=1,3} A_{13}^i u_0''(x)v_0''(x) - \sum_{i=1,3} A_{39}^i \theta''(x)\zeta_z'(x) - \sum_{i=2,4} B_7^i w_0''(x)\theta''(x) - \sum_{i=1,3} B_1^i \zeta_z'(x)v_0''(x) - \sum_{i=1,3} B_8^i \theta''(x)\chi''(x) \end{aligned} \right], \\
M_{\psi_y} &= E \left[\begin{aligned} &\left(\sum_{i=1,3} A_2^i - \sum_{i=2,4} B_{13}^i \right) w_0''(x) - \sum_{i=2,4} B_{14}^i \chi''(x) - \sum_{i=1,3} A_{37}^i (\zeta_z'(x) + u_0'(x)\zeta_z'(x)) \\ &\sum_{i=1,3} A_7^i v_0''(x)\theta''(x) + \left(\sum_{i=1,3} A_8^i + \sum_{i=2,4} B_8^i \right) \theta''(x)\chi''(x) \end{aligned} \right], \\
M_R &= E \left[\begin{aligned} &\left(\sum_{i=1,3} A_2^i + \sum_{i=2,4} B_2^i + \sum_{i=1,3} A_5^i + \sum_{i=2,4} B_5^i \right) \left[u_0'(x) + \frac{1}{2} (u_0'(x)^2 + v_0'(x)^2 + u_0'(x)^2) \right] + \frac{1}{2} \left(\sum_{i=1,3} A_{28}^i + \sum_{i=2,4} B_{28}^i \right) \theta(x)^2 \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_{29}^i + \sum_{i=2,4} B_{29}^i \right) \chi'(x)^2 + \frac{1}{2} \left(\frac{5}{2} \sum_{i=1,3} A_{36}^i + \sum_{i=1,3} A_{45}^i \right) \zeta_z'(x)^2 - \left(\sum_{i=1,3} A_{30}^i + \sum_{i=2,4} B_{30}^i \right) \theta'(x)\chi^2(x) \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i + \sum_{i=1,3} A_4^i + \sum_{i=2,4} B_{32}^i \right) w_0''(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i + \sum_{i=1,3} A_{32}^i + \sum_{i=2,4} B_4^i \right) v_0''(x)^2 \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_8^i + \sum_{i=2,4} B_8^i + \sum_{i=1,3} A_9^i + \sum_{i=2,4} B_9^i \right) [\theta''(x)^2 + \chi''(x)^2] - \sum_{i=1,3} A_{46}^i w_0''(x)\zeta_z'(x) \end{aligned} \right], \\
M_\psi &= E \left[\begin{aligned} &\left(\sum_{i=1,3} A_{19}^i + \sum_{i=2,4} B_{19}^i + \sum_{i=1,3} A_{20}^i + \sum_{i=2,4} B_{20}^i \right) \left[u_0'(x) + \frac{1}{2} (u_0'(x)^2 + v_0'(x)^2 + w_0'(x)^2) \right] + \frac{1}{2} \sum_{i=1,3} A_{47}^i \zeta_z'(x)^2 \\ &- \left(\sum_{i=1,3} A_{41}^i + \sum_{i=2,4} B_{41}^i \right) [\theta''(x) + u_0'(x)\theta''(x)] - \sum_{i=1,3} A_{48}^i w_0''(x)\zeta_z'(x)\zeta_z'(x) - \sum_{i=1,3} A_{49}^i v_0''(x)\zeta_z'(x)\zeta_z'(x) \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_{29}^i + \sum_{i=2,4} B_{29}^i \right) \theta'(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_{42}^i + \sum_{i=2,4} B_{42}^i \right) \chi'(x)^2 - \left(\sum_{i=1,3} A_{17}^i + \sum_{i=2,4} B_{17}^i \right) \theta'(x)\chi'(x) \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_{43}^i + \sum_{i=2,4} B_{44}^i \right) w_0''(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_{44}^i + \sum_{i=2,4} B_{43}^i \right) v_0''(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_{52}^i + \sum_{i=2,4} B_{52}^i \right) [\theta''(x)^2 + \chi''(x)^2] \\ &+ \left(\sum_{i=2,4} B_{41}^i - \sum_{i=1,3} A_{41}^i \right) v_0''w_0''(x) + \left(\sum_{i=2,4} B_{52}^i - \sum_{i=1,3} A_{53}^i \right) w_0''(x)\chi''(x) + \left(\sum_{i=1,3} A_{52}^i + \sum_{i=2,4} B_{53}^i \right) v_0''(x)\chi''(x) \end{aligned} \right], \\
M_{\psi R} &= E \left[\begin{aligned} &\left(\sum_{i=1,3} A_{21}^i + \sum_{i=2,4} B_{21}^i \right) \left[u_0'(x) + \frac{1}{2} (u_0'(x)^2 + v_0'(x)^2 + w_0'(x)^2) \right] + \frac{1}{2} \left(\sum_{i=1,3} A_{30}^i + \sum_{i=2,4} B_{30}^i \right) \theta(x)^2 - \sum_{i=1,3} A_{58}^i v_0''(x)\zeta_z'(x) \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_{17}^i + \sum_{i=2,4} B_{17}^i \right) \chi'(x)^2 + \frac{1}{2} \sum_{i=1,3} A_{50}^i \zeta_z'(x)^2 - \left(\sum_{i=1,3} A_{54}^i + \sum_{i=2,4} B_{54}^i \right) \theta(x)\chi'(x) - \sum_{i=1,3} A_{51}^i w_0''(x)\zeta_z'(x) \\ &+ \sum_{i=1,3} A_{59}^i \chi_0''(x)\zeta_z' + \frac{1}{2} \left(\sum_{i=1,3} A_4^i - \sum_{i=1,3} A_{15}^i + \sum_{i=2,4} B_{55}^i \right) w_0''(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_{55}^i + \sum_{i=1,3} B_4^i - \sum_{i=2,4} B_{15}^i \right) v_0''(x)^2 \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_{56}^i + \sum_{i=2,4} B_{56}^i \right) [\theta''(x)^2 + \chi''(x)^2] + \left(\sum_{i=1,3} A_{57}^i + 2 \sum_{i=2,4} B_{18}^i \right) v_0''(x)w_0''(x) + \left(2 \sum_{i=1,3} A_{18}^i + \sum_{i=2,4} B_{57}^i \right) w_0''(x)\chi''(x) \end{aligned} \right],
\end{aligned}$$

$$\begin{aligned}
B_{\omega_a} &= E \left[\frac{1}{2} \left(\sum_{i=1,3} A_{41}^i + \sum_{i=2,4} B_{41}^i \right) \theta'(x)^2 - \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) [\theta''(x) + u_0'(x)\theta''(x)] + \sum_{i=1,3} A_{60}^i \chi_0''(x) \zeta'(x) \right. \\
&\quad \left. + \left(\sum_{i=2,4} B_6^i - \sum_{i=1,3} A_6^i \right) v_0''(x) w_0''(x) + \sum_{i=1,3} B_8^i v_0''(x) \chi_0''(x) - \sum_{i=1,3} A_{38}^i v_0''(x) \zeta'_z(x) \right], \\
B_{\omega_d} &= E \left[\sum_{i=1,3} A_{60}^i \theta''(x) \zeta'_z(x) - \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) [\chi_0''(x) + u_0'(x) \chi_0''(x)] - \sum_{i=1,3} A_6^i \left[v_0'(x) \theta'(x) + w_0'(x) \theta'(x) \right] - \sum_{i=1,3} B_8^i v_0''(x) \theta''(x) \right], \\
M_{\omega_{\zeta_z}^2} &= E \left[\sum_{i=1,3} A_{36}^i \left[u_0'(x) + \frac{1}{2} (u_0'(x)^2 + v_0'(x)^2 + w_0'(x)^2) \right] + \frac{1}{2} \left[2 \sum_{i=1,3} A_{36}^i + \sum_{i=1,3} A_{45}^i \right] \theta_0''(x)^2 + \frac{1}{2} \sum_{i=1,3} A_{47}^i \chi_0'(x)^2 + \sum_{i=1,3} A_{36}^i w_0''(x)^2 \right. \\
&\quad \left. - \sum_{i=1,3} A_{50}^i \theta_0'(x) \chi_0'(x) - \sum_{i=1,3} A_{37}^i \chi''(x) \zeta'_z(x) + \frac{1}{2} \sum_{i=1,3} A_{45}^i v_0''(x)^2 + \frac{6}{7} \sum_{i=1,3} A_{60}^i v_0''(x) \chi''(x) + \frac{1}{2} \sum_{i=1,3} A_{62}^i [\theta_0''(x)^2 + \chi_0''(x)^2] \right], \\
M_{z\omega_{\zeta_z}} &= E \left[\sum_{i=1,3} A_{37}^i \left[u_0'(x) + \frac{1}{2} (u_0'(x)^2 + v_0'(x)^2 + w_0'(x)^2) \right] + \frac{1}{2} \sum_{i=1,3} A_{46}^i \theta_0'(x)^2 + \frac{1}{2} \sum_{i=1,3} A_{48}^i \chi_0'(x)^2 \right. \\
&\quad - \sum_{i=1,3} A_{51}^i \theta_0'(x) \chi_0'(x) + \frac{1}{2} \sum_{i=1,3} A_{62}^i w_0''(x)^2 + \frac{1}{8} \sum_{i=1,3} B_{38}^i v_0''(x)^2 + \frac{7}{12} \sum_{i=1,3} A_{62}^i [\theta_0''(x)^2 + \chi_0''(x)^2] \\
&\quad \left. - \sum_{i=1,3} A_{60}^i [v_0''(x) \theta_0''(x) v_0''(x) \chi_0''(x)] - \frac{7}{6} \sum_{i=1,3} A_{62}^i \theta_0''(x) \chi_0''(x) - 2 \sum_{i=1,3} A_{36}^i w_0''(x) \zeta'_z(x) \right], \\
M_{\omega_a \omega_{\zeta_z}} &= E \left[\sum_{i=1,3} A_{39}^i w_0'(x) \chi_0'(x) - \sum_{i=1,3} A_{60}^i [\chi_0''(x) + u_0'(x) \chi_0''(x) - \theta_0''(x) \zeta'_z(x)] + \frac{1}{4} \sum_{i=1,3} A_{38}^i [v_0''(x) + u_0'(x) v_0''(x)] \right], \\
M_{\omega_d \omega_{\zeta_z}} &= E \left[\frac{1}{4} \sum_{i=1,3} A_{38}^i w_0'(x) \theta_0'(x) - \sum_{i=1,3} A_{59}^i \theta_0'(x) \chi_0'(x) + \frac{1}{2} \sum_{i=1,3} A_{49}^i \chi_0'(x)^2 + \frac{6}{7} \sum_{i=1,3} A_{60}^i v_0''(x) \zeta'_z(x) \right. \\
&\quad \left. - \sum_{i=1,3} A_{60}^i [\theta_0''(x) + u_0'(x) \theta_0''(x) + \chi_0''(x) \zeta'_z(x) + v_0''(x) w_0''(x)] \right], \\
M_{z^2} &= E \left[\left(\sum_{i=1,3} A_2^i + \sum_{i=2,4} B_5^i \right) \left[u_0'(x) + \frac{1}{2} (u_0'(x)^2 + v_0'(x)^2 + w_0'(x)^2) \right] + \sum_{i=1,3} A_{36}^i \zeta'_z(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_{43}^i + \sum_{i=2,4} B_{44}^i \right) \chi'(x)^2 \right. \\
&\quad + \frac{1}{2} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i + \sum_{i=1,3} A_4^i + \sum_{i=2,4} B_{32}^i \right) \theta'(x)^2 - \left(\sum_{i=1,3} A_4^i - \sum_{i=1,3} A_{15}^i + \sum_{i=2,4} B_{55}^i \right) \theta'(x) \chi'(x) - \sum_{i=1,3} A_4^i w_0'(x) \zeta'_z(x) \\
&\quad + \frac{1}{2} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) v_0''(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_9^i + \sum_{i=2,4} B_{33}^i \right) [\theta''(x)^2 + \chi''(x)^2] + \frac{1}{2} \left(\sum_{i=1,3} A_4^i + \sum_{i=2,4} B_{732}^i \right) w_0''(x)^2 \\
&\quad \left. - \frac{5}{2} \sum_{i=1,3} A_{82}^i w_0''(x) \zeta'_z(x) \right],
\end{aligned}$$

$$\begin{aligned}
M_{Y^2} &= E \left[\begin{aligned} &\left(\sum_{i=1,3} A_5^i + \sum_{i=2,4} B_2^i \right) \left[u_0'(x) + \frac{1}{2} (u_0'(x)^2 + v_0'(x)^2 + w_0'(x)^2) \right] + \frac{1}{2} \sum_{i=1,3} A_{45}^i \zeta_z'(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_{44}^i + \sum_{i=2,4} B_{43}^i \right) \chi'(x)^2 \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i + \sum_{i=1,3} A_{32}^i + \sum_{i=2,4} B_4^i \right) \theta'(x)^2 - \frac{1}{4} \sum_{i=1,3} A_5^i w_0'(x) \zeta_z'(x) \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_{32}^i + \sum_{i=2,4} B_4^i \right) v_0''(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_{33}^i + \sum_{i=2,4} B_9^i \right) [\theta''(x)^2 + \chi''(x)^2] + \frac{1}{2} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) w_0''(x)^2 \\ &+ \sum_{i=1,3} A_{64}^i v_0''(x) \chi''(x) - \left(\sum_{i=1,3} A_{55}^i + \sum_{i=2,4} B_4^i - \sum_{i=2,4} B_{15}^i \right) \theta'(x) \chi'(x) \end{aligned} \right], \\
M_{\omega_d^2} &= E \left[\begin{aligned} &\left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) \left[u_0'(x) + \frac{1}{2} (u_0'(x)^2 + v_0'(x)^2 + w_0'(x)^2) \right] + \frac{1}{2} \left(\sum_{i=1,3} A_{62}^i \zeta_z'(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_{52}^i + \sum_{i=2,4} B_{52}^i \right) \right) \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_8^i + \sum_{i=2,4} B_8^i + \sum_{i=1,3} A_9^i + \sum_{i=2,4} B_9^i \right) \theta'(x)^2 + \left(\sum_{i=1,3} A_{56}^i + \sum_{i=2,4} B_{56}^i \right) \theta'(x) \chi'(x) - \frac{7}{6} \sum_{i=1,3} A_{62}^i w_0''(x) \zeta_z'(x)^2 \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_{33}^i + \sum_{i=2,4} B_9^i \right) v_0''(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_{35}^i + \sum_{i=2,4} B_{35}^i \right) [\theta''(x)^2 + \chi''(x)^2] + \frac{1}{2} \left(\sum_{i=1,3} A_9^i + \sum_{i=2,4} B_{33}^i \right) w_0''(x)^2 \\ &- \sum_{i=1,3} A_{34}^i [v_0''(x) \theta''(x) + v_0''(x) \chi''(x)] + \sum_{i=1,3} B_{34}^i w_0''(x)^2 \chi''(x) \end{aligned} \right], \\
M_{\omega_d^2} &= E \left[\begin{aligned} &\left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) \left[u_0'(x) + \frac{1}{2} (u_0'(x)^2 + v_0'(x)^2 + w_0'(x)^2) \right] + \frac{1}{2} \sum_{i=1,3} A_{62}^i \zeta_z'(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_{52}^i + \sum_{i=2,4} B_{52}^i \right) \chi'(x)^2 \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_8^i + \sum_{i=2,4} B_8^i + \sum_{i=1,3} A_9^i + \sum_{i=2,4} B_9^i \right) \theta'(x)^2 - \frac{7}{6} \sum_{i=1,3} A_{62}^i w_0''(x) \zeta_z'(x) + \frac{1}{2} \left(\sum_{i=1,3} A_{33}^i + \sum_{i=2,4} B_9^i \right) v_0''(x)^2 \\ &+ \frac{1}{2} \left(\sum_{i=1,3} A_{35}^i + \sum_{i=2,4} B_{35}^i \right) [\theta''(x)^2 + \chi''(x)^2] + \frac{1}{2} \left(\sum_{i=1,3} A_9^i + \sum_{i=2,4} B_{35}^i \right) w_0''(x)^2 - \sum_{i=1,3} A_{34}^i v_0''(x) \chi''(x) + \sum_{i=2,4} B_{34}^i w_0''(x) \chi''(x) \end{aligned} \right], \\
M_{ZY} &= E \left[\begin{aligned} &\frac{1}{2} \left(\sum_{i=1,3} A_{41}^i + \sum_{i=2,4} B_{41}^i \right) \chi'(x)^2 - \frac{1}{2} \left(\sum_{i=1,3} A_{17}^i + \sum_{i=2,4} B_{17}^i \right) \theta'(x) \chi'(x) + \left(\sum_{i=1,3} A_6^i - \sum_{i=2,4} B_6^i \right) [\theta''(x) + u_0'(x) \theta''(x)] \\ &- \frac{1}{4} \sum_{i=1,3} A_{38}^i [v_0''(x) \zeta_z'(x) + \chi''(x) \zeta_z'(x)] + \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) v_0''(x) w_0''(x) + \sum_{i=1,3} A_8^i w_0''(x) \chi''(x) + \sum_{i=2,4} B_8^i v_0''(x) \chi''(x) \end{aligned} \right], \\
M_{Z\omega_a} &= E \left[\left(\sum_{i=2,4} B_7^i - \sum_{i=1,3} A_7^i \right) [v_0''(x) + u_0'(x) v_0''(x)] - \left(\sum_{i=1,3} A_{33}^i + \sum_{i=2,4} B_{28}^i \right) w_0''(x) \theta''(x) - \sum_{i=2,4} B_8^i \theta''(x) \chi''(x) - \sum_{i=2,4} B_7^i w_0'(x) \chi'(x) \right], \\
M_{\omega_a \omega_d} &= E \left[\begin{aligned} &-\sum_{i=1,3} A_{66}^i [\zeta_z'(x) + u_0'(x) \zeta_z'(x)] + \frac{7}{6} \sum_{i=1,3} A_{62}^i w_0''(x) \zeta_z'(x) + \sum_{i=2,4} B_8^i w_0''(x) \theta''(x) \\ &+ \sum_{i=1,3} A_8^i [w_0''(x) + u_0'(x) w_0''(x) + v_0''(x) \theta''(x)] + \left(\sum_{i=2,4} B_{35}^i + \sum_{i=1,3} A_{35}^i \right) \theta''(x) \chi''(x) \end{aligned} \right], \\
M_{Z\omega_d} &= E \left[\begin{aligned} &-\sum_{i=2,4} B_6^i \left[u_0'(x) + \frac{1}{2} (u_0'(x)^2 + v_0'(x)^2 + w_0'(x)^2) \right] - \frac{1}{2} \sum_{i=2,4} B_{40}^i \theta_0'(x)^2 + \frac{1}{2} \left(\sum_{i=1,3} A_{53}^i - \sum_{i=2,4} B_{52}^i \right) \chi'(x)^2 \\ &- \left(2 \sum_{i=1,3} A_{18}^i + \sum_{i=2,4} B_{57}^i \right) \theta_0'(x) \chi'(x) + \sum_{i=1,3} A_{60}^i v_0''(x) \zeta_z'(x) - \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) w_0'(x) \theta_0'(x) - \frac{1}{2} \sum_{i=2,4} B_8^i v_0''(x)^2 \\ &- \frac{1}{2} \sum_{i=2,4} B_{34}^i \theta''(x)^2 - \sum_{i=1,3} A_{34}^i v_0''(x) w_0''(x) - \left(\sum_{i=1,3} A_{33}^i + \sum_{i=2,4} B_9^i \right) w_0''(x) \chi_0''(x) \end{aligned} \right],
\end{aligned}$$

$$\begin{aligned}
M_{Y\omega_a} &= E \left[\begin{aligned} &\left(\sum_{i=2,4} B_7^i - \sum_{i=1,3} A_7^i \right) [w_0''(x) + u_0'(x)w_0''(x)] \frac{1}{4} \sum_{i=1,3} A_{38}^i [\zeta_z'(x) + u_0'(x)\zeta_z'(x)] - \sum_{i=1,3} A_{60}^i w_0''(x)\zeta_z'(x) \\ &+ \sum_{i=2,4} B_8^i [\chi''(x) + u_0'(x)\chi''(x)] - \left(\sum_{i=2,4} B_{33}^i + \sum_{i=1,3} A_9^i \right) v_0''(x)\theta_0''(x) - \sum_{i=1,3} A_8^i \theta_0''(x)\chi''(x) - \sum_{i=1,3} A_2^i v_0'(x)\chi'(x) \end{aligned} \right], \\
M_{Y\omega_d} &= E \left[\begin{aligned} &-\sum_{i=1,3} A_6^i \left[u_0'(x) + \frac{1}{2} (u_0'(x)^2 + v_0'(x)^2 + w_0'(x)^2) \right] - \frac{1}{2} \sum_{i=1,3} A_{40}^i \theta_0(x)^2 - \frac{1}{2} \left(\sum_{i=1,3} A_{52}^i + \sum_{i=2,4} B_{53}^i \right) \chi'(x)^2 \\ &+ \left(2 \sum_{i=1,3} A_{18}^i + \sum_{i=2,4} B_{57}^i \right) \theta_0'(x)\chi'(x) + \sum_{i=2,4} B_8^i [\theta''(x) + u_0'(x)\theta''(x)] - \frac{3}{7} \sum_{i=1,3} A_{60}^i \zeta_z'(x)^2 \end{aligned} \right], \\
Q_{\omega_{\zeta_z}'} &= G \left[\begin{aligned} &\sum_{i=1,3} A_F^i \zeta_z(x) - 2 \sum_{i=1,3} A_{37}^i [v_0'(x)\theta''(x) + v_0''(x)\theta'(x)] - \frac{4}{5} \sum_{i=1,3} A_{36}^i [\theta'(x)\chi''(x) + \theta''(x)\chi'(x)] \\ &+ \sum_{i=1,3} A_8^i w_0'(x)\chi(x) - \sum_{i=1,3} A_{63}^i u_0'(x)\zeta_z(x) \end{aligned} \right], \\
M_{sv} &= G \left[\begin{aligned} &-2 \sum_{i=1,3} A_2^i [u_0'(x)\theta_0'(x) + \chi'(x) + v_0'(x)w_0''(x)] + \left(\sum_{i=1,3} A_{65}^i - \sum_{i=2,4} B_{65}^i + \sum_{i=1,3} A_{66}^i + \sum_{i=2,4} B_{66}^i \right) \theta_0(x)\chi_0'(x) \\ &+ 4 \sum_{i=1,3} A_{11}^i \theta_0'(x)\chi_0(x) + \left(10 \sum_{i=2,4} B_{13}^i - 10 \sum_{i=1,3} A_{13}^i - 2 \sum_{i=1,3} A_{67}^i - \sum_{i=1,3} A_{78}^i - \sum_{i=2,4} B_{78}^i \right) \chi_0(x)\chi_0'(x) \\ &- \sum_{i=1,3} A_3^i w_0''(x)\chi_0'(x) + \sum_{i=2,4} B_3^i v_0''(x)\chi_0'(x) - \frac{4}{5} \sum_{i=1,3} A_{36}^i \chi_0''(x)\zeta_z(x) - 4 \sum_{i=1,3} A_{37}^i v_0''(x)\zeta_z(x) \end{aligned} \right], \\
M_{sv\psi} &= G \left[\begin{aligned} &\sum_{i=2,4} B_1^i [2w_0'(x) + u_0'(x)w_0'(x) + v_0'(x)\theta_0(x)] + \sum_{i=1,3} A_1^i [2v_0'(x) + u_0'(x)v_0'(x) + v_0'(x)w_0'(x)] \\ &+ \left(\sum_{i=1,3} A_2^i - \sum_{i=2,4} B_2^i \right) \theta_0'(x) + 2 \sum_{i=1,3} A_2^i u_0'(x)\theta_0'(x) + \left(\sum_{i=1,3} A_{19}^i + \sum_{i=2,4} B_{19}^i \right) \chi'(x) + \sum_{i=1,3} A_3^i w_0''(x)\theta_0'(x) \\ &- \sum_{i=1,3} A_{66}^i w_0'(x)\chi_0(x) - \sum_{i=2,4} B_{66}^i v_0'(x)\chi_0(x) - 2 \left(\sum_{i=1,3} A_{11}^i + \sum_{i=2,4} B_{11}^i \right) \theta_0'(x)\chi_0(x) - \sum_{i=2,4} B_3^i v_0''(x)\theta_0'(x) \\ &- \left(\sum_{i=1,3} A_{66}^i + \sum_{i=2,4} B_{66}^i \right) \theta_0(x)\chi'(x) + \left(\sum_{i=1,3} A_{94}^i + \sum_{i=2,4} B_{94}^i + \sum_{i=1,3} A_{78}^i + \sum_{i=2,4} B_{78}^i \right) \chi_0(x)\chi_0'(x) - \sum_{i=1,3} A_{37}^i v_0'(x)\zeta_z'(x) \\ &+ \sum_{i=1,3} A_2^i v_0'(x)w_0''(x) + \sum_{i=2,4} B_2^i v_0''(x)w_0'(x) + 2 \sum_{i=1,3} A_{37}^i v_0''(x)\zeta_z(x) + \frac{4}{5} \sum_{i=1,3} A_{36}^i [\chi_0''(x)\zeta_z(x) + \theta_0''(x)\zeta_z(x)] \end{aligned} \right], \\
M_1 &= G \left[\begin{aligned} &\left(\sum_{i=1,3} A_5^i + \sum_{i=2,4} B_5^i \right) \theta_0(x)\theta_0'(x) - \left(\sum_{i=1,3} A_{13}^i + \sum_{i=2,4} B_{13}^i \right) \theta_0(x)\chi_0'(x) - 3 \left(\sum_{i=1,3} A_{51}^i + \sum_{i=2,4} B_{51}^i \right) \theta_0'(x)\chi_0(x) \\ &+ \left(\sum_{i=2,4} B_{76}^i - \sum_{i=1,3} A_{76}^i \right) \chi_0(x)\chi_0'(x) + \sum_{i=1,3} A_5^i v_0'(x)v_0''(x) + \sum_{i=2,4} B_5^i w_0'(x)w_0''(x) \\ &+ 2 \sum_{i=1,3} A_{37}^i w_0'(x)\zeta_z(x) + \sum_{i=1,3} A_{36}^i \zeta_z(x)\zeta_z'(x) + \sum_{i=1,3} A_6^i [v_0'(x)w_0''(x) + v_0''(x)\chi_0'(x)] \\ &+ \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) [\theta_0'(x)\theta_0''(x) + \chi_0'(x)\chi_0''(x)] + \left(\sum_{i=1,3} A_6^i + \sum_{i=2,4} B_6^i \right) w_0'(x)\chi_0''(x) \end{aligned} \right],
\end{aligned}$$

$$\begin{aligned}
M_2 = G & \left[\left(\sum_{i=1,3} A_{65}^i - \sum_{i=2,4} B_{65}^i \right) [\theta_0'(x) + u_0'(x)\theta_0'(x)] + \sum_{i=2,4} B_{13}^i w_0'(x)w_0''(x) + \sum_{i=2,4} B_{65}^i w_0'(x)v_0''(x) + \sum_{i=2,4} B_{79}^i w_0'(x)\chi_0''(x) \right. \\
& + \sum_{i=1,3} A_{13}^i v_0'(x)v_0''(x) + \sum_{i=1,3} A_{65}^i v_0'(x)w_0''(x) - \sum_{i=1,3} A_{73}^i v_0'(x)\zeta_z'(x) - \sum_{i=1,3} A_{79}^i v_0'(x)\chi_0''(x) + \sum_{i=1,3} A_{84}^i w_0'(x)\zeta_z'(x) \\
& + \sum_{i=1,3} A_{86}^i \zeta_z(x)\zeta_z'(x) - \sum_{i=1,3} A_{88}^i \chi_0''(x)\zeta_z(x) + \left(\frac{1}{2} \sum_{i=1,3} A_{12}^i + \sum_{i=2,4} B_{79}^i \right) w_0''(x)\chi_0'(x) \\
& + \left(\sum_{i=1,3} A_{79}^i + \frac{1}{2} \sum_{i=2,4} B_{12}^i \right) v_0''(x)\chi_0'(x) - \sum_{i=1,3} A_{39}^i \chi_0'(x)\zeta_z'(x) - \left(\sum_{i=1,3} A_{76}^i + \sum_{i=2,4} B_{76}^i \right) \theta_0'(x)\chi_0(x) \\
& + \left(\sum_{i=1,3} A_7^i - \sum_{i=1,3} A_{15}^i + \sum_{i=2,4} B_7^i + \sum_{i=2,4} B_{15}^i \right) [\theta_0'(x)\theta_0''(x) + \chi'(x)\chi''(x)] - \left(\sum_{i=1,3} A_{97}^i + \sum_{i=2,4} B_{97}^i \right) \theta_0(x)\chi'(x) \\
& + \left(\sum_{i=1,3} A_{94}^i - \sum_{i=1,3} A_{98}^i + \sum_{i=2,4} B_{94}^i + \sum_{i=2,4} B_{98}^i \right) \chi_0(x)\chi_0'(x) + \left(\sum_{i=1,3} A_{13}^i + \sum_{i=2,4} B_{13}^i \right) \theta_0(x)\theta_0'(x) \\
& \left. - \left(\sum_{i=1,3} A_{67}^i + \sum_{i=2,4} B_{67}^i \right) \theta_0(x) + \left(10 \sum_{i=1,3} A_{14}^i - 10 \sum_{i=2,4} B_{14}^i + \sum_{i=1,3} A_{67}^i - \sum_{i=2,4} B_{67}^i \right) [\theta_0'(x) + u_0'(x)\theta_0'(x)] \right] \\
M_3 = G & \left[- \left(\sum_{i=1,3} A_{78}^i + \sum_{i=2,4} B_{78}^i \right) \theta_0'(x) + \left(\sum_{i=1,3} A_{78}^i - \sum_{i=2,4} B_{78}^i + \sum_{i=1,3} A_{94}^i - \sum_{i=2,4} B_{94}^i \right) \chi_0'(x) + \left(\sum_{i=2,4} B_{76}^i - \sum_{i=1,3} A_{76}^i \right) \theta_0(x)\theta_0'(x) \right. \\
& + \left(\sum_{i=1,3} A_{93}^i + \sum_{i=2,4} B_{93}^i + 2 \sum_{i=1,3} A_{96}^i + 2 \sum_{i=2,4} B_{96}^i + \sum_{i=1,3} A_{31}^i + \sum_{i=2,4} B_{31}^i \right) \chi_0(x)\chi_0'(x) - \left(\sum_{i=1,3} A_{85}^i + \frac{7}{2} \sum_{i=1,3} A_{187}^i \right) \chi_0'(x)\zeta_z'(x) \\
& - \left(\sum_{i=1,3} A_{94}^i - \sum_{i=1,3} A_{98}^i + \sum_{i=2,4} B_{94}^i + \sum_{i=2,4} B_{98}^i \right) \theta_0(x)\chi_0'(x) + \frac{34}{91} \sum_{i=1,3} A_{83}^i \zeta_z(x)\zeta_z'(x) + \left(\sum_{i=1,3} A_{83}^i + \sum_{i=1,3} A_{85}^i \right) w_0''(x)\zeta_z(x) \\
& - \left(\sum_{i=1,3} A_{95}^i + \sum_{i=1,3} A_{100}^i + \sum_{i=2,4} B_{95}^i - \sum_{i=2,4} B_{100}^i \right) \theta_0'(x)\chi_0(x) - \frac{21}{2} \sum_{i=1,3} A_{87}^i [\theta_0''(x)\zeta_z(x) - \chi_0''(x)\zeta_z(x)] \\
& + \left(\sum_{i=2,4} B_{68}^i - \sum_{i=2,4} B_{76}^i \right) w_0'(x)w_0''(x) + \left(\sum_{i=2,4} B_{68}^i - \sum_{i=2,4} B_{77}^i \right) w_0'(x)\chi_0''(x) + \left(\sum_{i=1,3} A_{68}^i + \sum_{i=1,3} A_{76}^i \right) v_0'(x)v_0''(x) \\
& + \left(\sum_{i=1,3} A_{67}^i + \sum_{i=1,3} A_{77}^i \right) v_0'(x)w_0''(x) - \left(\sum_{i=1,3} A_{72}^i + \sum_{i=1,3} A_{75}^i \right) v_0'(x)\zeta_z'(x) + \left(\sum_{i=1,3} A_{69}^i + \sum_{i=1,3} A_{77}^i \right) v_0'(x)\chi_0''(x) \\
& + \left(\sum_{i=1,3} A_{89}^i + \sum_{i=1,3} A_{77}^i + \sum_{i=2,4} B_{69}^i + 10 \sum_{i=2,4} B_{15}^i \right) w_0''(x)\chi_0'(x) + \left(\sum_{i=1,3} A_{90}^i + \sum_{i=2,4} B_{70}^i \right) v_0''(x)\chi_0'(x) \\
& + \left(\sum_{i=1,3} A_{69}^i + \sum_{i=2,4} B_{69}^i + \sum_{i=1,3} A_{95}^i + \sum_{i=2,4} B_{95}^i \right) [\theta_0'(x)\theta_0''(x) + \chi_0'(x)\chi_0''(x)] + \sum_{i=2,4} B_{78}^i w_0'(x)v_0''(x) \\
& \left. \right]
\end{aligned}$$

$$\begin{aligned}
M_4 = G & \left[\begin{aligned} & - \left(\sum_{i=1,3} A_{92}^i + \sum_{i=2,4} B_{92}^i + \sum_{i=1,3} A_{76}^i \sum_{i=2,4} B_{76}^i \right) \theta_0(x) \chi_0'(x) + 2 \left(\sum_{i=1,3} A_{11}^i - \sum_{i=2,4} B_{11}^i \right) [\theta_0'(x) + u_0'(x) \theta_0'(x)] \\ & + 3 \sum_{i=2,4} B_{13}^i w_0'(x) w_0''(x) + 2 \sum_{i=2,4} B_{11}^i w_0'(x) v_0''(x) + 3 \sum_{i=2,4} B_{79}^i [w_0'(x) \chi_0''(x) - v_0'(x) \chi_0''(x)] \\ & + 3 \sum_{i=1,3} A_{13}^i v_0'(x) v_0''(x) + 2 \sum_{i=1,3} A_{11}^i v_0'(x) w_0''(x) + 2 \sum_{i=1,3} A_{73}^i [3 v_0''(x) \zeta_z(x) - v_0'(x) \zeta_z''(x)] + 3 \sum_{i=1,3} A_{84}^i w_0''(x) \zeta_z(x) \\ & + 6 \sum_{i=1,3} A_{88}^i [3 \chi_0''(x) \zeta_z(x) - \chi_0'(x) \zeta_z(x)] + \left(\sum_{i=1,3} A_{12}^i + 3 \sum_{i=1,3} A_{13}^i \right) w_0''(x) \chi_0'(x) + \sum_{i=1,3} A_{62}^i \zeta_z(x) \zeta_z'(x) \\ & + 3 \left(\sum_{i=1,3} A_{15}^i + \sum_{i=2,4} B_{15}^i \right) [\theta_0'(x) \theta_0''(x) + \chi'(x) \chi''(x)] + \left(3 \sum_{i=1,3} A_{13}^i + \sum_{i=2,4} B_{12}^i \right) v_0''(x) \chi_0'(x) \\ & + \left(\sum_{i=1,3} A_{95}^i + \sum_{i=1,3} A_{100}^i + \sum_{i=2,4} B_{95}^i + \sum_{i=2,4} B_{100}^i \right) \chi_0(x) \chi_0'(x) + \left(\sum_{i=2,4} B_{76}^i + \sum_{i=1,3} A_{76}^i \right) \theta_0(x) \theta_0'(x) \\ & - \left(\sum_{i=1,3} A_{99}^i + \sum_{i=2,4} B_{99}^i \right) \theta_0'(x) \chi_0(x) - 2 \left(\sum_{i=1,3} A_{11}^i + \sum_{i=2,4} B_{11}^i \right) [\theta_0'(x) - \chi_0'(x)] \end{aligned} \right], \\
M_{\omega_0' \omega_0' \zeta_z} = G & \left[2 \sum_{i=1,3} A_{36}^i \theta_0'(x) \zeta_z'(x) - \sum_{i=1,3} A_{37}^i [2 v_0'(x) + w_0'(x) \theta_0'(x)] - 5 \sum_{i=1,3} A_{82}^i [w_0''(x) \theta_0'(x) + w_0''(x) \chi_0'(x)] - 6 \sum_{i=1,3} A_{88}^i w_0'(x) \chi_0(x) \right], \\
M_{\omega_0' \omega_0' \zeta_z} = G & \left[\begin{aligned} & \sum_{i=1,3} A_{38}^i v_0'(x) \zeta_z'(x) + 2 \sum_{i=1,3} A_{36}^i \chi_0'(x) \zeta_z'(x) + \sum_{i=1,3} A_{39}^i \theta_0(x) \chi_0'(x) - 6 \sum_{i=1,3} A_{88}^i \theta_0'(x) \chi_0(x) \\ & \sum_{i=1,3} A_{82}^i [w_0''(x) \zeta_z'(x) - \chi_0''(x) \zeta_z'(x) + 5 w_0''(x) \chi_0'(x)] + \left(\frac{7}{2} \sum_{i=1,3} A_{87}^i - \sum_{i=1,3} A_{85}^i \right) \chi_0(x) \chi_0'(x) \end{aligned} \right], \\
M_5 = G & \left[\begin{aligned} & \left(\sum_{i=1,3} A_{69}^i + \sum_{i=1,3} A_{95}^i + \sum_{i=2,4} B_{69}^i + \sum_{i=2,4} B_{95}^i \right) \chi_0(x) \chi_0'(x) + \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) \theta_0(x) \theta_0'(x) \\ & - \left(\sum_{i=1,3} A_{15}^i + \sum_{i=2,4} B_{15}^i \right) [\theta_0(x) \chi_0'(x) + 3 \theta_0'(x) \chi_0(x)] + \sum_{i=1,3} A_6^i v_0'(x) v_0''(x) + \sum_{i=1,3} A_7^i w_0'(x) w_0''(x) \\ & + \sum_{i=2,4} B_8^i [w_0'(x) \chi_0''(x) + w_0''(x) \chi_0'(x)] + \sum_{i=1,3} A_{82}^i \left[\frac{5}{2} w_0''(x) \zeta_z(x) - \zeta_z(x) \zeta_z'(x) \right] \\ & + \left(\sum_{i=1,3} A_9^i + \sum_{i=2,4} B_9^i \right) [\theta_0'(x) \theta_0''(x) + \chi'(x) \chi''(x)] \end{aligned} \right], \\
M_6 = G & \left[\begin{aligned} & \sum_{i=1,3} A_7^i w_0'(x) w_0''(x) + \sum_{i=2,4} B_8^i w_0'(x) \chi''(x) + \sum_{i=1,3} A_6^i v_0'(x) v_0''(x) + \sum_{i=1,3} A_3^i v_0'(x) \chi''(x) \\ & + \sum_{i=1,3} A_{82}^i \left[\frac{5}{2} w_0''(x) \zeta_z(x) - \zeta_z(x) \zeta_z'(x) \right] + \sum_{i=2,4} B_8^i [w_0''(x) \chi_0'(x) - v_0''(x) \chi_0'(x)] \\ & + \left(\sum_{i=1,3} A_9^i + \sum_{i=2,4} B_9^i \right) [\theta_0'(x) \theta_0''(x) + \chi'(x) \chi''(x)] + \left(\sum_{i=1,3} A_{69}^i + \sum_{i=1,3} A_{95}^i + \sum_{i=2,4} B_{69}^i + \sum_{i=2,4} B_{95}^i \right) \chi_0(x) \chi_0'(x) \\ & + \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) \theta_0(x) \theta_0'(x) - \left(\sum_{i=1,3} A_{15}^i + \sum_{i=2,4} B_{15}^i \right) [\theta_0(x) \chi_0'(x) + 3 \theta_0'(x) \chi_0(x)] \end{aligned} \right], \\
M_7 = G & \left[\frac{4}{5} \sum_{i=1,3} A_{36}^i [\zeta_z(x) + u_0'(x) \zeta_z(x)] + 2 \sum_{i=2,4} B_8^i w_0'(x) \theta_0''(x) + \sum_{i=1,3} A_3^i v_0'(x) \theta_0''(x) - \sum_{i=1,3} A_{12}^i w_0'(x) \chi_0(x) + \sum_{i=2,4} B_{12}^i v_0'(x) \chi_0(x) \right], \\
M_8 = G & \left[\begin{aligned} & \frac{4}{5} \sum_{i=1,3} A_{36}^i [\zeta_z(x) + u_0'(x) \zeta_z(x)] + \sum_{i=1,3} A_3^i v_0'(x) \theta_0''(x) - \sum_{i=1,3} A_{12}^i w_0'(x) \chi_0(x) + \sum_{i=2,4} B_{12}^i v_0'(x) \chi_0(x) \\ & + \sum_{i=2,4} B_8^i [v_0''(x) \theta_0'(x) + 2 w_0'(x) \theta_0''(x)] - 3 \left(\sum_{i=1,3} A_{15}^i + \sum_{i=2,4} B_{15}^i \right) \theta_0'(x) \chi_0(x) + \left(\sum_{i=1,3} A_{95}^i + \sum_{i=2,4} B_{95}^i \right) \theta_0(x) \chi_0'(x) \end{aligned} \right], \tag{A.1}
\end{aligned}$$

in which A_i ($i = 1, \dots, 100$) and B_i ($i = 1, \dots, 100$) are parameters of beam cross section constants (b, h, t_i). B_i functions are calculated in the same way as A_i , except that b is replaced by h .

$$A_F^i = bt_i,$$

$$A_1^i = \left(\frac{h}{2} - z_0\right) bt_i,$$

$$A_2^i = \left(\frac{h}{2} - z_0\right)^2 bt_i,$$

$$A_3^i = \left(\frac{h}{2} - z_0\right)^3 bt_i,$$

$$A_4^i = \left(\frac{h}{2} - z_0\right)^4 bt_i,$$

$$A_5^i = \frac{b^3 t_i}{12},$$

$$A_6^i = \left(\frac{h}{2} - z_0\right) \frac{b^3 t_i}{12},$$

$$A_7^i = \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i}{12},$$

$$A_8^i = \left(\frac{h}{2} - z_0\right)^3 \frac{b^3 t_i}{12},$$

$$A_9^i = \left(\frac{h}{2} - z_0\right)^4 \frac{b^3 t_i}{12},$$

$$A_{10}^i = \frac{1}{2} \frac{b^3 t_i}{(b+h)},$$

$$A_{11}^i = \frac{1}{2} \left(\frac{h}{2} - z_0\right) \frac{b^3 t_i}{(b+h)},$$

$$A_{12}^i = \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i}{2(b+h)},$$

$$A_{13}^i = \frac{1}{20} \frac{h^4 t_i}{(b+h)},$$

$$A_{14}^i = \left(\frac{h}{2} - z_0\right) \frac{b^4 t_i}{20(b+h)},$$

$$A_{15}^i = -\left(\frac{h}{2} - z_0\right)^2 \frac{b^4 t_i}{20(b+h)},$$

$$A_{16}^i = \frac{b^5 t_i}{60(b+h)},$$

$$\begin{aligned} A_{17}^i &= \frac{17}{35} \left(\frac{h}{2} - z_0\right)^2 \frac{b^5 t_i}{(b+h)^2} + \left(\frac{3}{140} - \frac{17}{11 \cdot 2^3}\right) \frac{b^8 t_i}{(b+h)^3} \\ &\quad + \left(\frac{h}{2} - z_0\right)^4 bt_i - \left(\frac{h}{2} - z_0\right)^2 \frac{23}{20} \frac{b^3 t_i^3}{(b+h)^2} \\ &\quad - \left(\frac{h}{2} - z_0\right)^2 \frac{b^4 t_i}{20} bt_i - \frac{251}{15 \cdot 7 \cdot 2^5} \frac{b^5 t_i^3}{(b+h)^2}, \end{aligned}$$

$$A_{18}^i = \left(\frac{h}{2} - z_0\right)^2 \frac{b^5 t_i}{60(b+h)},$$

$$A_{19}^i = \left(\frac{h}{2} - z_0\right)^2 bt_i + \frac{23}{20} \frac{b^3 t_i^3}{(b+h)^2},$$

$$A_{20}^i = \frac{17}{35} \frac{b^3 t_i}{(b+h)^2},$$

$$A_{21}^i = \left(\frac{h}{2} - z_0\right)^2 bt_i + \frac{1}{2} \frac{b^4 t_i}{(b+h)},$$

$$A_{22}^i = \left(\frac{h}{2} - z_0\right) \left[\left(\frac{h}{2} - z_0\right)^2 bt_i + \frac{b^3 t_i}{12} \right],$$

$$A_{23}^i = \left(\frac{h}{2} - z_0\right) \left[\left(\frac{h}{2} - z_0\right)^2 bt_i + \frac{23}{20} \frac{b^3 t_i^3}{(b+h)^2} + \frac{17}{35} \frac{b^5 t_i}{(b+h)^2} \right],$$

$$A_{24}^i = \left(\frac{h}{2} - z_0\right) \left[\left(\frac{h}{2} - z_0\right)^2 bt_i + \frac{23}{20} \frac{b^3 t_i^3}{(b+h)^2} + \frac{1}{20} \frac{b^4 t_i}{(b+h)} \right],$$

$$A_{25}^i = \frac{1}{2} \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i}{(b+h)} + \frac{1}{120} \frac{b^5 t_i}{(b+h)},$$

$$A_{26}^i = \frac{1}{2} \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i}{(b+h)} + \frac{59}{280} \frac{b^5 t_i^3}{(b+h)^3} - \frac{27}{10} \frac{b^3 t_i}{(b+h)^3},$$

$$A_{27}^i = \frac{1}{2} \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i}{(b+h)} + \frac{1}{140} \frac{b^6 t_i}{(b+h)},$$

$$A_{28}^i = \left(\frac{h}{2} - z_0\right)^4 \frac{b^5 t_i}{80} + \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i}{6},$$

$$\begin{aligned} A_{29}^i &= \left(\frac{h}{2} - z_0\right)^4 bt_i + \frac{23}{20} \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i^3}{(b+h)^2} \\ &\quad + \frac{17}{35} \left(\frac{h}{2} - z_0\right)^2 \frac{b^5 t_i}{(b+h)^2} + \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i}{12} \\ &\quad + \frac{99}{560} \frac{b^5 t_i^3}{(b+h)^2} - \frac{71}{280} \frac{b^7 t_i}{(b+h)^2}, \end{aligned}$$

$$\begin{aligned} A_{30}^i &= \left(\frac{h}{2} - z_0\right)^4 bt_i + \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i}{12} - \frac{1}{20} \left(\frac{h}{2} - z_0\right)^2 \frac{b^4 t_i}{b+h} \\ &\quad - \frac{1}{7 \cdot 2^4} \frac{b^6 t_i}{b+h}, \end{aligned}$$

$$A_{31}^i = \frac{8557}{35 \cdot 11} \frac{b^9 t_i}{(b+h)^4},$$

$$A_{32}^i = \frac{b^5 t_i}{80},$$

$$A_{33}^i = \left(\frac{h}{2} - z_0\right) \frac{b^5 t_i}{80},$$

$$A_{34}^i = \left(\frac{h}{2} - z_0\right) \frac{b^5 t_i}{80},$$

$$A_{35}^i = \left(\frac{h}{2} - z_0\right) \frac{b^5 t_i}{80},$$

$$A_{36}^i = \frac{1}{15} h^2 b t_i,$$

$$A_{37}^i = \frac{1}{6} b h t_i,$$

$$A_{38}^i = \frac{1}{30} h^3 b t_i,$$

$$A_{39}^i = \frac{1}{280} \frac{h b^4 t_i}{(b+h)},$$

$$A_{40}^i = \left(\frac{h}{2} - z_0\right) \left[\left(\frac{h}{2} - z_0\right) \frac{b^3 t_i}{12} + \frac{b^5 t_i}{80} \right],$$

$$A_{41}^i = \left(\frac{h}{2} - z_0\right) \left(-\frac{1}{140} \frac{b^6 t_i}{(b+h)^2} + \frac{3}{10} \frac{b^4 t_i^3}{(b+h)^2} \right),$$

$$A_{42}^i = \left(\frac{h}{2} - z_0\right)^4 b t_i + \frac{36^2 b^5 t_i^5}{5(b+h)^4} \left(\frac{1}{5 \cdot 2^4} + \frac{1}{9 \cdot 2^8} + \frac{3}{7 \cdot 2^5} \right) \\ + \frac{b^9 t_i}{(b+h)^4} \left(\frac{1}{2^4} + \frac{3}{13 \cdot 2^5} - \frac{1}{70} + \frac{27}{11 \cdot 2^3} \right) \\ + \frac{34}{35} \left(\frac{h}{2} - z_0\right)^2 \frac{b^5 t_i}{(b+h)^4} + \frac{b^7 t_i^3}{(b+h)^4} \left(\frac{61}{140} + \frac{199}{3 \cdot 11 \cdot 2^3} \right),$$

$$A_{43}^i = \left(\frac{h}{2} - z_0\right)^2 \left[\left(\frac{h}{2} - z_0\right)^2 b t_i + \frac{23}{20} \frac{b^3 t_i^3}{(b+h)^2} + \frac{17}{35} \frac{b^3 t_i}{(b+h)^2} \right],$$

$$A_{44}^i = \frac{b^7 t_i}{(b+h)^2} \left(\frac{1}{12} + \frac{9}{7 \cdot 2^4} + \frac{1}{9 \cdot 2^4} - \frac{3}{20} \right) + \left(\frac{h}{2} - z_0\right)^3 \frac{b^3 t_i}{12} \\ + \frac{b^5 t_i^3}{(b+h)^2} \left(\frac{3}{7 \cdot 2^4} + \frac{3}{20} \right),$$

$$A_{45}^i = \frac{b^3 h^2 t_i}{210},$$

$$A_{46}^i = \frac{b h^3 t_i}{12} + \frac{b^3 h^2 t_i}{12},$$

$$A_{47}^i = \frac{949}{140 \cdot 9 \cdot 11} \frac{h^2 b^5 t_i}{(b+h)^2} + \left(\frac{h}{2} - z_0\right)^2 \frac{2 h^2 b t_i}{15} + \frac{h^2 b^3 t_i^3}{140 \cdot 3},$$

$$A_{48}^i = \left(\frac{h}{2}\right)^3 \frac{b t_i}{2} + \frac{b h^3 t_i}{48} + \frac{127}{1260} \frac{h b^5 t_i}{(b+h)^2} - \frac{31}{280} \frac{h b^3 t_i^3}{(b+h)^2},$$

$$A_{49}^i = \frac{1}{840} \frac{h b^6 t_i}{(b+h)^2} - \frac{3}{70} \frac{h b^4 t_i^3}{(b+h)^2},$$

$$A_{50}^i = \left(\frac{h}{2} - z_0\right)^2 \frac{2 h^2 b t_i}{15} + \frac{1}{630} \frac{h^2 b^4 t_i}{(b+h)},$$

$$A_{51}^i = \frac{h^4 b t_i}{24} + \frac{h b^4 t_i}{140},$$

$$A_{52}^i = \left(\frac{h}{2} - z_0\right) \left[\left(\frac{h}{2} - z_0\right) \frac{b^3 t_i}{12} - \frac{37}{630} \frac{b^7 t_i}{(b+h)^2} + \frac{33}{560} \frac{b^3 t_i^3}{(b+h)^2} \right] +,$$

$$A_{53}^i = \left(\frac{h}{2} - z_0\right)^2 \left(-\frac{1}{140} \frac{b^4 t_i}{(b+h)^2} + \frac{3}{10} \frac{b^3 t_i^3}{(b+h)^2} \right),$$

$$A_{54}^i = \left(\frac{h}{2} - z_0\right)^4 b t_i + \frac{23}{20} \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i^3}{(b+h)^2} - \frac{1}{10} \frac{b^4 t_i}{(b+h)} \left(\frac{h}{2} - z_0\right)^2 \\ + \frac{13}{40 \cdot 63} \frac{b^7 t_i}{(b+h)^2},$$

$$A_{55}^i = \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i}{12} - \frac{1}{7 \cdot 2^4} \frac{b^7 t_i}{(b+h)},$$

$$A_{56}^i = \left(\frac{h}{2} - z_0\right)^2 \left[\left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i}{12} + \frac{1}{7 \cdot 2^4} \frac{b^6 t_i}{b+h} \right],$$

$$A_{57}^i = \left(\frac{h}{2} - z_0\right) \left[\left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i}{12} + \frac{1}{7 \cdot 2^4} \frac{b^6 t_i}{b+h} \right],$$

$$A_{58}^i = \frac{h b^5 t_i}{420},$$

$$A_{59}^i = \frac{h^2 b^5 t_i}{840},$$

$$A_{60}^i = \frac{b^3 h^3 t_i}{240},$$

$$A_{61}^i = -\frac{89 b^3 h^2 t_i}{420(b+h)},$$

$$A_{62}^i = \frac{b^3 h^4 t_i}{840},$$

$$A_{63}^i = \frac{h^2 t_i}{3b},$$

$$A_{64}^i = \frac{1}{2} \frac{b^2 t_i}{(b+h)},$$

$$A_{65}^i = \frac{1}{2} \left(\frac{h}{2} - z_0\right) \frac{b^2 t_i}{(b+h)},$$

$$A_{66}^i = -\frac{1}{2} \left(\frac{h}{2} - z_0\right) \frac{b^2 t_i}{(b+h)},$$

$$A_{67}^i = \frac{1}{36} \left(\frac{h}{2} - z_0\right) \frac{b^3 t_i^3}{(b+h)},$$

$$A_{68}^i = \frac{13}{10} \frac{b^3 t_i^3}{(b+h)^2},$$

$$A_{69}^i = \frac{13}{10} \left(\frac{h}{2} - z_0\right) \frac{b^3 t_i^3}{(b+h)^2},$$

$$A_{70}^i = \frac{13}{10} \left(\frac{h}{2} - z_0\right)^2 \frac{b^3 t_i^3}{(b+h)^2},$$

$$\begin{aligned}
A_{71}^i &= \frac{4bht_i}{b+h}, & A_{87}^i &= \frac{3}{35} \frac{h^2b^2t_i}{(b+h)^2}, \\
A_{72}^i &= \frac{3}{5} \frac{hb^2t_i^3}{(b+h)^2}, & A_{88}^i &= \frac{1}{60} \frac{h^2b^3t_i}{(b+h)}, \\
A_{73}^i &= \frac{1}{10} \frac{hb^3t_i}{(b+h)}, & A_{89}^i &= -\frac{34}{35} \left(\frac{h}{2} - z_0\right)^2 \frac{h^2bt_i^3}{(b+h)^2}, \\
A_{74}^i &= \frac{1}{5} \frac{hb^2t_i}{(b+h)}, & A_{90}^i &= \left(\frac{h}{2} - z_0\right) \frac{9b^4t_i^3}{20(b+h)}, \\
A_{75}^i &= \frac{38}{35} \frac{hb^5t_i}{(b+h)^2}, & A_{91}^i &= \frac{16bt_i^3}{(b+h)^2}, \\
A_{76}^i &= \frac{1}{140} \frac{b^6t_i}{(b+h)^2}, & A_{92}^i &= \left(\frac{h}{2} - z_0\right)^2 \frac{16bt_i^3}{(b+h)^2}, \\
A_{77}^i &= \frac{1}{140} \left(\frac{h}{2} - z_0\right) \frac{b^6t_i}{(b+h)^2}, & A_{93}^i &= \frac{27 \cdot 438}{245} \frac{h^3t_i^5}{(b+h)^2}, \\
A_{78}^i &= -\left(\frac{h}{2} - z_0\right) \frac{b^5t_i}{2(b+h)^2}, & A_{94}^i &= \left(\frac{h}{2} - z_0\right) \frac{6b^2t_i^3}{(b+h)^2}, \\
A_{79}^i &= \frac{1}{20} \left(\frac{h}{2} - z_0\right) \frac{h^4t_i}{(b+h)}, & A_{95}^i &= \left(\frac{h}{2} - z_0\right)^2 \frac{b^6t_i}{140(b+h)^2}, \\
A_{80}^i &= \frac{h^4t_i}{3b}, & A_{96}^i &= \frac{53 \cdot 6}{35} \frac{b^6t_i^3}{(b+h)^4}, \\
A_{81}^i &= \frac{2h^4t_i}{15}, & A_{97}^i &= \frac{17}{35} \frac{b^5t_i^3}{(b+h)^2}, \\
A_{82}^i &= \frac{h^4bt_i}{60}, & A_{98}^i &= \frac{5}{3} \frac{b^7t_i}{(b+h)^3}, \\
A_{83}^i &= \frac{13}{5} \frac{h^2bt_i^3}{(b+h)^2}, & A_{99}^i &= \frac{33.9}{140} \frac{b^5t_i}{(b+h)^2}, \\
A_{84}^i &= \frac{1}{10} \frac{h^2b^2t_i}{(b+h)}, & A_{100}^i &= \frac{251}{140} \frac{b^7t_i}{(b+h)^3}, \\
A_{85}^i &= \frac{1}{70} \frac{h^2b^4t_i}{(b+h)^2}, \\
A_{86}^i &= \frac{1}{70} \frac{h^2b^2t_i}{(b+h)},
\end{aligned} \tag{A.2}$$

B. Algebraic System

$$P_1 = -\sum_{i=4} F_i,$$

$$P_2 = -\frac{8\pi}{3L^2} \sum_{i=1,3} A_{37}^i,$$

$$P_3 = \frac{3\pi^4}{4L^4} \left(\sum_{i=1,3} A_2^i + \sum_{i=2,4} B_5^i \right) - \frac{3\pi^2}{4L^2} \sum_{i=4} F_i,$$

$$P_4 = \frac{\pi^2}{L^2} \left(\sum_{i=1,3} A_5^i + \sum_{i=2,4} B_2^i \right),$$

$$P_5 = -\frac{\pi^2}{L^2} \sum_{i=2,4} A_6^i,$$

$$P_6 = \frac{4\pi}{3L^2} \left(4 \sum_{i=2,4} B_2^i + 4 \sum_{i=1,3} A_5^i + \sum_{i=1,3} A_2^i + \sum_{i=2,4} B_5^i \right) - \frac{8\pi^3}{3L^4} \left(\sum_{i=1,3} A_7^i - \sum_{i=2,4} B_7^i - \sum_{i=1,3} A_6^i + \sum_{i=2,4} B_6^i \right),$$

$$P_7 = \frac{4\pi}{8L^2} \left(\sum_{i=1,3} A_2^i - \sum_{i=2,4} B_5^i - 4 \sum_{i=1,3} A_{11}^i - 4 \sum_{i=2,4} B_2^i \right),$$

$$P_8 = \frac{\pi^4}{8L^4} \left(\sum_{i=1,3} A_2^i + \sum_{i=2,4} B_5^i \right) - \frac{3\pi^6}{8L^6} \left(2 \sum_{i=1,3} A_4^i + 2 \sum_{i=2,4} B_{32}^i + \sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right),$$

$$P_9 = \frac{\pi^2}{L^2} \left(\sum_{i=1,3} A_2^i + \sum_{i=2,4} B_5^i \right),$$

$$P_{10} = \frac{\pi^2}{L^2} \sum_{i=2,4} B_6^i,$$

$$P_{11} = -\frac{20\pi}{3L^2} \sum_{i=1,3} A_{11}^i,$$

$$P_{12} = \frac{\pi^4}{8L^4} \left(\sum_{i=1,3} A_5^i + \sum_{i=2,4} B_2^i \right) + \frac{3\pi^6}{8L^6} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right),$$

$$P_{13} = \frac{E\pi^2}{L^2} \left(\sum_{i=1,3} A_5^i - \sum_{i=2,4} B_7^i \right),$$

$$P_{14} = G \sum_{i=1,3} A_2^i,$$

$$P_{15} = \frac{4E\pi}{3L^2} \left(\sum_{i=1,3} A_2^i + \sum_{i=2,4} B_5^i \right) + \frac{8E\pi^3}{3L^4} \left(\sum_{i=2,4} B_6^i - \sum_{i=1,3} A_6^i \right) - \frac{8G\pi}{3L^2} \sum_{i=1,3} A_2^i,$$

$$\begin{aligned}
P_{16} &= \frac{8E\pi^3}{3L^4} \left(\sum_{i=2,4} B_8^i - \sum_{i=1,3} A_8^i \right) + \frac{4G\pi}{3L^2} \sum_{i=1,3} A_3^i, \\
P_{17} &= \frac{8E\pi^3}{3L^4} \left(\sum_{i=2,4} B_7^i - \sum_{i=1,3} A_7^i \right), \\
P_{18} &= \frac{3E\pi^2}{3L^2} \left(\sum_{i=1,3} A_2^i + \sum_{i=1,3} A_5^i + \sum_{i=2,4} B_2^i + \sum_{i=2,4} B_5^i \right) + \frac{2E\pi}{3L^2} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_5^i + \sum_{i=2,4} A_4^i + \sum_{i=2,4} B_{32}^i \right) \\
&\quad + \frac{E\pi^4}{8L^4} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) + \frac{3E\pi^6}{8L^6} \left(\sum_{i=1,3} A_9^i + \sum_{i=2,4} B_{33}^i \right) - \frac{G\pi^2}{2L^2} \sum_{i=2,4} B_5^i + \frac{G\pi^4}{2L^4} \sum_{i=1,3} A_7^i, \\
P_{19} &= \frac{3E\pi^2}{8L^2} \left(\sum_{i=1,3} A_2^i + \sum_{i=2,4} B_{21}^i \right) - \frac{E\pi^4}{4L^4} \left(\sum_{i=1,3} A_4^i - \sum_{i=1,3} A_{15}^i + \sum_{i=2,4} B_{55}^i \right), \\
P_{20} &= -2G \sum_{i=2,4} B_1^i, \\
P_{21} &= -2G \sum_{i=1,3} A_1^i, \\
P_{22} &= -G \left(\sum_{i=1,3} A_1^i - \sum_{i=2,4} B_2^i \right), \\
P_{23} &= -\frac{E\pi^2}{L^2} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right) - G \left(\sum_{i=1,3} A_{19}^i - \sum_{i=2,4} B_{19}^i \right), \\
P_{24} &= \frac{4G\pi}{3L^2} \left(\sum_{i=1,3} A_2^i + \sum_{i=2,4} B_2^i \right), \\
P_{25} &= -\frac{16E\pi}{3L^2} \sum_{i=1,3} A_6^i + \frac{8E\pi^3}{4L^4} \left(\sum_{i=2,4} B_8^i - \sum_{i=1,3} A_8^i \right) - \frac{4G}{3\pi} \sum_{i=1,3} A_1^i + \frac{4G\pi}{3L^2} \sum_{i=1,3} A_3^i, \\
P_{26} &= \frac{4E\pi}{3L^2} \sum_{i=1,3} A_{14}^i + \frac{4G}{3\pi} \sum_{i=1,3} A_{66}^i, \\
P_{27} &= \frac{4E\pi}{3L^2} \sum_{i=1,3} A_{11}^i w_0(x)^2, \\
P_{28} &= \frac{3E\pi^2}{8L^2} \left(\sum_{i=1,3} A_{21}^i + \sum_{i=2,4} B_{21}^i \right) + \frac{E\pi^4}{4L^4} \left(\sum_{i=1,3} A_4^i - 2 \sum_{i=2,4} B_7^i - \sum_{i=1,3} A_{15}^i + \sum_{i=2,4} B_{2,4}^i \right) + \frac{G\pi^2}{L^2} \sum_{i=2,4} B_{13}^i, \\
P_{29} &= -\frac{3E\pi^2}{8L^2} \left(\sum_{i=1,3} A_{19}^i + \sum_{i=2,4} B_{19}^i + \sum_{i=1,3} A_{20}^i + \sum_{i=2,4} B_{20}^i \right) + \frac{3E\pi^6}{8L^6} \left(\sum_{i=1,3} A_9^i + \sum_{i=2,4} B_{33}^i \right) \frac{E\pi^4}{8L^4} \left(\sum_{i=1,3} A_7^i + \sum_{i=2,4} B_7^i \right. \\
&\quad \left. + 2 \sum_{i=1,3} A_{43}^i + 2 \sum_{i=2,4} B_{44}^i \right) - \frac{G\pi^2}{2L^2} \left(\sum_{i=2,4} B_{68}^i + \sum_{i=2,4} B_{76}^i \right) + \frac{3G\pi^4}{4L^4} \sum_{i=1,3} A_7^i, \\
P_{30} &= \frac{GL^2}{\pi^2} \sum_{i=1,3} A_F^i.
\end{aligned}$$

(B.1)

Data Availability

All data and models generated or used during the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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