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Research Article

A Stochastic Differential Equation Driven by Poisson Random Measure and Its Application in a Duopoly Market

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We investigate a stochastic differential equation driven by Poisson random measure and its application in a duopoly market for a finite number of consumers with two unknown preferences. The scopes of pricing for two monopolistic vendors are illustrated when the prices of items are determined by the number of buyers in the market. The quantity of buyers is proved to obey a stochastic differential equation (SDE) driven by Poisson random measure which exists a unique solution. We derive the Hamilton-Jacobi-Bellman (HJB) about vendors' profits and provide a verification theorem about the problem. When all consumers believe a vendor's guidance about their preferences, the conditions that the other vendor's profit is zero are obtained. We give an example of this problem and acquire approximate solutions about the profits of the two vendors.

1. Introduction

Consider two vendors that provide different goods for different types of consumers in a duopoly market. For instance, in the pharmaceutical market, two vendors provide different drugs for different patients with different diseases. As the commodity price is proportional to the number of consumers, pricing strategies are especially important for vendors. We study this problem in a nondurable goods duopoly market.

Consumers' preferences or types take key roles in the market and affect the pricing strategies determined by vendors. Eeckhout and Weng [1] assume that there are $N \geq 2$ consumers who have the same type either H or L. Two vendors provide two different kinds of goods for the two types, respectively. In this article, we assume that consumers' types are diverse. This assumption is different from the assumption that all consumers' types are identical in [1]. In general, there exist different types of consumers who need to buy the same kind of goods in the market, just like people with high and low fever who need antipyretics simultaneously. Thus, it is reasonable to assume that all consumers' preferences are different in the market. In this situation, some consumers choose one vendor and others choose the other vendor, i.e., all consumers do not choose the same vendor at the first time.

Furthermore, we assume that the price of goods is a function of the number of consumers instead of the consumers' posterior beliefs in [1] as the price of goods is affected by the quantity of supply and demand. Since technological content of nondurable goods is lower than that of other goods in general, prices of nondurable goods fluctuate more obviously with the number of consumers. Therefore, the assumption that price of goods depends on the number of consumers is logical.

Vendors need to know how many consumers choose their own goods at each time $t \in [0, +\infty)$ in order to implement pricing strategies. Whether a consumer changes his decision is related to the judgment of his type and the prices of two goods. We find ranges of commodity prices and prove that the quantity of buyers who purchase one of the vendor's goods obeys a SDE and verifies existence and uniqueness of its solutions, which is a main contribution in this article.

Generally speaking, consumers do not know their own types whereas the vendors recognize. Under the setting of asymmetric information, consumers who are informed of their own types tend to buy commodity, which brings benefits for vendors. Consumers need to make choices of the types based on the guidance given by the two vendors. In certain cases, consumers are willing to follow the authoritative vendor's guidance. We call this vendor as the type-leader vendor and the other as the following vendor, which is a new setting evolved from the Stackelberg leadership model [2]. The type-leader model contains only one type-leader vendor. In the type-leader model, the vendor directly leads to consumers' preferences rather than the prices of goods. In some markets, such as the pharmaceutical market, consumers are more concerned about the efficacy of goods than its price. This makes it important for vendors to guide consumers' types. Thus, the type-leader model is more reasonable than the Stackelberg leadership model. Pricing rules for goods are obtained in the type-leader model. Moreover, we derive the conditions to ensure that the following vendor's profit is zero.

The main contributions of this paper are mentioned as follows. Compared with the assumption in [1] where there exists only one type of consumers, the situation that there exist two different types of consumers in the market is explored. Considering the impact of commodity prices, we assume that the price of goods is decided by the number of consumers. As a comparison, Eeckhout and Weng [1] define that the price is a function of consumers' posterior beliefs. Observing that the effectiveness of goods is more important than its price in our model, we use the type-leader model instead of the Stackelberg leadership model to study vendors' optimal strategies. An example of this problem is given and approximate solutions about the profit of the two vendors are acquired.

Several literatures investigate multiarmed bandit problems. Robbins [3] describes the problem as a decision-maker facing M slot machines (called arms), and the participator has to choose one of the arms at each instantaneous time. The value of pulling an arm in discrete time is calculated by Gittins and Jones [4] and Michael et al. [5]. Comparing the value to the Gittins index of all other arms, Michael et al. [5] present that the value pulling each arm itself does not depend on the cutoff. This problem is transformed into a standard optimal stopping problem in [6, 7]. Bolton and Harris [8] and Bergemann and Valimaki [9] show that choosing the products from the same vendor is the optimal strategy of consumers when there are $K \geq 2$ vendors who offer different products and M consumers whose preferences are the same (but unknown) in the market. The necessary and sufficient conditions for the existence of only two vendors in the market are obtained by Gao et al. [10]. Two-armed bandit problems in the continuous time with the property of Le vy processes are studied by Cohen and Solan [11] (Lévy process is described in [12, 13]). Cohen and Solan [11] conclude that the optimal strategy is a cutoff strategy when the arms have two types. The problem that multiple arms can be chosen by the decision-maker is studied by Kuksov et al. [14] and Doval [15]. It is discovered that the decision-maker is indifferent to search an alternative arm which does not have the highest reservation price. When a Bayesian decision-maker makes a selection from multiple arms with uncertain payoffs and an outside arm with known payoff, maximizing his expected profit is studied in Ke et al. [16]. For other optimal strategies and control approaches, the reader is referred to [17, 18] and the references therein.

The remainder of the paper is organized as follows. In Section 2, we introduce a two-period example and show the scopes of prices. In Section 3, the definition of the Poisson integral is used to prove that the quantity of buyers obeys an SDE and verifies the existence and uniqueness of the solutions for the SDE in global space. In Section 4, based on the dynamic programming principle, we derive HJB equations for the vendors' utility functions and give the verification theorem for the type-leader model. Using solutions of HJB equations, we obtain the optimal strategy for the type-leader vendor. In Section 5, we give an example of this problem and acquire approximate solutions about the profit of the two vendors.

2. A Two Period Example

There are two vendors who offer different nondurable goods, indexed by $j = 1, 2$, and M consumers whose preferences are high or low in the market, where M is positive integer. However, consumers do not know their preferences. If the type is high, a consumer gets expected value ζ_{1H} from buying goods 1 and ζ_{2H} from buying goods 2. Otherwise, a consumer gets expected value ζ_{1L} from buying goods 1 and ζ_{2L} from buying goods 2. We assume that $\zeta_{1H} > \zeta_{2H}$ and $\zeta_{1L} < \zeta_{2L}$, where ζ_{jH} and ζ_{jL} belong to $(0, +\infty)$.

At any time, all market participants observe all previous outcomes. Because of the influence caused by uncertain external factors, the flow utility $u_{ji}(t)\,(i \in \{H, L\})$ has a noisy signal of the true value (for detailed discussion, refer to [1]).

$$\mathrm{d}u_{ji}(t) = \zeta_{ji}\mathrm{d}t + \sigma_j \mathrm{d}\widetilde{B}_j(t), \tag{1}$$

where $\widetilde{B}_1(t)$ and $\widetilde{B}_2(t)$ are independent Brownian motions. In the market, besides the types of consumers, there are many uncertain factors affecting the effectiveness of products to consumers. For example, there exist some subtle and unavoidable differences in the quality of the goods and these differences affect consumers' utilities. The noisy signal of the true value is used to characterize the effects of these factors on consumers' utilities.

Assume that $x_t \in [0, 1]$ is the belief that the type is high if consumers choose the first vendor and $y_t \in [0, 1]$ is the belief that the type is high if consumers choose the other vendor; that is to say, $x_t \coloneqq \Pr(i = H \mid u_{1i}^t)$ and $y_t \coloneqq \Pr(i = H \mid u_{2i}^t)$, where $u_{ji}^t \coloneqq \{u_{ji}(\tau)\}_{\tau=0}^{t}$ is a realized path. From [19], we have

$$\begin{aligned}
\mathrm{d}x_t &= x_t(1 - x_t)s_1 \frac{\mathrm{d}u_{1i}(t) - \zeta_{1H}\mathrm{d}t - \zeta_{1L}\mathrm{d}t}{\sigma_1} \\
&\coloneqq x_t(1 - x_t)s_1 \mathrm{d}B_1(t), \\
\mathrm{d}y_t &= y_t(1 - y_t)s_2 \frac{\mathrm{d}u_{2i}(t) - \zeta_{2H}\mathrm{d}t - \zeta_{2L}\mathrm{d}t}{\sigma_2} \\
&\coloneqq y_t(1 - y_t)s_2 dB_2(t),
\end{aligned} \tag{2}$$

where $s_j = (\zeta_{jH} - \zeta_{jL})/\sigma_j$, $j = 1, 2$, $B_1(t)$ and $B_2(t)$ are independent Brownian motions.

If a consumer chooses the first vendor, his expected utility is $f_1(x_t) := x_t\zeta_{1H} + (1 - x_t)\zeta_{1L}$. Letting $a_1 := \zeta_{1H} - \zeta_{1L}$ and $b_1 := \zeta_{1L}$, the utility is represented by

$$f_1(x) = a_1 x_t + b_1. \quad (3)$$

Similarly, if a consumer chooses the second vendor, his expected utility is represented by

$$f_2(x) = a_2 y_t + b_2, \quad (4)$$

where $a_2 := \zeta_{2H} - \zeta_{2L}$ and $b_2 := \zeta_{2L}$. The definitions of a_j and b_j imply $a_1 + b_1 > a_2 + b_2$, $b_1 < b_2$ and $a_1 > a_2$.

Let n_t be the number of consumers who choose the first vendor and $M - n_t$ be the number of consumers who choose the second vendor, $n_t \in \{0, 1, \ldots, M\}$. $P_1(n_t)$ is the price of goods from the first vendor and $P_2(n_t)$ is the price of goods from the second one. Formally, the price of goods is a measurable function $P_j: \{0, 1, \ldots, M\} \longrightarrow \mathbb{R}^+ \cup \{0\}$. We assume that the change of price has hysteresis(the product price of the second vendor is a function with respect to n_t when M fixed).

As consumers' beliefs change, the lower the beliefs they have, the lower the profits they earn if they choose the first vendor. For a consumer who chooses the first vendor, if his belief is low enough at certain time, he gives up the vendor to choose the second one. Denote $\alpha_t \in [0, 1]$ as the belief, i.e., if $x_t < \alpha_t$, a consumer transforms his choice from the first to the second vendor. Similarly, the consumer gives up buying goods from the second vendor and chooses to buy goods from the first vendor if $y_t > \beta_t \in [0, 1]$.

For nondurable goods, if a consumer chooses the first vendor at first time, the common belief for high type is x_t at time t. The utility of the consumer is $x_t\zeta_{1H} + (1 - x_t)\zeta_{1L} - P_1(n_t)$. If the consumer gives up the first vendor to choose second one, the prices for the vendors do not change due to the hysteresis of the change of price. The utility of the consumer is $x_t\zeta_{2H} + (1 - x_t)\zeta_{2L} - P_2(n_t)$. If the consumer voluntarily gives up the first vendor to choose the second vendor, it has

$$x_t\zeta_{1H} + (1 - x_t)\zeta_{1L} - P_1(n_t) < x_t\zeta_{2H} + (1 - x_t)\zeta_{2L} - P_2(n_t). \quad (5)$$

Inequality (5) is rewritten as

$$x_t < \frac{b_2 - b_1 + P_1(n_t) - P_2(n_t)}{a_1 - a_2}. \quad (6)$$

From the definition of α_t, it has $\alpha_t = ((b_2 - b_1 + P_1(n_t) - P_2(n_t))/(a_1 - a_2))$. In the same way, we obtain $\beta_t = ((b_2 - b_1 + P_1(n_t) - P_2(n_t))/(a_1 - a_2))$. After the above discussion, α_t and β_t satisfy

$$\alpha_t = \beta_t = \frac{b_2 - b_1 + P_1(n_t) - P_2(n_t)}{a_1 - a_2}. \quad (7)$$

Equation (7) shows that the common belief which makes a consumer change his choice from the first vendor to the second one is equal to that which makes the changes from the second vendor to the first one. It is called cutoff in [1]. The cutoff increases as $P_1(n_t)$ increases and decreases as $P_2(n_t)$ increases. The cutoff is linear with both $P_1(n_t)$ and $P_2(n_t)$. In a market, as $P_1(n_t)$ increases, the utility of the consumer who chooses the first vendor is smaller. The consumers who have the same common belief tend to choose the second vendor. Thus, the cutoff cuts down. If $P_2(n_t)$ decreases, the relative price for the second vendor increases. Similarly, the cutoff reduces. In the following, we use α_t to denote the cutoff. The ranges of pricing for two vendors obtained from equation (7) are shown in Proposition 1.

Proposition 1. *Suppose that $P_1(\cdot)$ and $P_2(\cdot)$ are the prices of goods from the first and second vendors, respectively. Then $P_1(\cdot)$ and $P_2(\cdot)$ satisfy*

$$0 \le P_1(n_t) \le (a_1 + b_1) - (a_2 + b_2) + P_2(n_t), \quad (8)$$

$$0 \le P_2(n_t) \le b_2 - b_1 + P_1(n_t), \quad (9)$$

where one of the second signs of inequalities (8) or (9) is sign of strict inequality.

Proof. $P_1(n_t)$ and $P_2(n_t)$ are greater than zero as the vendors' costs of goods are zero. In the following, we prove $P_1(n_t) \le (a_1 + b_1) - (a_2 + b_2) + P_1(n_t)$ and $P_2(n_t) \le b_2 - b_1 + P_2(n_t)$.

If $\alpha_t > 1$, i.e., $x_t < \alpha_t$, all consumers give up the first vendor to choose the second one. For the first vendor, the price of his goods satisfies $\alpha_t \le 1$. It has

$$\frac{b_2 - b_1 + P_1(n_t) - P_2(n_t)}{a_1 - a_2} \le 1, \quad (10)$$

i.e.,

$$P_1(n_t) \le (a_1 + b_1) - (a_2 + b_2) + P_2(n_t). \quad (11)$$

In the same way, the second vendor makes the price of goods satisfy $\alpha_t \ge 0$. We have

$$P_2(n_t) \le b_2 - b_1 + P_2(n_t). \quad (12)$$

Inequalities (8) and (9) are proved.

From inequalities (8) and (9), we obtain

$$\begin{aligned} P_1(n_t) &\le (a_1 + b_1) - (a_2 + b_2) + P_2(n_t) \\ &\le (a_1 + b_1) - (a_2 + b_2) + b_2 - b_1 + P_1(n_t) \\ &= a_1 - a_2 + P_1(n_t). \end{aligned} \quad (13)$$

From inequality (13), we obtain $a_2 \le a_1$ which contradicts the assumption $a_2 < a_1$. Thus, one of the second signs of inequalities in (8) and (9) is sign of strict inequality.

Proposition 1 shows the scopes of commodity prices which are decided by two monopolistic vendors. We find that the supremum of the price for a vendor increases as the price for the other vendor increases. The supremum of the price for a vendor and the price for the other vendor are linear dependence. In the duopoly market, an increase in the price of goods means a decrease of benefit to the number of consumers. Due to the substitution effect between goods, the benefit to someone who chooses the other vendor increases.

The supremum of the price for the other vendor aggrandizes. □

Corollary 1. *$P_1(n_t) < +\infty$ if and only if $P_2(n_t) < +\infty$ while $P_1(n_t) = \infty$ if and only if $P_2(n_t) = \infty$.*

Corollary 1 is easily proved from Proposition 1 and reveals the commodity prices under periods of economic prosperity and economic crisis. During periods of economic prosperity, the commodity prices are bounded while the prices are infinite on account of discontinued sale of goods during the economic crisis. In next discussion, we ignore the situation of economic crisis.

3. Duopoly Market

If there exists $\{t_\Theta\}_{0\le\Theta\le d}$ such that $0 = t_0 < t_1 < \cdots < t_d = t$, a consumer chooses a vendor, denoted by $j, t \in (t_{2k}, t_{2k+1})$ and chooses the other vendor, denoted by $-j$, $t \in (t_{2k+1}, t_{2k+2})$, $k = 0, 1, \ldots, (d/2) - 1$. Let

$$p_t := \Pr\left\{i = H \mid u_{ji}^{(t_0,t_1)}, u_{-j,i}^{(t_1,t_2)}, \ldots, u_{ji}^{(t_{2k},t_{2k+1})}, u_{-j,i}^{(t_{2k+1},t_{2k+2})}, \ldots, u_{ji}^{(t_{d-2},t_{d-1})}, u_{-j,i}^{(t_{d-1},t)}\right\}, \tag{14}$$

where $u_{ji}^{(\pi_1,\pi_2)} := \left\{u_{ji}(\tau)\right\}_{\tau=\pi_1}^{\pi_2}$. Using the arguments in [8, 9], we know that p_t satisfies

$$\begin{aligned} \mathrm{d}p_t &= \sqrt{n_t}\, p_t (1 - p_t) s_1 \mathrm{d}W_1(t) \\ &\quad + \sqrt{M - n_t}\, p_t (1 - p_t) s_2 \mathrm{d}W_2(t), \end{aligned} \tag{15}$$

where $W_1(t)$ and $W_2(t)$ are independent Winner processes.

For any time l and small $\varepsilon > 0$, a consumer gives up the second vendor and chooses the first vendor when

$$\begin{aligned} l &\in \{s > 0 \mid p_s = \alpha_s, \alpha_{s-\varepsilon} - p_{s-\varepsilon} \in A_2, \alpha_{s+\varepsilon} - p_{s+\varepsilon} \in A_1\} \\ &\subseteq \{s > 0 \mid p_s = \alpha_s, \Delta(\alpha_s - p_s) \in A_1\} \\ &\subseteq \{s > 0 \mid \Delta(\alpha_s - p_s) \in A_1\}, \end{aligned} \tag{16}$$

where $A_1 := \{x \le 0\}/\{0\}$, $A_2 := \{x \ge 0\}/\{0\}$ and $\Delta(X_s) := X_s - X_{s-}$ with $X_{s-} := \lim_{t\longrightarrow s-} X_t$. Similarly, a consumer replaces the goods of the first vendor with the goods of the second vendor if

$$\begin{aligned} l &\in \{s > 0 \mid p_s = \alpha_s, \alpha_{s-\varepsilon} - p_{s-\varepsilon} \in A_1, \alpha_{s+\varepsilon} - p_{s+\varepsilon} \in A_2\} \\ &\subseteq \{s > 0 \mid p_s = \alpha_s, \Delta(\alpha_s - p_s) \in A_2\} \\ &\subseteq \{s > 0 \mid \Delta(\alpha_s - p_s) \in A_2\}. \end{aligned} \tag{17}$$

We assume that the number of consumers who choose one vendor to replace the other one is related to $\Delta(\alpha_l - p_l)$, p_l and n_l at time l. All of them reflect the relationship between α_t and p_t at $t \in (l - \varepsilon, l + \varepsilon)$. Let $G_j(\Delta(\alpha_l - p_l), p_l, n_l)$ $(\in \{0, 1, \ldots, M\})$ be the number of consumers who give up the other vendor to choose the vendor j, $j = 1, 2$. If n_t is known, the number of consumers who choose the first vendor at $(t + \mathrm{d}t)$ is expressed by

$$\begin{aligned} n_{t+\mathrm{d}t} &= n_t \\ &\quad + \sum_{0\le u\le dt} G_1(\Delta(\alpha_u - p_u), p_u, n_u) \sharp\{0 \le s \le u \mid p_s = \alpha_s, \alpha_{s-\varepsilon} - p_{s-\varepsilon} \in A_2, \alpha_{s+\varepsilon} - p_{s+\varepsilon} \in A_1\} \\ &\quad - \sum_{0\le u\le dt} G_2(\Delta(\alpha_u - p_u), p_u, n_u) \sharp\{0 \le s \le u \mid p_s = \alpha_s, \alpha_{s-\varepsilon} - p_{s-\varepsilon} \in A_1, \alpha_{s+\varepsilon} - p_{s+\varepsilon} \in A_2\}, \end{aligned} \tag{18}$$

where $\sharp$ represents the number of elements in a set. Let

$$\begin{aligned} \varpi_1(\Delta(\alpha_u - p_u), p_u) &= \frac{\sharp\{0 \le s \le u \mid p_s = \alpha_s, \alpha_{s-\varepsilon} - p_{s-\varepsilon} \in A_2, \alpha_{s+\varepsilon} - p_{s+\varepsilon} \in A_1\}}{\sharp\{0 \le s \le u \mid \Delta(\alpha_u - p_u) \in A_1\}}, \\ \varpi_2(\Delta(\alpha_u - p_u), p_u) &= \frac{\sharp\{0 \le s \le u \mid p_s = \alpha_s, \alpha_{s-\varepsilon} - p_{s-\varepsilon} \in A_1, \alpha_{s+\varepsilon} - p_{s+\varepsilon} \in A_2\}}{\sharp\{0 \le s \le u \mid \Delta(\alpha_u - p_u) \in A_2\}}. \end{aligned} \tag{19}$$

If $\{0 \le s \le u \mid p_s = \alpha_s, \alpha_{s-\varepsilon} - p_{s-\varepsilon} \in A_2, \alpha_{s+\varepsilon} - p_{s+\varepsilon} \in A_1\} \neq \phi$, we require that $\gamma_1(\Delta(\alpha_u - p_u), p_u)$ equals to $\varpi_1(\Delta(\alpha_u - p_u), p_u)$ while $\gamma_1(\Delta(\alpha_u - p_u), p_u) := 0$ for another case. Similarly, if $\{0 \le s \le u \mid p_s = \alpha_s, \alpha_{s-\varepsilon} - p_{s-\varepsilon} \in A_1, \alpha_{s+\varepsilon} - p_{s+\varepsilon} \in A_2\}$ is not empty, let $\gamma_2(\Delta(\alpha_u - p_u), p_u)$ be equal to $\varpi_2(\Delta(\alpha_u - p_u), p_u)$ and zero otherwise. Defining

$$g_1(\Delta(\alpha_u - p_u), p_u, n_u) = G_1(\Delta(\alpha_u - p_u), p_u, n_u)\gamma_1 \cdot (\Delta(\alpha_u - p_u), p_u), \quad (20)$$

$$g_2(\Delta(\alpha_u - p_u), p_u, n_u) = G_2(\Delta(\alpha_u - p_u), p_u, n_u)\gamma_2 \cdot (\Delta(\alpha_u - p_u), p_u), \quad (21)$$

we have

$$n_{t+dt} = n_t + \sum_{0\le u\le dt} g_1(\Delta(\alpha_u - p_u), p_u, n_u)\sharp\{0\le s\le u \mid \Delta(\alpha_u - p_u) \in A_1\} - \sum_{0\le u\le dt} g_2(\Delta(\alpha_u - p_u), p_u, n_u)\sharp\{0\le s\le u \mid \Delta(\alpha_u - p_u) \in A_2\}, \quad (22)$$

where $g_j(\Delta(\alpha_u - p_u), p_u, n_u) \le G_j(\Delta(\alpha_u - p_u), p_u, n_u)$ due to $\gamma_j(\cdot,\cdot) \in [0,1]$, i.e., $g_j(\Delta(\alpha_u - p_u), p_u, n_u)$ is bounded. Proposition 2 can be obtained by using the definition of Poisson stochastic integral as $\mathrm{d}t \longrightarrow 0$ and $s \longrightarrow u$.

Proposition 2. *If $(\alpha_t - p_t)$ is a stochastic process which has independent and stationary increments with ca dla g paths6 (paths are continuous on the right and have limits on the left [20, 21]), then*

$$dn_t = \int_{A_1} g_1(h, p_{t-}, n_{t-})N(dt, dh) - \int_{A_2} g_2(h, p_{t-}, n_{t-})N(dt, dh), \quad (23)$$

where $g_1(h, p_{t-}, n_{t-})$ and $g_2(h, p_{t-}, n_{t-})$ are defined in Equations (20) and (21). Moreover, SDE (23) has a unique solution.

Proof. As $g_1(\cdot,\cdot,\cdot)$ and $g_2(\cdot,\cdot,\cdot)$ are bounded, A_1 and A_2 are bounded below, from [22], we obtain that n_t is convergent in L^2 for any measurable random variable n_0. Besides, $g_1(\cdot,\cdot,\cdot)$ and $g_2(\cdot,\cdot,\cdot)$ can be verified to satisfy

(i) Lipschitz condition:tThere exists $K_1 > 0$, for any y_1, $y_2 \in \{0, 1, \ldots, M\}$ and $x \in [0,1]$ such that

$$\left|\int_{A_1} |g_1(h, x, y_1) - g_1(h, x, y_2)|^2 \nu(\mathrm{d}h)\right| \le K_1 |y_1 - y_2|^2,$$
$$\left|\int_{A_2} |g_2(h, x, y_1) - g_2(h, x, y_2)|^2 \nu(\mathrm{d}h)\right| \le K_1 |y_1 - y_2|^2. \quad (24)$$

(ii) Growth condition: there exists $K_2 > 0$, for any $y \in \{0, 1, \ldots, M\}$ and $x \in [0,1]$, such that

$$2y\int_{A_1} g_1(h, x, y)\nu(\mathrm{d}h) + \int_{A_1} |g_1(h, x, y)|^2 \nu(\mathrm{d}h) \le K_2\left(1 + |y|^2\right),$$
$$2y\int_{A_1} g_1(h, x, y)\nu(\mathrm{d}h) + \int_{A_2} |g_2(h, x, y)|^2 \nu(\mathrm{d}h) \le K_2\left(1 + |y|^2\right). \quad (25)$$

Equation (23) is proved to exist a unique solution by using the results in [23, 24].

Proposition 2 shows the existence and uniqueness of global solutions for the SDE (23) which describes the quantity of buyers who choose the first vendor in the duopoly market. It is presented as a pure jump process. $g_j(\cdot,\cdot,\cdot)$ is defined as the rate of change of the number of consumers who give up the other vendor and choose the jth one. □

4. The Optimal Strategy for Vendors

As consumers do not know their types, they choose one of two vendors referred by the prices of goods at initial time. We assume that there are n consumers who choose the first vendor, i.e., $n_0 = n$. After that, vendors inform consumers' types, denoted by $p := p_0$, to guide consumers to make subsequent choices. Ignoring the interaction between the two vendors, denoting $V_j(\cdot)$ as the jth vendor's optimal utility and r as risk-free interest rate, for the first vendor, we have

$$V_1(n) = \max_{p\in[0,1]} \mathbb{E}\left[\int_0^{+\infty} e^{-rt} n_t P_1(n_t)\mathrm{d}t\right]$$
$$s.t\ \mathrm{d}n_t = \int_{A_1} g_1(h, p_{t-}, n_{t-})N(\mathrm{d}t, \mathrm{d}h) - \int_{A_2} g_2(h, p_{t-}, n_{t-})N(\mathrm{d}t, \mathrm{d}h). \quad (26)$$

The HJB equation is obtained as follows:

$$rV_1(n) = nP_1(n) + \max_{p\in[0,1]}\left\{\int_{A_1} [V_1(n + g_1(h, p, n)) - V_1(n)]\nu(\mathrm{d}h) - \int_{A_2} [V_1(n + g_2(h, p, n)) - V_1(n)]\nu(\mathrm{d}h)\right\}, \quad (27)$$

where ν, the intensity measure of N, is the finite intensity measure as A_1 and A_2 are bounded below. Suppose that $w(n)$ is a solution of equation (27), the verification result is obtained as follows.

Proposition 3. *If there exists an integrable function $\phi(\cdot)$ such that $|w(\cdot)| \le \phi(\cdot)$.*

(i) Suppose that

$$\begin{aligned} rw(n) - nP_1(n) - \max_{p\in[0,1]} \Bigg\{ \int_{A_1} [w(n + g_1(h, p, n)) - w(n)]\nu(\mathrm{d}h) \\ - \int_{A_2} [w(n + g_2(h, p, n)) - w(n)]\nu(\mathrm{d}h) \Bigg\} \ge 0, \end{aligned} \tag{28}$$

$$\limsup_{T\longrightarrow+\infty} e^{-rT}\mathbb{E}[w(n_T)] \ge 0. \tag{29}$$

Then $w(n) \ge v(n)$ if $n \in \{1, 2, \ldots, M\}$.

(ii). Suppose that for all $n \in \{1, 2, \ldots, M\}$, there exists a p^* such that

$$\begin{aligned} rw(n) - nP_1(n) - \Bigg\{ \int_{A_1} [w[n + g_1[h, p^*, n]] - w[n]]\nu(\mathrm{d}h) \\ - \int_{A_2} [w[n + g_2[h, p^*, n]] - w[n]]\nu(\mathrm{d}h) \Bigg\} = 0, \end{aligned} \tag{30}$$

the stochastic differential equation

$$\begin{aligned} \mathrm{d}n_t = \int_{A_1} g_1(h, p^*_{t-}, n_{t-})N(\mathrm{d}t, \mathrm{d}h) \\ - \int_{A_2} g_2(h, p^*_{t-}, n_{t-})N(\mathrm{d}t, \mathrm{d}h), \end{aligned} \tag{31}$$

admits a unique solution and

$$\liminf_{t\longrightarrow\infty} e^{-rT}\mathbb{E}[w(n_T)] \le 0, \tag{32}$$

then

$$w(n) = v(n), \quad n \in \{1, 2, \ldots, M\}. \tag{33}$$

Proof. Three steps are divided to prove Proposition 3. □

Step 1. We prove $\mathbb{E}[\int_0^{+\infty} e^{-rt} n_t P_1(n_t)\mathrm{d}t] < +\infty$. From the assumptions, it has

$$\mathbb{E}\left[\int_0^{+\infty} e^{-rt} n_t P_1(n_t)\mathrm{d}t\right] \le M\mathbb{E}\left[\int_0^{+\infty} e^{-rt} P_1(n_t)\mathrm{d}t\right]. \tag{34}$$

Hence, from $\int_0^{+\infty} e^{-rt}\mathrm{d}t < +\infty$ and $P_1(n_t)$ which is a monotone bounded function, we derive

$$\mathbb{E}\left[\int_0^{+\infty} e^{-rt} P_1(n_t)\mathrm{d}t\right] < +\infty. \tag{35}$$

Thus, $\mathbb{E}[\int_0^{+\infty} e^{-rt} n_t P_1(n_t)\mathrm{d}t] < +\infty$ as $M < +\infty$.

Step 2. The proof of (i) in Proposition 3. Let

$$\begin{aligned} \tau_k := \inf\Bigg\{ t \ge 0 \mid \int_{A_1} e^{-rt} [w(n_{t-} + g_1(h, p_t, n_t)) \\ - w(n_{t-})]\nu(\mathrm{d}h) \ge k \\ \text{or} \int_{A_2} e^{-rt} [w(n_{t-} + g_1(h, p_t, n_t)) - w(n_{t-})]\nu(\mathrm{d}h) \ge k \Bigg\}. \end{aligned} \tag{36}$$

Using Itô's lemma for $e^{-r(T\wedge\tau_k)} w(n_{T\wedge\tau_k})$, where $T \wedge \tau_k := \min\{T, \tau_k\}$, we obtain

$$\begin{aligned} &e^{-r(T\wedge\tau_k)}w\left(n_{T\wedge\tau_k}\right)\\ &\quad = w(n) - \int_0^{T\wedge\tau_k} re^{-rs}w(n_s)\mathrm{d}s\\ &\qquad + \int_0^{T\wedge\tau_k}\int_{A_1} e^{-rs}\left[w\left(n_{s-}+g_1(h,p_s,n_s)\right)-w\left(n_{s-}\right)\right]N(\mathrm{d}s,\mathrm{d}h)\\ &\qquad - \int_0^{T\wedge\tau_k}\int_{A_2} e^{-rs}\left[w\left(n_{s-}+g_2(h,p_s,n_s)\right)-w\left(n_{s-}\right)\right]N(\mathrm{d}s,\mathrm{d}h). \end{aligned} \tag{37}$$

Introducing compensation Poisson random measure, equation (37) is rewritten as

$$\begin{aligned} &e^{-r(T\wedge\tau_k)}w\left(n_{T\wedge\tau_k}\right)\\ &\quad = w(n) - \int_0^{T\wedge\tau_k} re^{-rs}w(n_s)\mathrm{d}s\\ &\qquad + \int_0^{T\wedge\tau_k}\int_{A_1} e^{-rs}\left[w\left(n_{s-}+g_1(h,p_s,n_s)\right)-w\left(n_{s-}\right)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}h)\\ &\qquad - \int_0^{T\wedge\tau_k}\int_{A_2} e^{-rs}\left[w\left(n_{s-}+g_2(h,p_s,n_s)\right)-w\left(n_{s-}\right)\right]\widetilde{N}(\mathrm{d}s,\mathrm{d}h)\\ &\qquad + \int_0^{T\wedge\tau_k} ds\int_{A_1} e^{-rs}\left[w\left(n_{s-}+g_1(h,p_s,n_s)\right)-w\left(n_{s-}\right)\right]\nu(\mathrm{d}h)\\ &\qquad - \int_0^{T\wedge\tau_k}\int_{A_2} e^{-rs}\left[w\left(n_{s-}+g_2(h,p_s,n_s)\right)-w\left(n_{s-}\right)\right]\nu(\mathrm{d}h), \end{aligned} \tag{38}$$

where the compensation Poisson random measure $\widetilde{N}(s,\mathrm{d}h) \coloneqq N(s,\mathrm{d}h) - s\cdot\nu(\mathrm{d}h)$ is a martingale. Taking expectation for both sides of equation (38) yields

$$\begin{aligned} &\mathbb{E}\left[e^{-r(T\wedge\tau_k)}w\left(n_{T\wedge\tau_k}\right)\right]\\ &\quad = w(n) - \mathbb{E}\left[\int_0^{T\wedge\tau_k} re^{-rs}w(n_s)\mathrm{d}s\right]\\ &\qquad + \mathbb{E}\left[\int_0^{T\wedge\tau_k}\mathrm{d}s\int_{A_1} e^{-rs}\left[w\left(n_{s-}+g_1(h,p_s,n_s)\right)-w\left(n_{s-}\right)\right]\nu(\mathrm{d}h)\right]\\ &\qquad - \mathbb{E}\left[\int_0^{T\wedge\tau_k}\int_{A_2} e^{-rs}\left[w\left(n_{s-}+g_2(h,p_s,n_s)\right)-w\left(n_{s-}\right)\right]\nu(\mathrm{d}h)\right]. \end{aligned} \tag{39}$$

From inequality (28), we obtain

$$\mathbb{E}\left[e^{-r(T\wedge\tau_k)}w\left(n_{T\wedge\tau_k}\right)\right] \le w(n) - \mathbb{E}\left[\int_0^{T\wedge\tau_k} e^{-rs}n_sP_1(n_s)\right]. \tag{40}$$

Letting $k \longrightarrow +\infty$, using the dominated convergence theorem, inequality (40) becomes

$$\mathbb{E}\left[e^{-rT}w(n_T)\right] \le w(n) - \mathbb{E}\left[\int_0^T e^{-rs}n_sP_1(n_s)\mathrm{d}s\right]. \tag{41}$$

As $T \longrightarrow +\infty$, in accordance with inequality (29), for any $p \in [0,1]$, the inequality

$$w(n) \ge \mathbb{E}\left[\int_0^T e^{-rs}n_sP_1(n_s)\mathrm{d}s\right], \tag{42}$$

is obtained. Hence,

$$w(n) \geq \max_{p\in[0,1]} \mathbb{E}\left[\int_0^T e^{-rs} n_s P_1(n_s)\mathrm{d}s\right] = V(n). \tag{43}$$

The (i) in Proposition 3 is proved by Steps 1 and 2.

Step 3. The proof of (ii) in Proposition 3. As p^* is the optimal choice for the vendor, we obtain

$$\mathbb{E}\left[e^{-rT} w(n_T)\right] = w(n) - \mathbb{E}\left[\int_0^T e^{-rs} n_s P_1(n_s)\mathrm{d}s\right], \tag{44}$$

where

$$\begin{aligned}\mathrm{d}n_t = &\int_{A_1} g_1(h, p^*_{t-}, n_{t-})N(\mathrm{d}t, \mathrm{d}h)\\ &- \int_{A_2} g_2(h, p^*_{t-}, n_{t-})N(\mathrm{d}t, \mathrm{d}h),\end{aligned} \tag{45}$$

has a unique solution. Letting $T \longrightarrow +\infty$, from inequality (32), we have

$$w(n) \leq \mathbb{E}\left[\int_0^T e^{-rs} n_s P_1(n_s)\mathrm{d}s\right] = V(n). \tag{46}$$

From inequalities (43) and (46), we have $w(n) = V(n)$. The proof is completed.

Proposition 3 shows that the solution of HJB equation (27) is $V_1(n)$.

Similarly, for the second vendor, it has

$$\begin{aligned}rV_2(n) = (M-n)P_2(n) + \max_{p\in[0,1]}\Bigg\{&\int_{A_1}[V_2(n+g_1(h,p,n)) - V_2(n)]\nu(\mathrm{d}h)\\ &-\int_{A_2}[V_2(n+g_2(h,p,n)) - V_2(n)]\nu(\mathrm{d}h)\Bigg\}.\end{aligned} \tag{47}$$

We assume that one of the two vendors is a type-leader vendor. Without loss of generality, denote the first vendor as the type-leader vendor. In this situation, we acquire

$$\begin{aligned}\kappa V_1(n) = nP_1(n) + &\int_{A_1} V_1(n+g_1(h,p^*,n))\nu(\mathrm{d}h)\\ &- \int_{A_2} V_1(n+g_2(h,p^*,n))\nu(\mathrm{d}h).\end{aligned} \tag{48}$$

For the second vendor, his payoff function satisfies

$$\begin{aligned}\kappa V_2(n) = (M-n)P_2(n) + &\int_{A_1} V_2(n+g_1(h,p^*,n))\nu(\mathrm{d}h)\\ &- \int_{A_2} V_2(n+g_2(h,p^*,n))\nu(\mathrm{d}h),\end{aligned} \tag{49}$$

where $\kappa \coloneqq r - [\int_{A_1}\nu(\mathrm{d}h) - \int_{A_1}\nu(\mathrm{d}h)] < \infty$.

As the second vendor considers that the first vendor informs all consumers whose types are H and consumers tend to buy goods from the first vendor, the second vendor's goods are not sold. From Corollary 2.1, if the vendor wants more consumers to buy his goods, he prices his goods $P_2(n_t)$, based on the price $P_1(n_t)$ of the first vendor, such that $P_1(n_t) = (a_1+b_1) - (a_2+b_2) + P_2(n_t)$., i.e.,

$$P_2(n_t) = (a_2+b_2) - (a_1+b_1) + P_1(n_t). \tag{50}$$

From equation (50), we find that $P_2(n_t) < 0$ if $P_1(n_t) < (a_1+b_1) - (a_2+b_2)$. Thus, the price of the second vendor's goods is zero. For the type-leader vendor, the price of their goods is

$$P_1(n) = (a_1+b_1) - (a_2+b_2) - \varepsilon, \tag{51}$$

where $\varepsilon > 0$ is an arbitrary small number. Denoted by P_1^ε as $(a_1+b_1) - (a_2+b_2) - \varepsilon$, then $P_1^\varepsilon < (a_1+b_1) - (a_2+b_2) + 0$ and $0 = P_2(n_t) \leq a_1 - a_2 - \varepsilon$ satisfy pricing ranges for both vendors in Proposition 1. Equations (48) and (49) are equivalent to

$$\begin{aligned}\kappa V_1(n) = nP_1^\varepsilon + &\int_{A_1} V_1(n+g_1(h,p^*,n))\nu(\mathrm{d}h)\\ &- \int_{A_2} V_1(n+g_2(h,p^*,n))\nu(\mathrm{d}h),\end{aligned} \tag{52}$$

$$\begin{aligned}\kappa V_2(n) = &\int_{A_1} V_2(n+g_1(h,p^*,n))\nu(\mathrm{d}h)\\ &- \int_{A_2} V_2(n+g_2(h,p^*,n))\nu(\mathrm{d}h),\end{aligned} \tag{53}$$

respectively. In this situation, the payoffs of the leader and the following vendors are shown in Proposition 4.

Proposition 4. *If there exist P_1^ε, the rate of change $g_j(\cdot,\cdot,\cdot)$, a finite intensity measure ν and $p^* \in [0,1]$ such that the equation*

$$\begin{aligned}\kappa V_1(n) = nP_1^\varepsilon + &\int_{A_1} V_1(n+g_1(h,p^*,n))\nu(\mathrm{d}h)\\ &- \int_{A_2} V_1(n+g_2(h,p^*,n))\nu(\mathrm{d}h),\end{aligned} \tag{54}$$

has solutions; then its solution is unique if and only if $V_2(n) \equiv 0$, i.e., the type-leader vendor makes the other vendor's profit be zero with pricing strategy.

Proof. Firstly, we prove that if there exists P_1^{ε}, the rate of change $g_j(\cdot,\cdot,\cdot)$, a finite intensity measure ν and $p^* \in [0,1]$ such that the equation

$$\kappa V_1(n) = nP_1^{\varepsilon} + \int_{A_1} V_1(n + g_1(h, p^*, n))\nu(dh) - \int_{A_2} V_1(n + g_2(h, p^*, n))\nu(dh), \tag{55}$$

has a unique solution, then $V_2(n) \equiv 0$.

As $V_1(n)$ and $V_2(n)$ are the maximum utilities of the first and second vendor, we have $V_1(n) \geq 0$ and $V_2(n) \geq 0$. Denote $V(\cdot) = V_1(\cdot) + V_2(\cdot)$, where $V(\cdot)$ represents the sum of two vendors' profits. It is straightforward to verify $V(n) \geq V_1(n)$ and $V(n) \geq V_2(n)$. From equations (52) and (53), we have

$$\begin{aligned}\kappa V(n) &= nP_1^{\varepsilon} + \int_{A_1} V_1(n + g_1(h, p^*, n))\nu(dh) - \int_{A_2} V_1(n + g_2(h, p^*, n))\nu(dh)\\ &\quad + \int_{A_1} V_2(n + g_1(h, p^*, n))\nu(dh) - \int_{A_2} V_2(n + g_2(h, p^*, n))\nu(dh)\\ &= nP_1^{\varepsilon} + \int_{A_1} [V_1(n + g_1(h, p^*, n)) + V_2(n + g_1(h, p^*, n))]\nu(dh)\\ &\quad - \int_{A_2} [V_1(n + g_2(h, p^*, n)) + V_2(n + g_2(h, p^*, n))]\nu(dh).\end{aligned} \tag{56}$$

From the definition of $V(\cdot)$, we derive that

$$\kappa V(n) = nP_1^{\varepsilon} + \int_{A_1} V(n + g_1(h, p^*, n))\nu(dh) - \int_{A_2} V(n + g_2(h, p^*, n))\nu(dh). \tag{57}$$

Comparing Eqs. (54) and (57) finds that $V_1(\cdot)$ and $V(\cdot)$ have the same structure. If there exists a unique solution to equation (57), we have $V(\cdot) \equiv V_1(\cdot)$. Therefore, $V_2(\cdot) \equiv 0$. The necessity of Proposition 3 is proved.

Now, we prove the sufficiency. If $V_2(\cdot) \equiv 0$ and the solution of equation (54) exists, then it is unique. Assume that there exist two solutions v_1 and v_2, $v_1 \equiv v_2$, i.e., there exists $m \in \{0, 1, \ldots, M\}$ such that $v_1(m) \neq v_2(m)$. Without loss of generality, we assume that $v_1(m) < v_2(m)$. From $V(\cdot) > V_1(\cdot)$, $v_2(m) = V(m)$ and $v_1(m) = V_1(m)$ are obtained. Moreover, $V_2(m) = V(m) - V_1(m) \neq 0$, contradictory with $V_2 \equiv 0$. Therefore, if the solution of equation (54) exists, its solution is unique when $V_2(\cdot) \equiv 0$. The sufficiency of Proposition 3 is proved.

Proposition 4 shows the condition that the type-leader vendor obtains surplus of all producers. In this case, the following vendor is unprofitable and gradually withdraws from the market. Eventually, the market will be monopolized by the type-leader vendor. □

5. Example

For the convenience of explanation, we add two assumptions.

Assumption 1. Let $g_j(h, p, n) := \psi_j(h)\phi_j(p, n)$ where $\psi_j(\cdot)$ and $\phi_j(\cdot,\cdot)$ are bounded and $\phi_j(\cdot,\cdot)$ is twice continuously differentiable for the first variable.

Let $\xi_j := \int_{A_j} \nu(dh)$, $\omega_j := \int_{A_j} \psi_j(h)\nu(dh)$ and $\eta_j := \int_{A_j} \psi_j^2(h)\nu(dh)$. ξ_j, ω_j and η_j are finite as A_1 and A_2 are bounded below. The first order condition for p from HJB equation (27) is

$$\int_{A_1} V_1'(n + \psi_1(h)\phi_1(p, n))\frac{\partial\phi_1(p, n)}{\partial p}\psi_1(h)\nu(dh) - \int_{A_2} V_2'(n + \psi_2(h)\phi_2(p, n))\frac{\partial\phi_2(p, n)}{\partial p}\psi_2(h)\nu(dh) = 0. \tag{58}$$

As the exact analytic solution of HJB equation (27) is difficult to be obtained, we consider the approximate solution of equation (27). Let $V_j(n + \psi_1(h)\phi_1(p, n))$ be approximately equal to $V_j(n) + V_j'(n)\psi_j(h)\phi_j(p, n) + 1/2V_j''(n)\psi_j^2(h)\phi_j^2(p, n)$ and $V_j'(n + \psi_1(h)\phi_1(p, n))$ be approximately equal to $V_j'(n) + V_j''(n)\psi_j(h)\phi_j(p, n)$. Substituting them into (58) yields

$$\widetilde{V}_1'(n)\left[\omega_1\frac{\partial\phi_1(p, n)}{\partial p} - \omega_2\frac{\partial\phi_2(p, n)}{\partial p}\right] + \widetilde{V}_1''(n)\left[\eta_1\phi_1(p, n)\frac{\partial\phi_1(p, n)}{\partial p} - \eta_2\phi_2(p, n)\frac{\partial\phi_2(p, n)}{\partial p}\right] = 0, \tag{59}$$

where $\widetilde{V}_j(\cdot)$ is an approximate solution of $V_j(\cdot)$. For acquiring the type-leader vendor's optimal strategy, Assumption 2 is given.

Assumption 2. There exist ω_j, η_j, p^*, $\phi_j(p,n)$ to satisfy Assumption 1, for any $p \in [0,1]$, such that

$$\begin{aligned}
&\widetilde{V}_1'(n)\left[\omega_1\frac{\partial^2\phi_1(p,n)}{\partial p^2}-\omega_2\frac{\partial^2\phi_2(p,n)}{\partial p^2}\right]+\widetilde{V}_1''(n)\left[\eta_1\phi_1(p,n)\frac{\partial^2\phi_1(p,n)}{\partial p^2}\right.\\
&\left.+\eta_1\left(\frac{\partial\phi_1(p,n)}{\partial p}\right)^2-\eta_2\phi_2(p,n)\frac{\partial\phi_2(p,n)}{\partial p}-\eta_2\left(\frac{\partial\phi_2(p,n)}{\partial p}\right)^2\right]<0\\
&\left.\frac{\partial\phi_1(p,n)}{\partial p}\right|_{p=p^*}=\left.\frac{\partial\phi_2(p,n)}{\partial p}\right|_{p=p^*}=0.
\end{aligned}\tag{60}$$

Assumption 2 implies that p^* is a unique optimal strategy for the type-leader vendor. Combining with pricing strategy, HJB equation (27) can be approximately written by

$$q_1\widetilde{V}_1''(n)+q_2\widetilde{V}_1'(n)+q_3\widetilde{V}_1(n)+nP_1^{\varepsilon}=0,\tag{61}$$

where $q_1 := 1/2(\eta_1\phi_1^2(p^*,n)-\eta_2\phi_2^2(p^*,n))$, $q_2 := \omega_1\phi_1(p^*,n)-\omega_2\phi_2(p^*,n)$, and $q_3 := \xi_1-\xi_2-\kappa$. Similarly,

$$q_1\widetilde{V}_2''(n)+q_2\widetilde{V}_2'(n)+q_3\widetilde{V}_2(n)=0.\tag{62}$$

In order to simplify the analytical solution of equations (61) and (62), we assume that q_1 and q_2 are independent to n. Denote C_Y as constants which do not depend on n $(Y=1,2,\ldots,11)$. Four cases are divided to solve equations (61) and (62).

Case 1. Consider $q_1\neq 0$ and $q_2^2-4q_1q_3>0$. If $q_3\neq 0$, Solving equations (61) and (62) yields

$$\begin{aligned}
\widetilde{V}_1(n)&=-\frac{nP_1^{\varepsilon}}{q_3}+\frac{q_2P_1^{\varepsilon}}{q_3^2}+C_1e^{((-q_2+\sqrt{q_2^2-4q_1q_3})/2q_1)n}+C_2e^{((-q_2+\sqrt{q_2^2-4q_1q_3})/2q_1)n},\\
\widetilde{V}_2(n)&=C_1e^{((-q_2+\sqrt{q_2^2-4q_1q_3})/2q_1)n}+C_2e^{((-q_2+\sqrt{q_2^2-4q_1q_3})/2q_1)n}.
\end{aligned}\tag{63}$$

If $q_3=0$, the solutions of $\widetilde{V}_1(n)$ and $\widetilde{V}_2(n)$ are

$$\begin{aligned}
\widetilde{V}_1(n)&=C_3e^{-(q_2/q_1)n}+C_4-\frac{nP_1^{\varepsilon}}{q_2}\\
\widetilde{V}_2(n)&=C_3e^{-(q_2/q_1)n}+C_4,
\end{aligned}\tag{64}$$

respectively.

Case 2. Consider $q_1\neq 0$ and $q_2^2-4q_1q_3=0$. If $q_2\neq 0$, solving equations (61) and (62), we have

$$\begin{aligned}
\widetilde{V}_1(n)&=-\frac{nP_1^{\varepsilon}}{q_3}+\frac{q_2P_1^{\varepsilon}}{q_3^2}+(C_5+nC_6)e^{-(q_2/q_1)n},\\
\widetilde{V}_2(n)&=(C_5+nC_6)e^{-(q_2/q_1)n}.
\end{aligned}\tag{65}$$

If $q_2=0$, $\widetilde{V}_1(n)$ and $\widetilde{V}_2(n)$ are verified to satisfy

$$\begin{aligned}
\widetilde{V}_1(n)&=-\frac{n^3P_1^{\varepsilon}}{6q_1}+C_7n+C_8,\\
\widetilde{V}_2(n)&=C_7n+C_8.
\end{aligned}\tag{66}$$

Case 3. Consider $q_1\neq 0$ and $q_2^2-4q_1q_3<0$. In this case, we solve equations (61) and (62) to obtain

$$\begin{aligned}
\widetilde{V}_1(n)&=-\frac{nP_1^{\varepsilon}}{q_3}+\frac{q_2P_1^{\varepsilon}}{q_3^2}+e^{-(q_2/2q_1)n}\left(C_9\cos\frac{\sqrt{4q_1q_3-q_2^2}}{2q_1}n+C_{10}\sin\frac{\sqrt{4q_1q_3-q_2^2}}{2q_1}n\right),\\
\widetilde{V}_2(n)&=e^{-(q_2/2q_1)n}\left(C_9\cos\frac{\sqrt{4q_1q_3-q_2^2}}{2q_1}n+C_{10}\sin\frac{\sqrt{4q_1q_3-q_2^2}}{2q_1}n\right).
\end{aligned}\tag{67}$$

Case 4. Consider $q_1 = 0$. If $q_2 \neq 0$, Solving equations (61) and (62) yields

$$\widetilde{V}_1(n) = C_{11} e^{-(q_3/q_2)n} - \frac{P_1^{\varepsilon}}{q_3}\left(n - \frac{q_2}{q_3}\right),$$
$$\widetilde{V}_2(n) = C_{11} e^{-(q_3/q_2)n}. \tag{68}$$

If $q_2 = 0$, $\widetilde{V}_1(n)$ and $\widetilde{V}_2(n)$ satisfy

$$\widetilde{V}_1(n) = -\frac{n}{q_3} P_1^{\varepsilon}, \tag{69}$$

$$\widetilde{V}_2(n) \equiv 0. \tag{70}$$

It is easy to verify that the solutions in the four cases satisfy the verification theorem in Proposition 3. In particular, from equations (69) and (70), $\widetilde{V}_1(\cdot)$ has a unique solution and $\widetilde{V}_2(n) \equiv 0$ in this situation, which satisfies the conclusion in Proposition 4.

6. Conclusion

This paper explores a stochastic differential equation driven by Poisson random measure and its application in a duopoly market which exists two different types of consumers. We assume that prices of goods are decided by the number of consumers. To study vendors' optimal pricing strategies, scopes of goods prices are obtained from the cutoff. In addition, we prove that the quantity of buyers obeys a SDE resorting to the definition of Poisson stochastic measure and maximizing the vendor payoff. We also verify that the SDE exists a unique solution. Given the SDE, the corresponding HJB equation reflecting the profits of vendor is derived by using the dynamic programming principle.

In certain markets where the effectiveness of goods is more important than its price, we introduce the type-leader model. In the type-leader model, we find vendors' price strategies and verify that the commodity prices are in the price ranges in Proposition 1. The conditions that the type-leader vendor obtains surplus of all producers are acquired by existence and uniqueness of solutions of HJB equations. An example of this problem is given and approximate solutions for the profit of the two vendors are obtained.

Data Availability

No data were used in this manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interests.

Authors' Contributions

The article is a joint work of two authors who contributed equally to the final version of the paper. Both authors read and approved the final manuscript.

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