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Research Article

Partially Observed Nonzero-Sum Differential Game of BSDEs with Delay and Applications

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A class of partially observed nonzero-sum differential games for backward stochastic differential equations with time delays is studied, in which both game system and cost functional involve the time delays of state variables and control variables under each participant with different observation equations. A necessary condition (maximum principle) for the Nash equilibrium point to this kind of partially observed game is established, and a sufficient condition (verification theorem) for the Nash equilibrium point is given. A partially observed linear quadratic game is taken as an example to illustrate the application of the maximum principle.

1. Introduction

Game theory has penetrated into many fields of economics and attracted more and more attention. A series of studies on game theory was given by [1–9]. There have been many papers about the differential games driven by backward stochastic differential equations (BSDEs), such as [10, 11]. The prospective progress of many systems relies as much on their past history as on their current state. The optimal control problem of stochastic systems with time delays is studied by [12–22]. The game problem of stochastic systems with the time-delayed generator is discussed by [23–25].

Nevertheless, in the aforementioned control and game problems, it is assumed that the information is fully observed. This does not make sense in real life. In general, only partial information is available for controllers in most cases. The latest research studies on the partially observed optimal control issues of stochastic differential systems were given by [26–32]. The partially observed game issues of stochastic systems were studied by [33–36].

As far as we know, the results in regard to partially observed differential games corresponding to backward stochastic systems with time delays (BSDDE) are few. This problem will be investigated in this paper. Comparing the above results, our work differs in several aspects. Firstly, we research such a kind of partially observed differential game problem corresponding to the BSDDE, which enriches the game theory of backward stochastic systems. Secondly, under the circumstance of different observation equations for each participant, our controlled systems and utility functions include the delays of state variables and control variables. Thirdly, we study a class of linear quadratic (LQ) game corresponding to backward stochastic systems with the time-delayed generator and give the specific expression of the Nash equilibrium point.

The outline of this article is as follows. We present the main hypotheses and the partially observed differential game problem of BSDDE in Section 2. In Section 3, we obtain the necessary optimality conditions of the partially observed game of BSDDE. Section 4 is devoted to the sufficient maximum principle. In Section 5, we take a partially observed LQ game as an example to illustrate the application of our maximum principle. Section 6 is the conclusion of this paper.

2. Statement of the Problems

Throughout our article, $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete filtered probability space, on which three mutually independent one-dimensional standard Brownian motions $W(t), Y_1(t)$, and $Y_2(t)$ are defined. Let $\mathscr{F}_t^W, \mathscr{F}_t^1$, and $\mathscr{F}_t^2$ be the natural filtrations generated by $W(\cdot), Y_1(\cdot)$, and $Y_2(\cdot)$, respectively. We set $\mathscr{F}_t = \mathscr{F}_t^W \otimes \mathscr{F}_t^1 \otimes \mathscr{F}_t^2$. For all $t \in [-\delta, 0]$, $\mathscr{F}_t \equiv \mathscr{F}_0$, which is the trivial σ-field, and $\mathscr{F} := \mathscr{F}_{T+\delta}$. Set $\mathscr{F}_t^i = \sigma\{Y_i(s); 0 \leq s \leq t\}$, $(i = 1, 2)$ and $\mathscr{F}_t^i \equiv \mathscr{F}_0^i \neq \phi$, $\forall t \in [-\delta, 0]$. The finite time duration is defined by $T > 0$, and the constant time delays are defined by $0 < \delta, \delta_1, \delta_2 < T$, respectively. The expectation on $(\Omega, \mathscr{F}, P)$ is denoted by $\mathbb{E}$, and the conditional expectation under $\mathscr{F}_t$ is denoted by $\mathbb{E}^{\mathscr{F}_t} := \mathbb{E}[.|\mathscr{F}_t]$. In $\mathbb{R}$ and $\mathbb{R}^{n\times d}$, $\langle\cdot,\cdot\rangle$ is the usual inner product and $|\cdot|$ is the Euclidean norm. The symbol "$\top$" that appears in the superscript represents the transpose of the matrix. In this article, all of the equalities and inequalities are in the sense of $dt \times d\mathbb{P}$ almost surely on $[0, T] \times \Omega$.

We introduce the following notations:

$$
\begin{aligned}
&L^2(\mathscr{F}_T; \mathbb{R}) = \left\{\xi\colon \xi \text{ is an } \mathbb{R} - \text{valued, } \mathscr{F}_T - \text{measurable random variable satisfying } \mathbb{E}|\xi|^2 < \infty\right\},\\
&L^2_{\mathscr{F}}(s, r; \mathbb{R}) = \left\{v(t), s \leq t \leq r\colon v(t) \text{ is an } \mathbb{R} - \text{valued, } \mathscr{F}_t - \text{adapted process satisfying } \mathbb{E}\int_s^r |v(t)|^2 dt < \infty\right\}.
\end{aligned} \tag{1}
$$

Let the nonempty set $U_i \in \mathbb{R}\,(i = 1, 2)$ be convex and the admissible control set be defined as the following:

$$
\mathscr{U}_i[0, T] = \left\{v_i\colon [0, T] \times \Omega \to U_i \mid v_i \text{ is } \mathscr{F}_t^i - \text{adapted, } \mathbb{E}\int_0^T |v_i(t)|^4 dt < \infty\right\}, \quad (i = 1, 2). \tag{2}
$$

Every element in $\mathscr{U}_i$ is known as an admissible control to Player $i\,(i = 1, 2)$. And $\mathscr{U}_1 \times \mathscr{U}_2$ is known as the admissible control set to the players.

This work pays attention to a kind of partially observed games of BSDDE, which stems from some attractive financial scenarios. Now let us elaborate on the problem. Take into account the following BSDDE:

$$
\begin{cases}
-dy^{v_1,v_2}(t) = f(\Theta^{v_1,v_2}(t))dt - z^{v_1,v_2}(t)dW(t), & t \in [0, T],\\
y^{v_1,v_2}(T) = \xi,\ y^{v_1,v_2}(t) = \psi_0(t), & t \in [-\delta, 0],\\
v_1(t) = \psi_1(t), & t \in [-\delta_1, 0],\\
v_2(t) = \psi_2(t), & t \in [-\delta_2, 0],
\end{cases} \tag{3}
$$

where

$$
\begin{aligned}
\Theta^{v_1,v_2}(t) = (&t, y^{v_1,v_2}(t), y^{v_1,v_2}(t-\delta), z^{v_1,v_2}(t), v_1(t),\\
&v_1(t-\delta_1), v_2(t), v_2(t-\delta_2)),
\end{aligned} \tag{4}
$$

and $f\colon [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $\xi \in L^2(\mathscr{F}_T; \mathbb{R})$, and $\psi_0(\cdot) \in L^2_{\mathscr{F}}(-\delta, 0; \mathbb{R})$, $\psi_1(\cdot) \in L^2_{\mathscr{F}}(-\delta_1, 0; \mathbb{R})$, and $\psi_2(\cdot) \in L^2_{\mathscr{F}}(-\delta_2, 0; \mathbb{R})$ are the initial paths of y, v_1, andv_2, respectively. We suppose that $\psi_1(t) \in \mathscr{F}_0^1$ and $\psi_2(t) \in \mathscr{F}_0^2$ are measurable continuous functions such that $\mathbb{E}\int_0^T |\psi_1(t)|^4 dt < \infty$ and $\mathbb{E}\int_0^T |\psi_2(t)|^4 dt < \infty$. The control processes for Player 1 and Player 2 are $v_1(\cdot)$ and $v_2(\cdot)$, and $v(\cdot) = (v_1(\cdot), v_2(\cdot))$. Subscript 1 presents the variables to Player 1, and subscript 2 presents the variables to Player 2, respectively. BSDDE game system (3) means that the two players complete a common target ξ in the end time T.

Suppose that the two participants cannot directly observe the state processes $y^{v_1,v_2}(\cdot)$, but they can be aware of related noisy processes $Y_1(\cdot)$ and $Y_2(\cdot)$ of $y^{v_1,v_2}(\cdot)$, which are described as follows:

$$
\begin{cases}
dY_i(t) = h_i(t, y^{v_1,v_2}(t), y^{v_1,v_2}(t-\delta), z^{v_1,v_2}(t), v_1(t), v_2(t))dt + dW_i(t),\\
Y_i(0) = 0, \quad (i = 1, 2),
\end{cases} \tag{5}
$$

where $W_1(\cdot)$ and $W_2(\cdot)$ are $\mathbb{R}$-valued stochastic processes depending on $v_1(\cdot)$ and $v_2(\cdot)$ and $h_i\colon [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1, 2$, are continuous functions.

We assume

$(H1)$ (i) f and $h_i, i = 1, 2$, are continuously differentiable with respect to $(y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta})$

(ii) $f_y, f_{y_\delta}, f_z, f_{v_1}, f_{v_{1\delta}}, f_{v_2}, f_{v_{2\delta}}, h_{iy}, h_{iy_\delta}, h_{iz}, h_{iv_1}, h_{iv_2}, i = 1, 2$, are bounded by $c > 0$

Now, if $(H1)$ is true and both $v_1(\cdot)$ and $v_2(\cdot)$ are admissible controls, then BSDDE (3) has a unique solution $(y^{v_1,v_2}(\cdot), z^{v_1,v_2}(\cdot)) \in L^2_{\mathscr{F}}(-\delta, T; \mathbb{R}) \times L^2_{\mathscr{F}}(-\delta, T; \mathbb{R})$ (see [14]).

Define $d\mathbb{P}^{v_1,v_2} \doteq Z^{v_1,v_2}(t) d\mathbb{P}$, where

$$Z^{v_1,v_2}(t) = \exp\left\{\sum_{j=1}^{2}\int_0^t h_j(s, y^{v_1,v_2}(s), y^{v_1,v_2}(s-\delta), z^{v_1,v_2}(s), v_1(s), v_2(s)) dY_j(s) - \frac{1}{2}\sum_{j=1}^{2}\int_0^t \left|h_j(s, y^{v_1,v_2}(s), y^{v_1,v_2}(s-\delta), z^{v_1,v_2}(s), v_1(s), v_2(s))\right|^2 ds\right\}. \tag{6}$$

Obviously, $Z^{v_1,v_2}(t)$ satisfies the subsequent SDE:

$$\begin{cases} dZ^{v_1,v_2}(t) = \sum_{j=1}^{2} h_j(t, y^{v_1,v_2}(t), y^{v_1,v_2}(t-\delta), z^{v_1,v_2}(t), v_1(t), v_2(t)) Z^{v_1,v_2}(t) dY_j(t), \\ Z^{v_1,v_2}(0) = 1. \end{cases} \tag{7}$$

Hence, by $(H1)$ and Girsanov's theorem, we obtain a three-dimensional Brownian motion $(W(\cdot), W_1(\cdot), W_2(\cdot))$ built on the probability space $(\Omega, \mathscr{F}, \mathbb{P}^{v_1,v_2})$, in which $\mathbb{P}^{v_1,v_2}$ is a probability measure.

Making sure to accomplish the target ξ, each player owns his individual interest, which is the cost functional as follows:

$$J_i(v_1(\cdot), v_2(\cdot)) = \mathbb{E}^{v_1,v_2}\left\{\int_0^T l_i(\Theta^{v_1,v_2}(t)) dt + \Phi_i(y^{v_1,v_2}(0))\right\}, \quad (i = 1, 2), \tag{8}$$

where $\mathbb{E}^{v_1,v_2}$ is the expectation on $(\Omega, \mathscr{F}, \mathbb{P}^{v_1,v_2})$ and

$$\begin{aligned} l_i\colon f\colon [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R}, \\ \Phi_i\colon \mathbb{R} \longrightarrow \mathbb{R}, \quad (i = 1, 2). \end{aligned} \tag{9}$$

We also assume for $i = 1, 2$,

$(H2)$ (i) l_i are continuously differentiable with respect to $(y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta})$, and their partial derivatives are continuous in $(y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta})$ and bounded by $c(1 + |y| + |y_\delta| + |z| + |v_1| + |v_{1\delta}| + |v_2| + |v_{2\delta}|)$

(ii) Φ_i are continuously differentiable, and Φ_{iy} are bounded by $c(1 + |y|)$

Assume that every player wants to minimize the cost functional $J_i(v_1(\cdot), v_2(\cdot))$ by picking the appropriate admissible control $v_i(\cdot)$ $(i = 1, 2)$. Then, our partially observed nonzero-sum stochastic differential game problem is to find out a pair of admissible controls $(u_1(\cdot), u_2(\cdot)) \in \mathscr{U}_1 \times \mathscr{U}_2$ such that

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) = \min_{v_1(\cdot) \in \mathscr{U}_1} J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) = \min_{v_2(\cdot) \in \mathscr{U}_2} J_2(u_1(\cdot), v_2(\cdot)). \end{cases} \tag{10}$$

Obviously, cost functional (8) can be converted to

$$J_i(v_1(\cdot), v_2(\cdot)) = \mathbb{E}\left\{\int_0^T Z^{v_1,v_2}(t) l_i(\Theta^{v_1,v_2}(t)) dt + \Phi_i(y^{v_1,v_2}(0))\right\}, \quad (i = 1, 2). \tag{11}$$

So, the original problem (10) is the same thing as minimizing (11) over $(v_1(\cdot), v_2(\cdot)) \in \mathscr{U}_1 \times \mathscr{U}_2$ subject to (3) and (7). For the sake of convenience, we refer to the above game problem as Problem (POBNZ). If an admissible control $u(\cdot) = (u_1(\cdot), u_2(\cdot))$ which satisfied (10) can be found, then it is called as an equilibrium point of Problem (POBNZ), and the corresponding state trajectory is denoted by $(y(\cdot), z(\cdot)) = (y^u(\cdot), z^u(\cdot))$.

3. A Partially Observed Necessary Maximum Principle

In the case of a convex admissible control set, the convex perturbation method is the classical method to obtain the necessary optimality condition. Let the equilibrium point of Problem (POBNZ) be $u(\cdot) = (u_1(\cdot), u_2(\cdot))$, and the corresponding optimal trajectory is $(y(\cdot), z(\cdot))$. Let $(\tilde{v}_1(\cdot), \tilde{v}_2(\cdot))$

be such that $(u_1(\cdot)+\tilde{v}_1(\cdot), u_2(\cdot)+\tilde{v}_2(\cdot)) \in \mathscr{U}_1 \times \mathscr{U}_2$. Since $\mathscr{U}_1$ and $\mathscr{U}_2$ are convex, for any $0 \le \rho \le 1$, $(u_1^\rho(\cdot), u_2^\rho(\cdot)) = (u_1(\cdot)+\rho\tilde{v}_1(\cdot), u_2(\cdot)+\rho\tilde{v}_2(\cdot))$ is also in $\mathscr{U}_1 \times \mathscr{U}_2$. For the controls $(u_1^\rho(\cdot), u_2(\cdot))$ and $(u_1(\cdot), u_2^\rho(\cdot))$, the corresponding state trajectories of game system (3) are denoted by $(y^{u_1^\rho}(\cdot), z^{u_1^\rho}(\cdot))$ and $(y^{u_2^\rho}(\cdot), z^{u_2^\rho}(\cdot))$.

We introduce the subsequent symbols:

$$\begin{aligned}
\varphi(t) &= \varphi(t, y(t), y(t-\delta), z(t), u_1(t), u_1(t-\delta_1), u_2(t), u_2(t-\delta_2)),\\
\varphi^{v_1,v_2}(t) &= \varphi(t, y(t), y(t-\delta), z(t), v_1(t), v_1(t-\delta_1), v_2(t), v_2(t-\delta_2)),\\
\varphi^{u_1^\rho,u_2}(t) &= \varphi(t, y(t), y(t-\delta), z(t), u_1^\rho(t), u_1^\rho(t-\delta_1), u_2(t), u_2(t-\delta_2)),\\
\varphi^{u_1,u_2^\rho}(t) &= \varphi(t, y(t), y(t-\delta), z(t), u_1(t), u_1(t-\delta_1), u_2^\rho(t), u_2^\rho(t-\delta_2)),
\end{aligned} \tag{12}$$

where φ denotes one of $f, l_i, i = 1, 2$, and

$$\begin{aligned}
h_i(t) &= h_i(t, y(t), y(t-\delta), z(t), u_1(t), u_2(t)),\\
h_i^{v_1,v_2}(t) &= h_i(t, y(t), y(t-\delta), z(t), v_1(t), v_2(t)),\\
h_i^{u_1^\rho,u_2}(t) &= h_i(t, y(t), y(t-\delta), z(t), u_1^\rho(t), u_2(t)),\\
h_i^{u_1,u_2^\rho}(t) &= h_i(t, y(t), y(t-\delta), z(t), u_1(t), u_2^\rho(t)),\\
&\qquad (i = 1, 2).
\end{aligned} \tag{13}$$

The variational equations are as follows:

$$\begin{cases}
-\mathrm{d}y_i^1(t) = \left[f_y(t)y_i^1(t) + f_{y_\delta}(t)y_i^1(t-\delta) + f_z(t)z_i^1(t) + f_{v_i}(t)\tilde{v}_i(t) + f_{v_{i\delta}}(t)\tilde{v}_i(t-\delta_i)\right]\mathrm{d}t - z_i^1(t)\mathrm{d}W(t), \quad t \in [0, T],\\
y_i^1(T) = 0,\ y_i^1(t) = 0, \quad t \in [-\delta, 0],\\
\tilde{v}_1(t) = 0, \quad t \in [-\delta_1, 0],\\
\tilde{v}_2(t) = 0, \quad t \in [-\delta_2, 0],\ (i = 1, 2),
\end{cases} \tag{14}$$

$$\begin{cases}
\mathrm{d}Z_i^1(t) = \sum_{j=1}^{2}\left[Z_i^1(t)h_j(t) + Z(t)\left(h_{jy}(t)y_i^1(t) + h_{jy_\delta}(t)y_i^1(t-\delta) + h_{jz}(t)z_i^1(t) + h_{jv_i}(t)\tilde{v}_i(t)\right)\right]\mathrm{d}Y_j(t),\\
Z_i^1(0) = 0,\ (i = 1, 2).
\end{cases} \tag{15}$$

From $(H1)$, it is easy to see that (14) and (15) admit unique solutions $(y^{v_1,v_2}(\cdot), z^{v_1,v_2}(\cdot)) \in L^2_{\mathscr{F}}(-\delta, T; \mathbb{R}) \times L^2_{\mathscr{F}}(-\delta, T; \mathbb{R})$ and $Z^1(t) \in L^2_{\mathscr{F}}(0, T; \mathbb{R})$, respectively.

For $t \in [0, T]$ and $\rho > 0$, we set

$$\begin{aligned}
\tilde{y}_i^\rho(t) &= \frac{y^{u_i^\rho}(t) - y(t)}{\rho} - y_i^1(t),\\
\tilde{z}_i^\rho(t) &= \frac{z^{u_i^\rho}(t) - z(t)}{\rho} - z_i^1(t),\\
\tilde{Z}_i^\rho(t) &= \frac{Z^{u_i^\rho}(t) - Z(t)}{\rho} - Z_i^1(t),\\
&\qquad (i = 1, 2).
\end{aligned} \tag{16}$$

Similar to the arguments in Lemmas 3.1 and 3.2 in [34], it is easy to obtain subsequent Lemmas 1 and 2. Thus, we omit the details for simplicity.

Lemma 1. *Assume (H1) and (H2) are true. Then,*

$$\begin{aligned}
&\lim_{\rho\to 0}\sup_{0\le t\le T}\mathbb{E}\left|\tilde{y}_i^\rho(t)\right|^2 = 0,\\
&\lim_{\rho\to 0}\mathbb{E}\int_0^T\left|\tilde{z}_i^\rho(t)\right|^2\mathrm{d}t = 0,\\
&\lim_{\rho\to 0}\sup_{0\le t\le T}\mathbb{E}\left|\tilde{Z}_i^\rho(t)\right|^2 = 0,\\
&\qquad (i = 1, 2).
\end{aligned} \tag{17}$$

Since $(u_1(\cdot), u_2(\cdot))$ is a Nash equilibrium point, then

$$\begin{aligned}
&\rho^{-1}\left[J_1(u_1^\rho(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot))\right] \ge 0,\\
&\rho^{-1}\left[J_2(u_1(\cdot), u_2^\rho(\cdot)) - J_2(u_1(\cdot), u_2(\cdot))\right] \ge 0.
\end{aligned} \tag{18}$$

Let $\Gamma_i(t) = Z^{-1}(t)Z_i^1(t)$, $i = 1, 2$. From Itô's formula, we deduce

$$\begin{cases} d\Gamma_i(t) = \sum_{j=1}^{2} \left[h_{jy}(t)y_i^1(t) + h_{jz}(t)z_i^1(t) + h_{jv}(t)\tilde{v}_i(t)\right]dW_j(t), \\ \Gamma_i(0) = 0, \quad (i = 1, 2). \end{cases} \tag{19}$$

From this and Lemma 1, we have the following.

Lemma 2. *Assume (H1) and (H2) are true. Then, we get the following variational inequality:*

$$\begin{aligned} &\mathbb{E}^{u_1,u_2} \int_0^T \left[l_i(t)\Gamma_i(t) + l_{iy}(t)y_i^1(t) + l_{iy_\delta}(t)y_i^1(t-\delta) + l_{iz}(t)z_i^1(t) + l_{iv_i}(t)\tilde{v}_i(t) + l_{iv_{i\delta}}(t)\tilde{v}_i(t-\delta_i)\right]dt \\ &\quad + \mathbb{E}^{u_1,u_2}\left[\Phi_{iy}(y(0))y_i^1(0)\right] \geq 0, \quad (i = 1, 2). \end{aligned} \tag{20}$$

Our Hamiltonian function H_i: $[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$, is defined as follows:

$$\begin{aligned} H_i(t, y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta}, p_i, Q_{1i}, Q_{2i}) &= -\langle p_i(t), f(t, y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta})\rangle \\ &\quad + \sum_{j=1}^{2} \langle Q_{ji}(t), h_j(t, y, y_\delta, z, v_1, v_2)\rangle + l_i(t, y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta}), \quad (i = 1, 2). \end{aligned} \tag{21}$$

Denote $H_i(t) \equiv H_i(t, y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta}, p_i, Q_{1i}, Q_{2i})$ and its derivatives.

We note that the adjoint equation to (19) is a BSDE, whose solution is $(P_i(\cdot), Q_{1i}(\cdot), Q_{2i}(\cdot))$:

$$\begin{cases} -dP_i(t) = l_i(t)dt - \sum_{j=1}^{2} Q_{ji}(t)dW_j(t), \\ P_i(T) = 0, \quad (i = 1, 2), \end{cases} \tag{22}$$

and the adjoint equation to (14) is an SDE, whose solution is $p_i(\cdot)$:

$$\begin{cases} dp_i(t) = -\left\{H_{iy}(t) + \mathbb{E}^{\mathcal{F}_t}\left[H_{iy_\delta}(t+\delta)\right]\right\}dt - H_{iz}(t)dW(t), \\ p_i(0) = -\Phi_{iy}(y(0)), \quad (i = 1, 2). \end{cases} \tag{23}$$

Remark 1. It is easy to see that equation (23) is a linear anticipated SDE. Under ($H1$) and ($H2$), the unique solvability of equation (23) is assured by Theorem 2.2 in [14].

Based on variational inequality (20), we set out the main result of this section.

Theorem 1 (partially observed necessary maximum principle). *Assume (H1) and (H2) are true, an equilibrium point of Problem (POBNZ) is $(u_1(\cdot), u_2(\cdot))$, the optimal trajectory is $(y(\cdot), z(\cdot))$, and the solution of (7) is $Z(\cdot)$. Let $(P_i(\cdot), Q_{1i}(\cdot), Q_{2i}(\cdot))$, $i = 1, 2$, be the solution of (22) and $p_i(\cdot)$ be the solution of adjoint equation (23). Then, the following maximum principle*

$$\mathbb{E}^{u_1,u_2}\left[\left(H_{1v_1}(t) + \mathbb{E}^{u_1,u_2}\left[H_{1v_1}(t+\delta_1) \mid \mathcal{F}_t^1\right]\right)(v_1 - u_1(t)) \mid \mathcal{F}_t^1\right] \geq 0, \tag{24}$$

$$\mathbb{E}^{u_1,u_2}\left[\left(H_{2v_2}(t) + \mathbb{E}^{u_1,u_2}\left[H_{2v_2}(t+\delta_2) \mid \mathcal{F}_t^2\right]\right)(v_2 - u_2(t)) \mid \mathcal{F}_t^2\right] \geq 0, \tag{25}$$

for any $(v_1, v_2) \in U_1 \times U_2$, a.e., $t \in [0,T]$, in which the Hamiltonian function H is defined as (21).

Proof. For $i = 1$, using Itô's formula to $\langle y_1^1(t), p_1(t)\rangle + \langle \Gamma(t), P_1(t)\rangle$, from variational equations (14) and (15), variational inequality (20), and adjoint equations (22) and (23), we obtain

$$\begin{aligned}
&\mathbb{E}^{u_1,u_2}\int_0^T\Big[l_1(t)\Gamma_1(t)+l_{1y}(t)y_1^1(t)+l_{1y_\delta}(t)y_1^1(t-\delta)+l_{1z}(t)z_1^1(t)+l_{1v_1}(t)\tilde{v}_1(t)+l_{1v_{1\delta}}(t)\tilde{v}_1(t-\delta_1)\Big]dt\\
&+\mathbb{E}^{u_1,u_2}\big[\Phi_{1y}(y(0))y_1^1(0)\big]\\
&=\mathbb{E}^{u_1,u_2}\int_0^T\langle f_{v_1}^\top(t)p_1(t)+\sum_{j=1}^2 h_{jv_1}^\top(t)Q_{j1}(t)+l_{1v_1}(t)\\
&\quad+\mathbb{E}^{u_1,u_2}\Big[f_{v_1}^\top(t+\delta)p_1(t+\delta)+\sum_{j=1}^2 h_{jv_1}^\top(t+\delta)Q_{j1}(t+\delta)+l_{1v_1}(t+\delta)\,\Big|\,\mathcal{F}_t^1\Big],\tilde{v}_1(t)\rangle dt\\
&=\mathbb{E}^{u_1,u_2}\int_0^T\langle H_{1v_1}(t)+\mathbb{E}^{u_1,u_2}\Big[H_{1v_1}(t+\delta_1)\,\Big|\,\mathcal{F}_t^1\Big],\tilde{v}_1(t)\rangle dt\\
&\geq 0.
\end{aligned} \tag{26}$$

Because $\tilde{v}_1(t)$ satisfies $u_1(t)+\tilde{v}_1(t)\in U_1$, we have

$$\mathbb{E}^{u_1,u_2}\int_0^T\langle H_{1v_1}(t)+\mathbb{E}^{u_1,u_2}\Big[H_{1v_1}(t+\delta_1)\,\Big|\,\mathcal{F}_t^1\Big],v_1-u_1(t)\rangle dt\geq 0,\quad\forall v_1\in U_1. \tag{27}$$

This implies that

$$\mathbb{E}^{u_1,u_2}\langle H_{1v_1}(t)+\mathbb{E}^{u_1,u_2}\Big[H_{1v_1}(t+\delta_1)\,\Big|\,\mathcal{F}_t^1\Big],v_1-u_1(t)\rangle\geq 0,\quad\forall v_1\in U_1. \tag{28}$$

Now, assume that F is an arbitrary element of σ-algebra $\mathcal{F}_t^1$ and $v_1(t)\in U_1$ is a deterministic element. Let

$$w_1(t)=v_1(t)\mathbf{1}_F+u_1(t)\mathbf{1}_{\Omega-F}. \tag{29}$$

Obviously, w_1 is an admissible control.

Using the above inequality to w_1, we obtain

$$\mathbb{E}^{u_1,u_2}\Big[\mathbf{1}_F\langle H_{1v_1}(t)+\mathbb{E}^{u_1,u_2}\Big[H_{1v_1}(t+\delta_1)\,\Big|\,\mathcal{F}_t^1\Big],v_1-u_1(t)\rangle\Big]\geq 0,\quad\forall F\in\mathcal{F}_t^1, \tag{30}$$

which implies that

$$\mathbb{E}^{u_1,u_2}\Big[\langle H_{1v_1}(t)+\mathbb{E}^{u_1,u_2}\Big[H_{1v_1}(t+\delta_1)\,\Big|\,\mathcal{F}_t^1\Big],v_1-u_1(t)\rangle\,\Big|\,\mathcal{F}_t^1\Big]\geq 0,\quad\forall v_1\in U_1,\ \text{a.e.}t\in[0,T],\ \text{a.s.} \tag{31}$$

Similar to the aforementioned method, we can get the other inequality for any $v_2\in U_2$. The proof of Theorem 1 is completed. □

4. A Partially Observed Sufficient Maximum Principle

In this section, we explore a sufficient maximum principle to Problem (POBNZ). Let $(y(t),z(t),u_1(t),u_2(t))$ be a quintuple that satisfies (3), and assume that there is a solution $p_i(t)$ corresponding to adjoint SDE (23). We assume the following:

($H3$) For $i=1,2$, for all $t\in[0,T]$, $H_i(t,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,p_i,Q_{1i},Q_{2i})$ is convex in $(y,y_\delta,z,v_1,v_{1\delta},v_2,v_{2\delta})$, and $\Phi_i(y)$ is convex in y

For $i=1,2$, let

$$\begin{aligned}
H_i(t)&=H_i\big(t,y(t),y(t-\delta),z(t),u_1(t),u_1(t-\delta),u_2(t),u_2(t-\delta),p_i(t),Q_{1i}(t),Q_{2i}(t)\big),\\
H_i^{v_1}(t)&=H_i\big(t,y(t),y(t-\delta),z(t),v_1(t),v_1(t-\delta_1),u_2(t),u_2(t-\delta_2),p_i(t),Q_{1i}(t),Q_{2i}(t)\big),\\
H_i^{v_2}(t)&=H_i\big(t,y(t),y(t-\delta),z(t),u_1(t),u_1(t-\delta_1),v_2(t),v_2(t-\delta_2),p_i(t),Q_{1i}(t),Q_{2i}(t)\big),\\
h_i(t)&=h_i\big(t,y(t),y(t-\delta),z(t),u_1(t),u_2(t)\big),\\
h_i^{v_1}(t)&=h_i\big(t,y(t),y(t-\delta),z(t),v_1(t),u_2(t)\big),\\
h_i^{v_2}(t)&=h_i\big(t,y(t),y(t-\delta),z(t),u_1(t),v_2(t)\big),\\
\varphi(t)&=\varphi\big(t,y(t),y_\delta(t),z(t),u_1(t),u_1(t-\delta_1),u_2(t),u_2(t-\delta_2)\big),\\
\varphi^{v_1}(t,\cdot)&=\varphi\big(t,y(t),y_\delta(t),z(t),v_1(t),v_1(t-\delta_1),u_2(t),u_2(t-\delta_2)\big),\\
\varphi^{v_2}(t,\cdot)&=\varphi\big(t,y(t),y_\delta(t),z(t),u_1(t),u_1(t-\delta_1),v_2(t),v_2(t-\delta_2)\big),
\end{aligned} \tag{32}$$

where $\varphi = f, l_i, i = 1, 2$.

Theorem 2 (partially observed sufficient maximum principle). *Assume (H1)–(H3) are true. Moreover, the maximum conditions (24) and (25) of the partial observation are true; then, $(u_1(\cdot), u_2(\cdot))$ is the equilibrium point to Problem (POBNZ).*

Proof. For any $v_1(\cdot) \in \mathcal{U}_1$, we consider

$$\begin{aligned} &J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \\ &\quad = \mathbb{E}\int_0^T l_1(t)[Z^{v_1}(t) - Z(t)]dt + \mathbb{E}[\Phi_1(y^{v_1}(0)) - \Phi_1(y(0))] + \mathbb{E}^{v_1,u_2}\int_0^T [l_1^{v_1}(t) - l_1(t)]dt \\ &\quad = I_1 + I_2 + I_3. \end{aligned} \tag{33}$$

Using Itô's formula to $\langle P_1(t), Z^{v_1,u_2}(t) - Z(t)\rangle$ on $[0, T]$, we deduce

$$I_1 = \mathbb{E}^{v_1,u_2}\int_0^T \sum_{j=1}^2 Q_{j1}(t)\left[h_j^{v_1}(t) - h_j(t)\right]dt. \tag{34}$$

Using Itô's formula to $\langle p_1(t), y^{v_1}(t) - y(t)\rangle$ on $[0, T]$, from the convexity of Φ_1, we have

$$\begin{aligned} I_2 &\geq \mathbb{E}^{v_1,u_2}\langle \Phi_{1y}(y(0)), y^{v_1}(0) - y(0)\rangle \\ &= -\mathbb{E}^{v_1,u_2}\int_0^T \langle y^{v_1}(t) - y(t), H_{1y}(t) + \mathbb{E}^{\mathcal{F}_t}\left[H_{1y_\delta}(t+\delta)\right]\rangle dt \\ &\quad - \mathbb{E}^{v_1,u_2}\int_0^T \langle z^{v_1}(t) - z(t), H_{1z}(t)\rangle dt + \mathbb{E}^{v_1,u_2}\int_0^T \langle p_1(t), f^{v_1}(t) - f(t)\rangle dt. \end{aligned} \tag{35}$$

Then, we have

$$\begin{aligned} &J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \\ &\geq \mathbb{E}\int_0^T [H_1^{v_1}(t) - H_1(t)]dt - \mathbb{E}\int_0^T \langle y^{v_1}(t) - y(t), H_{1y}(t) + \mathbb{E}^{\mathcal{F}_t}\left[H_{1y_\delta}(t+\delta)\right]\rangle dt - \mathbb{E}\int_0^T \langle z^{v_1}(t) - z(t), H_{1z}(t)\rangle dt. \end{aligned} \tag{36}$$

By the virtue of convexity of H_1 to $(y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta})$, we deduce

$$\begin{aligned} H_1^{v_1}(t) - H_1(t) &\geq \langle y^{v_1}(t) - y(t), H_{1y}(t)\rangle + \langle y_\delta^{v_1}(t) - y_\delta(t), H_{1y_\delta}(t)\rangle \\ &\quad + \langle z^{v_1}(t) - z(t), H_{1z}(t)\rangle + \langle v_1(t) - u_1(t), H_{1v_1}(t)\rangle + \langle v_{1\delta}(t) - u_{1\delta}(t), H_{1v_{1\delta}}(t)\rangle. \end{aligned} \tag{37}$$

Notice the truth that

$$\begin{aligned} &\mathbb{E}\int_0^T \langle y_\delta^{v_1}(t) - y_\delta(t), H_{1y_\delta}(t)\rangle dt - \mathbb{E}\int_0^T \langle y^{v_1}(t) - y(t), \mathbb{E}^{\mathcal{F}_t}\left[H_{1y_\delta}(t+\delta)\right]\rangle dt \\ &\quad = \mathbb{E}\int_0^T \langle y_\delta^{v_1}(t) - y_\delta(t), H_{1y_\delta}(t)\rangle dt - \mathbb{E}\int_\delta^{T+\delta} \langle y_\delta^{v_1}(t) - y_\delta(t), H_{1y_\delta}(t)\rangle dt \\ &\quad = \mathbb{E}\int_0^\delta \langle y_\delta^{v_1}(t) - y_\delta(t), H_{1y_\delta}(t)\rangle dt - \mathbb{E}\int_T^{T+\delta} \langle y_\delta^{v_1}(t) - y_\delta(t), H_{1y_\delta}(t)\rangle dt \\ &\quad = 0. \end{aligned} \tag{38}$$

Then, we get

$$J_1(v_1(\cdot),u_2(\cdot)) - J_1(u_1(\cdot),u_2(\cdot)) \geq \mathbb{E}\int_0^T \langle H_{1v_1}(t) + \mathbb{E}^{\mathcal{F}_t}\left[H_{1v_{1\delta}}(t+\delta)\right], v_1(t) - u_1(t)\rangle dt. \quad (39)$$

Finally, by necessary optimality conditions (24), we obtain

$$J_1(v_1(\cdot),u_2(\cdot)) - J_1(u_1(\cdot),u_2(\cdot)) \geq 0. \quad (40)$$

Then, it implies

$$J_1(u_1(\cdot),u_2(\cdot)) = \min_{v_1(\cdot)\in\mathcal{U}_1} J_1(v_1(\cdot),u_2(\cdot)). \quad (41)$$

In the same way,

$$J_2(u_1(\cdot),u_2(\cdot)) = \min_{v_2(\cdot)\in\mathcal{U}_2} J_2(u_1(\cdot),v_2(\cdot)). \quad (42)$$

So, we come to the expected conclusion. The proof is completed. □

5. Application

In this section, we construct a partially observed LQ differential game with regard to backward stochastic systems with time delays. Using the classical filtering theory and the aforementioned theoretical results, we attempt to give a specific expression of the Nash equilibrium point. Let us think about the subsequent linear BSDDE:

$$\begin{cases} -dy^{v_1,v_2}(t) = \left(A(t)y^{v_1,v_2}(t) + \overline{A}(t)y^{v_1,v_2}(t-\delta) + B(t)z^{v_1,v_2}(t) + C_1(t)v_1(t) + C_2(t)v_2(t)\right)dt - z^{v_1,v_2}(t)dW(t), \quad t\in[0,T], \\ y^{v_1,v_2}(T) = \xi,\ y^{v_1,v_2}(t) = \psi(t), \quad t\in[-\delta,0], \end{cases} \quad (43)$$

and the observation

$$\begin{aligned} dY_i(t) &= D_i(t)dt + dW_i(t), \\ Y_i^{v_1,v_2}(0) &= 0, \\ &(i=1,2). \end{aligned} \quad (44)$$

We introduce the following cost functional:

$$J_i(v_1(\cdot),v_2(\cdot)) = \frac{1}{2}\mathbb{E}^{v_1,v_2}\left[\int_0^T M_i(t)v_i^2(t)dt + N_i\left(y^v(0)\right)^2\right], \quad (i=1,2), \quad (45)$$

where constant $N_i \geq 0$, functions $A(\cdot), \overline{A}(\cdot), B(\cdot), C_i(\cdot), D_i(\cdot), M_i(\cdot)$, $i=1,2$, are deterministic and bounded, and $M_i^{-1}(\cdot)$ is bounded. Our partially observed nonzero-sum LQ differential game is to find out a pair of admissible controls $(u_1(\cdot),u_2(\cdot)) \in \mathcal{U}_1\times\mathcal{U}_2$ satisfying

$$\begin{cases} J_1(u_1(\cdot),u_2(\cdot)) = \min_{v_1(\cdot)\in\mathcal{U}_1} J_1(v_1(\cdot),u_2(\cdot)), \\ J_2(u_1(\cdot),u_2(\cdot)) = \min_{v_2(\cdot)\in\mathcal{U}_2} J_2(u_1(\cdot),v_2(\cdot)). \end{cases} \quad (46)$$

Similarly to [34], with the help of the necessary maximum principle (Theorem 1), we have the explicit expression to a Nash equilibrium point with regard to the above LQ game problem.

Theorem 3. *For the above LQ game, we find out a Nash equilibrium point as*

$$\begin{aligned} (u_1(t),u_2(t)) = &\left(M_1^{-1}(t)C_1^\top(t)\mathbb{E}^{v_1,v_2}\left[p_1(t)\mid\mathcal{F}_t^1\right],\right. \\ &\left. M_2^{-1}(t)C_2^\top(t)\mathbb{E}^{v_1,v_2}\left[p_2(t)\mid\mathcal{F}_t^2\right]\right), \quad t\in[0,T], \end{aligned} \quad (47)$$

in which $(y(t),z(t),p_1(t),p_2(t))$ *satisfy the general FBSDE:*

$$\begin{cases} -dy^{v_1,v_2}(t) = \left(A(t)y^{v_1,v_2}(t) + \overline{A}(t)y^{v_1,v_2}(t-\delta) + B(t)z^{v_1,v_2}(t) + C_1(t)M_1^{-1}(t)C_1^\top(t)\mathbb{E}^{v_1,v_2}\left[p_1(t)\mid\mathcal{F}_t^1\right]\right. \\ \left. + C_2(t)M_2^{-1}(t)C_2^\top(t)\mathbb{E}^{v_1,v_2}\left[p_2(t)\mid\mathcal{F}_t^2\right]\right)dt - z^{v_1,v_2}(t)dW(t), \quad t\in[0,T], \\ dp_i(t) = \left\{A^\top(t)p_i(t) + \mathbb{E}^{\mathcal{F}_t}\left[\overline{A}^\top(t)p_{i\delta+}(t)\right]\right\}dt + B^\top(t)p_i(t)dW(t), \quad t\in[0,T],\ (i=1,2), \\ y^{v_1,v_2}(T) = \xi,\ y^{v_1,v_2}(t) = \psi(t), \quad t\in[-\delta,0], \\ p_i(0) = N_i y(0),\ p_i(t) = 0, \quad t\in[T,T+\delta],\ (i=1,2). \end{cases} \quad (48)$$

6. Conclusion

In this research, we have explored a class of partially observed game problem of the backward stochastic system with delay. More specially, based on the convex variational method, we establish the necessary and sufficient conditions with regard to Nash equilibrium in our game issue. The theoretical results of this paper are applied to an LQ game, for which the unique equilibrium point is expressed explicitly. On account of that the LQ model is usually used to depict many financial and economic phenomena, we expect that our LQ game result of BSDDEs can be widely used in these fields. As far as we know, the partially observed nonzero-sum backward game problem with the time-delay generator is firstly investigated in our paper.

Notwithstanding that we are committed to the above game problem, we are likewise able to progress some consequences of optimal control for BSDDEs, for example, [14, 34].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] J. Von Neumann and O. Morgenstern, *The Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ, USA, 1944.

[2] J. Nash, "Non-cooperative games," *The Annals of Mathematics*, vol. 54, no. 2, pp. 286–295, 1951.

[3] W. Yu, Y. Huang, and C. Cui, "The absolute ruin insurance risk model with a threshold dividend strategy," *Symmetry*, vol. 10, no. 9, p. 377, 2018.

[4] W. Yu, Y. Yong, G. Guan, Y. Huang, W. Su, and C. Cui, "Valuing guaranteed minimum death benefits by cosine series expansion," *Mathematics*, vol. 7, no. 9, p. 835, 2019.

[5] W. Yu, F. Wang, Y. Huang, and H. Liu, "Social optimal mean field control problem for population growth model," *Asian Journal of Control*, pp. 1–8, 2019.

[6] Q. Zhu, Y. Shi, and B. Teng, "Forward-backward doubly stochastic differential equations with random jumps and related games," *Asian Journal of Control*, pp. 1–16, 2020.

[7] W. Yu, P. Guo, Q. Wang et al., "On a periodic capital injection and barrier dividend strategy in the compound Poisson risk model," *Mathematics*, vol. 8, no. 4, p. 511, 2020.

[8] Z. Zhang, Y. Yong, and W. Yu, "Valuing equity-linked death benefits in general exponential Lévy models," *Journal of Computational and Applied Mathematics*, vol. 365, Article ID 112377, 2020.

[9] X. Peng, W. Su, W. Su, and Z. Zhang, "On a perturbed compound Poisson risk model under a periodic threshold-type dividend strategy," *Journal of Industrial & Management Optimization*, vol. 16, no. 4, pp. 1967–1986, 2020.

[10] Z. Yu and S. Ji, "Linear-quadratic non-zero sum differential game of backward stochstic differential equations," in *Proceedings of the 27th Chinese Control Conference*, Kunming, China, July 2008.

[11] G. Wang and Z. Yu, "A Pontryagin's maximum principle for nonzero sum differential games of BSDEs with applications," *IEEE Transactions on Automatic Control*, vol. 55, pp. 1742–1747, 2010.

[12] L. Chen and Z. Wu, "Maximum principle for the stochastic optimal control problem with delay and application," *Automatica*, vol. 46, no. 6, pp. 1074–1080, 2010.

[13] L. Chen and Z. Wu, "A type of general forward-backward stochastic differential equations and applications," *Chinese Annals of Mathematics, Series B*, vol. 32, no. 2, pp. 279–292, 2011.

[14] L. Chen and J. Huang, "Stochastic maximum principle for controlled backward delayed system via advanced stochastic differential equation," *Journal of Optimization Theory and Applications*, vol. 167, no. 3, pp. 1112–1135, 2015.

[15] M. Tang and Q. Zhang, "Optimal variational principle for backward stochastic control systems associated with Lévy processes," *Science China Mathematics*, vol. 55, no. 4, pp. 745–761, 2012.

[16] Q. Meng and P. Shi, "Stochastic optimal control for backward stochastic partial differential systems," *Journal of Mathematical Analysis and Applications*, vol. 402, no. 2, pp. 758–771, 2013.

[17] Y. Shen, Q. Meng, and P. Shi, "Maximum principle for mean-field jump-diffusion stochastic delay differential equations and its application to finance," *Automatica*, vol. 50, no. 6, pp. 1565–1579, 2014.

[18] Q. Meng and Y. Shen, "Optimal control of mean-field jump-diffusion systems with delay: a stochastic maximum principle approach," *Journal of Computational and Applied Mathematics*, vol. 279, pp. 13–30, 2015.

[19] Q. Meng and Y. Shen, "Optimal control for stochastic delay evolution equations," *Applied Mathematics & Optimization*, vol. 74, no. 1, pp. 53–89, 2016.

[20] M. Tang and Q. Meng, "Linear-quadratic optimal control problems for mean-field stochastic differential equations with jumps," *Asian Journal of Control*, vol. 21, pp. 1–15, 2019.

[21] Y. Dong and Q. Meng, "Second-order necessary conditions for optimal control with recursive utilities," *Journal of Optimization Theory and Applications*, vol. 182, no. 2, pp. 494–524, 2019.

[22] J. Xu, "Stochastic maximum principle for delayed doubly stochastic control systems and their applications," *International Journal of Control*, vol. 93, no. 6, pp. 1371–1380, 2020.

[23] L. Chen and Z. Yu, "Maximum principle for nonzero-sum stochastic differential game with delays," *IEEE Transactions on Automatic Control*, vol. 60, no. 5, pp. 1422–1426, 2015.

[24] J. Shi and G. Wang, "A nonzero sum differential game of BSDE with time-delayed generator and applications," *IEEE Transactions on Automatic Control*, vol. 61, no. 7, pp. 1959–1964, 2016.

[25] Y. Zhuang, "Non-zero sum differential games of anticipated forward-backward stochastic differential delayed equations under partial information and application," *Advances in Difference Equations*, vol. 2017, no. 1, p. 383, 2017.

[26] Z. Wu, "A maximum principle for partially observed optimal control of forward-backward stochastic control systems," *Science China Information Sciences*, vol. 53, no. 11, pp. 2205–2214, 2010.

[27] J. T. Shi and Z. Wu, "Maximum principle for partially-observed optimal control of fully-coupled forward-backward stochastic systems," *Journal of Optimization Theory and Applications*, vol. 145, no. 3, pp. 543–578, 2010.

[28] G. Wang and Z. Wu, "The maximum principle for stochastic recursive optimal control problems under partial information," *IEEE Transactions on Automatic Control*, vol. 54, pp. 1230–1242, 2009.

[29] G. C. Wang and Z. Wu, "General maximum principles for partially observed risk-sensitive optimal control problems and applications to finance," *Journal of Optimization Theory and Applications*, vol. 141, no. 3, pp. 677–700, 2009.

[30] H. Xiao, "The maximum principle for partially observed optimal control of forward-backward stochastic systems with random jumps," *Journal of Systems Science and Complexity*, vol. 24, no. 6, pp. 1083–1099, 2011.

[31] G. Wang, Z. Wu, and J. Xiong, "Maximum principles for forward-backward stochastic control systems with correlated state and observation noises," *SIAM Journal on Control and Optimization*, vol. 51, no. 1, pp. 491–524, 2013.

[32] Y. Shi and Q. Zhu, "Partially observed optimal controls of forward-backward doubly stochastic systems," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 19, no. 3, pp. 828–843, 2013.

[33] G. Wang and Z. Yu, "A partial information non-zero sum differential game of backward stochastic differential equations with applications," *Automatica*, vol. 48, no. 2, pp. 342–352, 2012.

[34] S. Wu and L. Shu, "Non-zero sum differential games of backward stochastic differential delay equations under partial information," *Asian Journal of Control*, vol. 19, no. 1, pp. 316–324, 2017.

[35] B. Yang, T. Guo, and J. Wu, "A partially observed nonzero-sum stochastic differential game with delays and its application to finance," *Asian Journal of Control*, vol. 21, pp. 1–12, 2019.

[36] J. Xiong, S. Zhang, and Y. Zhuang, "A partially observed non-zero sum differential game of forward-backward stochastic differential equations and its application in finance," *Mathematical Control & Related Fields*, vol. 9, no. 2, pp. 257–276, 2019.