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Research Article

Study of Dynamical Behavior and Stability of Iterative Methods for Nonlinear Equation with Applications in Engineering

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In this article, we first construct a family of optimal 2-step iterative methods for finding a single root of the nonlinear equation using the procedure of weight function. We then extend these methods for determining all roots simultaneously. Convergence analysis is presented for both cases to show that the order of convergence is 4 in case of the single-root finding method and is 6 for simultaneous determination of all distinct as well as multiple roots of a nonlinear equation. The dynamical behavior is presented to analyze the stability of fixed and critical points of the rational operator of one-point iterative methods. The computational cost, basins of attraction, efficiency, log of the residual, and numerical test examples show that the newly constructed methods are more efficient as compared with the existing methods in the literature.

1. Introduction

To solve the nonlinear equation

$$f(s) = 0, \tag{1}$$

is the oldest problem of engineering in general and in mathematics in particular. These nonlinear equations have diverse applications in many areas of science and engineering. To find the roots of (1), we look towards iterative schemes, which can be classified as to approximate single root and all roots of (1). In this article, we are going to work on both types of iterative schemes. A lot of iterative methods of different convergence orders already exist in the literature (see [1–11]) to approximate the roots of (1). Ostrowski [7] defined efficiency index I of these iterative methods in terms of their convergence order k and the number of function evaluations per iteration, say u, i.e.,

$$I = k^{1/u}. \tag{2}$$

An iterative method is said to be optimal according to Kung–Traub conjecture [1] if

$$k = 2^{u-1}, \tag{3}$$

holds. The aforementioned methods are used to approximate one root at a time. However, mathematicians are also interested in finding of all roots of (1) simultaneously. This is due to the fact that simultaneous iterative methods are very popular due to their wider region of convergence, are more stable as compared to single-root finding methods, and implemented for parallel computing as well. More detail on single as well as simultaneous determination of all roots can be found in [1, 12–24] and references cited therein.

The most famous of the single-root finding method is the classical Newton–Raphson method:

$$s_{i+1} = s_i - \frac{f(s_i)}{f'(s_i)}, \quad i = 1, 2, \ldots. \tag{4}$$

Method (4) requires one evaluation of the function and one of its first derivative to achieve optimal order 2 having efficiency 1.41 using the Traub conjecture. Using Weierstrass' correction [15],

$$\frac{f(s_i)}{f'(s_i)} = w(s) = \frac{f(s_i)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(s - s_j)}, \tag{5}$$

in (4), we get the classical Weierstrass—Dochive method to approximate all roots of nonlinear equation (1) given as

$$s_{i+1} = s_i - \frac{f(s_i)}{\prod_{\substack{j=1\\ j\neq i}}^{n}(s_i - s_j)}. \tag{6}$$

Method (6) has convergence order 2. Later, Aberth-Ehrlich [14] presented the 3rd-order simultaneous method given as

$$s_{i+1} = s_i - \frac{1}{(1/N(s_i)) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/(s_i - s_j)\right)}, \tag{7}$$

where $N(s_i) = f'(s_i)/f(s_i)$.

The main aim of this paper is to construct the family of optimal fourth-order single-root finding methods using the procedure of weight function and then convert them into simultaneous iterative methods for finding all distinct as well as multiple roots of nonlinear equation (1). Using the complex dynamical system, we will be able to choose those values of parameters of iterative methods which give a wider convergence area on initial approximations.

2. Construction of the Method and Convergence Analysis

King [25] presented the following optimal fourth-order method (abbreviated as MM1):

$$\begin{aligned} y_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right), \\ z_i &= y_i - \frac{f(s_i) + \beta f(y_i)}{f(s_i) + (\beta - 2) f(y_i)} \frac{f(y_i)}{f'(s_i)}. \end{aligned} \tag{8}$$

Chun [26] gave the fourth-order optimal method as (abbreviated as MM2)

$$\begin{aligned} y_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right), \\ z_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(1 + u + 2u^2\right), \\ \text{where } u &= \left(\frac{f(y_i)}{f(s_i)}\right). \end{aligned} \tag{9}$$

Cordero et al. [3] proposed the fourth-order optimal method as (abbreviated as MM3)

$$\begin{aligned} y_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right), \\ v_i &= s_i - \frac{f(s_i) + f(y_i)}{f'(s_i)}, \\ z_i &= v_i - \frac{(f(y_i))^2 (2f(s_i) + f(y_i))}{(f(s_i))^2 f'(s_i)}. \end{aligned} \tag{10}$$

Chun [27] introduced the fourth-order optimal method as (abbreviated as MM4)

$$\begin{aligned} y_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right), \\ z_i &= s_i - \left(\frac{fs_i)}{f's_i)}\right) - \left(\frac{fs_i)}{f's_i)}\right)\left(\frac{f(s_i)}{f(s_i) - f(y_i)}\right). \end{aligned} \tag{11}$$

Here, we propose the following family of iterative methods:

$$\begin{aligned} y_i &= s_i - \left(\frac{f(s_i)}{f'(s_i)}\right), \\ z_i &= y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\Gamma(u), \\ \text{where } u &= \frac{f(y_i)}{f(s_i)}. \end{aligned} \tag{12}$$

For iterative scheme (12), we have the following convergence theorem:

Theorem 1. *Let $\zeta \in I$ be a simple root of a sufficiently differential function $f: I \subseteq \mathfrak{R} \longrightarrow \mathfrak{R}$ in an open interval I. If s_0 is sufficiently close to ζ and Γ be a real-valued function satisfying $\Gamma(0) = 0, \Gamma'(0) = 1, \Gamma''(0) = 4$, and $\Gamma'''(0) < \infty$, then the convergence order of the family of iterative method (12) is 4 and satisfies the following error equation:*

$$e_{i+1} = \left(5c_2^3 - c_2c_3 - \frac{1}{6}\Gamma'''(0)c_2^3\right)e_i^4 + O\left(e_i^5\right), \tag{13}$$

where $c_m = f^m(\zeta)/m!f'(\zeta); m \geq 2$.

Proof. Let ζ be a simple root of f, and $s_i = \zeta + e_i$. By Taylor's series expansion of $f(s_i)$ about $s = \zeta$ taking $f(\zeta) = 0$, we get

$$f(s_i) = f'(\zeta)\left(e_i + c_2e_i^2 + c_3e_i^3 + c_4e_i^4 + O\left(e_i^5\right)\right), \tag{14}$$

$$f'(s_i) = f'(\zeta)\left(1 + 2c_2e_i + 3c_3e_i^2 + O\left(e_i^3\right)\right) \tag{15}$$

Dividing (14) by (10), we have

$$\frac{f(s_i)}{f'(s_i)} = e_i + c_2e_i^2 + \left(2c_2^2 - 4c_3\right)e_i^3 + O\left(e_i^4\right). \tag{16}$$

This gives

$$y_i = \zeta + c_2 e_i^2 + \left(-2c_2^2 + 2c_3\right)e_i^3 + \cdots$$
$$u = \left(\frac{f(y_i)}{f(s_i)}\right) = c_2 e_i + \left(-3c_2^2 + 2c_3\right)e_i^2 + \left(8c_2^3 - 10c_2c_3 + 3c_4\right)e_i^3 + \cdots. \tag{17}$$

Thus, using Taylor series, we have

$$\Gamma(u) = \Gamma(0) + u\Gamma'(0) + \frac{u^2}{2!}\Gamma''(0) + \cdots$$
$$\Gamma(u) = \Gamma(0) + c_2\Gamma'(0)e_i + \left\{\frac{1}{2}c_2^2\left(\Gamma''(0) - 3\Gamma'(0)\right) + 2c_3\Gamma\prime(0)\right\}e_i^2 + \cdots. \tag{18}$$

Now, taking $\Gamma q = 0, \Gamma(0) = 1$, and $\Gamma''(0) = 4$ in equation (12) and on simplification gives

$$e_{i+1} = \left(5c_2^3 - c_2c_3 - \frac{1}{6}\Gamma^{'''}(0)c_2^3\right)e_i^4 + O\left(e_i^5\right). \tag{19}$$

Hence, it proves fourth-order convergence. □

2.1. The Concrete Fourth-Order Family of Methods. We now construct some concrete forms of the family of fourth-order methods from the family of methods (12) by choosing weight function $\Gamma(u)$ containing an arbitrary real number β as provided in Table 1, satisfying the condition $\Gamma(0) = 0, \Gamma'(0) = 1, \Gamma''(0) = 4$ and $\Gamma^{'''}(0) < \infty$ of Theorem 1 with β as a real number. Therefore, we get the following new five families of iterative methods:

Method-1 (abbreviated as BB1):

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$
$$z_i = y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(u + 2u^2 + \beta u^4\right). \tag{20}$$

Method-2 (abbreviated as BB2):

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$
$$z_i = y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(\frac{(u)}{1 - 2(u) + \beta(u)^2}\right). \tag{21}$$

Method-3 (abbreviated as BB3):

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$
$$z_i = y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(u + +2u^2 + \beta u^3\right). \tag{22}$$

Method-4 (abbreviated as BB4):

Table 1: Concrete forms of methods.

S. no	β	$\Gamma(u), u = (f(y_i)/f(s_i))$
B1	$\Re$	$\Gamma(u) = u + 2u^2 + \beta u^4$
B2	$\Re$	$\Gamma(u) = u/1 - 2u + \beta u^2$
B3	$\Re$	$\Gamma(u) = u + 2u^2 + \beta u^3$
B4	$\Re$	$\Gamma(u) = (u/1 - \beta u^2) + 2u^2$
B5	$\Re$	$\Gamma(u) = ((u + 2u^2)/(1 - \beta u^3))$

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$
$$z_i = y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(\frac{u}{1 - \beta u^2} + 2u^2\right). \tag{23}$$

Method-5 (abbreviated as BB5):

$$y_i = s_i - \left(\frac{f(s_i)}{f'(s_i)}\right),$$
$$z_i = y_i - \left(\frac{f(s_i)}{f'(s_i)}\right)\left(\frac{u + 2u^2}{1 - \beta u^3}\right), \tag{24}$$

where $u = \left(\frac{f(y_i)}{f(s_i)}\right)$.

3. Construction of Simultaneous Methods

Suppose nonlinear equation (1) has n roots. Then, $f(s)$ and $f'(s)$ can be approximated as

$$f(s) = \prod_{j=1}^{n}\left(s - s_j\right) \text{ and } f'(s) = \sum_{k=1}^{n}\prod_{\substack{j\neq k\\ j=1}}^{n}\left(s - s_j\right). \tag{25}$$

This implies that

$$\frac{f'(s)}{f(s)} = \sum_{j=1}^{n}\left(\frac{1}{\left(s - s_j\right)}\right) = \frac{1}{\left(1/\left(s - s_i\right)\right) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/\left(s - s_j\right)\right)}. \tag{26}$$

Now, an approximation of $f(s_i)/f'(s_i)$ is formed by replacing s_j with z_{tj} as follows:

$$\frac{f(s_i)}{f'(s_i)} = \frac{1}{\left(1/N(s_i)\right) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/\left(s_i - z_{tj}\right)\right)}. \tag{27}$$

Using (27) in (4), we have

$$s_{i+1} = s_i - \frac{1}{\left(1/N(s_i)\right) - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(1/\left(s_i - z_{tj}\right)\right)}, \quad \text{for } t = 1, \ldots, 5. \tag{28}$$

Using corrections z_{tj} from BB1 to BB5, we get the following five simultaneous iterative methods for extracting all distinct as well as multiple roots of nonlinear equation (1):

$$s_{i+1} = s_i - \frac{\sigma_i}{(\sigma_i/N(s_i)) - \sum_{\substack{j=1 \\ j\neq i}}^{n}\left(\sigma_j/s_i - z_{tj}\right)}, \quad t = 1, \ldots, 5,$$

$$\text{where } z_{1j} = y_j - \left(\frac{f(s_j)}{f'(s_j)}\right)\left(u + 2u^2 + \beta u^4\right),$$

$$z_{2j} = y_j - \left(\frac{f(s_j)}{f'(s_j)}\right)\left(\frac{u}{1 - 2u + \beta u^2}\right),$$

$$z_{3j} = y_j - \left(\frac{f(s_j)}{f'(s_j)}\right)\left(u + 2u^2 + \beta u^3\right),$$

$$z_{4j} = y_j - \left(\frac{f(s_j)}{f'(s_j)}\right)\left(\frac{u}{1 - \beta u^2} + 2u^2\right),$$

$$z_{5j} = y_j - \left(\frac{f(s_j)}{f'(s_j)}\right)\left(\frac{u + 2u^2}{1 - \beta u^3}\right),$$

$$\text{where } y_j = s_j - \left(\frac{f(s_j)}{f'(s_j)}\right),$$

$$u_1 = \left(\frac{f(y_j)}{f(s_j)}\right). \tag{29}$$

Thus, we constructed new five simultaneous iterative methods (29) abbreviated as M1–M5.

3.1. Convergence Analysis. In this section, the convergence analysis of a family of simultaneous methods (M1–M5) is given in the form of the following theorem. Obviously, convergence for the methods (28) will follow from the convergence of the methods (29) from theorem (2) when the multiplicities of the roots are one.

Theorem 2. *Let $\zeta_1, \zeta_2, \ldots, \zeta_n$ be the simple roots of nonlinear equation (1). If $s_1^{(0)}, s_2^{(0)}, s_3^{(0)}, \ldots, s_n^{(0)}$. If be the initial approximations of the roots, respectively, and sufficiently close to actual roots, the order of convergence of method (58–62) is six.*

Proof. Let $\varepsilon_i = s_i - \zeta_i$and $\varepsilon_i' = s_{i+1} - \zeta_i$ be the errors in approximations s_i and s_{i+1}, respectively. Then, for distinct roots, we have

$$\frac{1}{N(s_i)} = \left(\frac{f'(s_i)}{f(s_i)}\right) = \sum_{j=1}^{n}\left(\frac{1}{(s_i - \zeta_j)}\right) = \frac{1}{(s_i - \zeta_i)} + \sum_{\substack{j\neq i \\ j=1}}^{n}\left(\frac{1}{(s_i - \zeta_j)}\right). \tag{30}$$

Thus, for multiple roots, we have from (29)

$$s_{i+1} = s_i - \frac{\sigma_i}{(\sigma_i/(s_i - \zeta_i)) + \sum_{j\neq i}^{n}\left(\sigma_j/(s_i - \zeta_j)\right) - \sum_{j\neq i}^{n}\left(\sigma_j/(s_i - z_{tj})\right)},$$

$$s_{i+1} - \zeta_i = s_i - \zeta_i - \frac{\sigma_i}{(\sigma_i/(s_i - \zeta_i)) + \sum_{j=1}^{n}\left(\sigma_j(s_i - z_{tj} - s_i + \zeta_j)/(s_i - \zeta_j)(s_i - z_{tj})\right)},$$

$$\text{this implies, } \varepsilon_i' = \varepsilon_i - \frac{\sigma_i\varepsilon_i}{\sigma_i + \varepsilon_i\sum_{j=1}^{n}\left(-\sigma_j(z_{tj} - \zeta_j)/(s_i - \zeta_j)(s_i - z_{tj})\right)},$$

$$= \varepsilon_i - \frac{\sigma_i\varepsilon_i}{\sigma_i + \varepsilon_i\sum_{j=1}^{n}\left(E_i\varepsilon_j^4\right)}, \tag{31}$$

$$\text{where } z_{tj} - \zeta_j = \varepsilon_j^4$$

$$E_i = \left(\frac{-\sigma_j}{(s_i - \zeta_j)(s_i - z_{tj})}\right).$$

$$\text{Thus, } \varepsilon_i' = \frac{\varepsilon_i^2\sum_{j=1}^{n}\left(E_i\varepsilon_j^4\right)}{\sigma_i + \varepsilon_i\sum_{j=1}^{n}\left(E_i\varepsilon_j^4\right)}.$$

If it is assumed that absolute values of all errors $\varepsilon_j\,(j = 1, 2, 3, \ldots)$ are of the same order as, say, $|\varepsilon_j| = O|\varepsilon|$, then from (30), we have

$$\varepsilon_i' = O(\varepsilon)^6. \tag{32}$$

Thus, (32) shows the convergence order of methods M1–M5 which is six. Hence, the theorem is proved. □

4. Complex Dynamical Study of Families of Iterative Methods

Here, we discuss stability of the family of iterative method (BB1) only in the background contexture of complex dynamics. Rational map arising due to iterative method (BB1) is

$$\mathfrak{R}_f = y - \left(\frac{f(s)}{f'(s)}\right)\left(u + 2u^2 + \beta u^4\right), \tag{33}$$

where $y = s - (f(s)/f'(s))$ and $u = (f(y)/f(s))$.

Recalling some basic concepts of this theory, detailed information can be found in [2, 4, 6, 8]. Taking a rational function $\mathfrak{R}_f\colon C \longrightarrow C$, where C denotes the Riemann sphere, the orbit $s_0 \in C$ defines a set such as $\mathrm{orb}(s) = \left\{s_0, \mathfrak{R}_f(s_0), \mathfrak{R}_f^2(s_0), \ldots, \mathfrak{R}_f^m(s_0), \ldots\right\}$. A point $s_0 \in \mathrm{C}$ is called a fixed point if $\mathfrak{R}_f(s_0) = s_0$. In particular, a fixed point s_0 is called the strange fixed point if $f(s^r) = 0$ when $s_0 \neq s^r$. A T-periodic point is defined as a point $s_T \in C$ satisfying $\mathfrak{R}^T(s_T) = s_T$ but $\mathfrak{R}\imath(s_T) \neq s_T$ for $t < T$. If s_0 is the fixed point of $\mathfrak{R}$, then it is

(i) Superattracting if $|\mathfrak{R}_f'(s_0)| = 0$

(ii) Attracting if $|\mathfrak{R}_f'(s_0)| < 1$

(iii) Repulsive if $|\mathfrak{R}_f'(s_0)| > 1$

(iv) Neutral if $|\mathfrak{R}_f'(s_0)| = 1$

(v) A strange fixed point if it not associated to any root of nonlinear equation (1)

An attracting point $s^* \in \mathrm{C}$ defines the basin of attraction, $\mathfrak{R}(s^*)$, as the set of starting points whose orbit tends to s^*.

Furthermore, the implementation of the dynamical plane of the rational operator corresponding to iterative methods divides the complex plane into a mesh of values of real part along the x-axis and imaginary on the y-axis. The initial estimates are depicted in color depending on where their orbit converges, and thus, basins of attraction of the corresponding iterative methods are obtained. The scaling theorem allows the suitable change of the coordinate to reduce dynamics of iteration of general maps to study the specific family of iteration of similar maps.

Theorem 3. *Let us take an analytic function $f(s)$ and $T(s) = \alpha s + \beta$ be an affine map, with $\alpha \neq 0$. Take $g(s) = goT(s)$; then, $To\mathfrak{R}_g oT^{-1} = \mathfrak{R}_f(s)$, i.e., $\mathfrak{R}_g = \mathfrak{R}_f$ affine conjugate by T (Scaling theorem).*

As iterative method (33), holds scaling theorem and thus allows the dynamic studies of iterative function (BB1) for the polynomial $f(s) = (s-a)(s-b)$, where $a, b \in \mathfrak{R}$. One-point iterative method (BB1) has a universal Julia set if a rational map exists which conjugates by Mobius transformation.

Theorem 4. *For a rational map $\Omega_g(s)$ arising from (33) applied to $f(s) = (s-a)(s-b)$, where $a, b \in \mathfrak{R}, \Omega_g(s)$ is conjugate via Möbius transformation by $M(v) = (v-a)/(v-b)$ to*

$$\Omega_g(s,\beta) = \frac{s^4\left(-s^6 - 8s^5 - 27s^4 + A1\right)}{-5s^6 + \beta s^5 - 24s^5 - 47s^4 + A2}, \tag{34}$$

where $A_1 = -48s^3 + \beta s - 47s^2 - 24s - 5$ and $A_2 = -48s^3 - 27s^2 - 8s - 1$.

Proof. Let $f(s) = (s-a)(s-b)$, where $a, b \in \mathfrak{R}$. Möbius transformation is given by

$$M(v) = \frac{v-a}{v-b}, \qquad \text{with inverse } [M(v)]^{-1} = \frac{vb-a}{v-1}, \tag{35}$$

which is considered as the map from $\mathrm{C} \cup \infty$. Then, we have

$$\Omega_g(s,\beta) = \frac{s^4\left(-s^6 - 8s^5 - 27s^4 + A_1\right)}{-5s^6 + \beta s^5 - 24s^5 - 47s^4 + A_2}, \tag{36}$$

where $A_1 = -48s^3 + \beta s - 47s^2 - 24s - 5$ and $A_2 = -48s^3 - 27s^2 - 8s - 1$.

Similarly, we can get the following conclusions. □

Theorem 5. *For a rational map $\Omega_g(s)$ arising from (BB2) to (BB5) applied to $f(s) = (s-a)(s-b)$, where $a, b \in \mathfrak{R}$, $\Omega_g(s)$ is conjugate via Möbius transformation by $M(v) = (v-a)/(v-b)$ to the following:*

$$\Omega_g(s,\beta) = \frac{s^4\left(s^2 + \beta + 2s + 1\right)}{(\beta+1)s^2 + 2s + 1}, \tag{37}$$

$$\Omega_g(s,\beta) = \frac{s^4\left(-s^4 - 6s^3 - 14s^2 + \beta - 14s - 5\right)}{(\beta-5)s^4 - 14s^3 - 14s^2 - 6s - 1}, \tag{38}$$

$$\Omega_g(s,\beta) = \frac{s^4\left(-s^6 - 8s^5 + (\beta-27)s^4 + B1\right)}{(\beta-5)s^6 + (2\beta-24)s^5 + B2}, \tag{39}$$

where $B_1 = -48s^3 + (6\beta-47)s^2 + (2\beta-24)s - 5$ and $B_2 = (6\beta-47)s^4 + (4\beta-48)s^3 + (\beta-27)s^2 - 8s - 1$.

$$\Omega_g(s,\beta) = \frac{s^4\left(-s^4 - 6s^3 + \beta s - 14s^2 - 14s - 5\right)}{-5s^4 + (\beta-14)s^3 - 14s^2 - 6s - 1}. \tag{40}$$

Theorem 6. *For a rational map $\Omega_g(s)$arising from (MM1) to (MM4) applied to $f(s) = (s-a)(s-b)$, where $a, b \in \mathfrak{R}$, $\Omega_g(s)$ is conjugate via Möbius transformation by $M(v) = (v-a)/(v-b)$ to the following:*

$$\Omega_g(s,\beta) = \frac{s^4\left(s^2+(\beta+2)s+2\beta+1\right)}{(2\beta+1)s^2+(\beta+2)s+1}, \tag{41}$$

$$\Omega_g(s,\ \beta) = \frac{s^4\left(s^2+4s+5\right)}{5s^2+4s+1}. \tag{42}$$

$$\Omega_g(s,\beta) = \frac{s^4\left(s^4+6s^3+14s^2+14s+4\right)}{4s^4+14s^3+14s^2+6s+1}, \tag{43}$$

$$\Omega_g(s,\beta) = \frac{s^4\left(s^2+2s+2\right)}{2s^2+2s+2}. \tag{44}$$

The unions of the respective stability functions of all the strange fixed points of rational maps (36)–(41) are graphed in Figures 1(a)–1(f).

The fixed points for rational function (36) are $s = 1, s = 0$, and $s = \infty$. For the stability of fixed points of iterative method (36), we calculate $\Omega_g'(s,\beta)$, i.e.,

$$\Omega_g'(s,\beta) = \frac{s^3(s+1)^8\left(A_3\right)}{\left(A_4\right)^2}, \tag{45}$$

where $A_3 = -20s^4 + (5\beta-80)s^3 + (-8\alpha^2-120)s^2 + (5\beta+80)s - 20$ and $A_4 = -\ 5s^6 + (\beta-24)s^5 - 47s^4 - 48s^3 - 27s^2 - 8s - 1$.

It is evident from (36) that $s = 0$ and ∞ are always superattractive fixed points, but stability of other fixed points depends on the value of parameter β which is present here. The operator $\Omega_g'(s,\beta)$ for $s = 1$ gives

$$\left|\Omega_g'(1,\beta)\right| = \left|\frac{512}{\beta-160}\right|. \tag{46}$$

Analyzing (46), as $\beta \longrightarrow \pm\infty$, we obtained horizontal asymptotes in $|\Omega_g'(1,\beta)| = 0$.

In the following result, we present the stability of the strange fixed point for $s = 1$.

Theorem 7. *The character of the strange point $s = 1$ is as follows:*

(i) If $|512/\beta - 160| < 1$, then $s = 1$ is an attractor, and it can be superattractor for $\beta = 1$

(ii) When $|512/\beta - 160| = 1$, $s = 1$ is a parabolic point

(iii) If $\beta \neq 0$ and $|512/\beta - 160| > 1$, then $s = 1$ is repulsive

Proof. From (45), we have

$$\Omega_g'(1,\beta) = \frac{512}{\beta-160}. \tag{47}$$

So,

$$\left|\frac{512}{\beta-160}\right| \leq 1 \text{ is equivalent to } |512| \leq |\beta-160|. \tag{48}$$

Let us consider $\beta = a + ib$ as an arbitrary complex number. Then, $(a-160)^2 + b^2 \geq 512^2$. Therefore, $|\Omega_g'(1,\beta)| \leq 1 \text{iff} |\beta - 160 \geq 512|$. Finally, if β varies, then $|\Omega_g'(1,\beta)| > 1$ and $s = 1$ is repulsive, except if $\beta = 1$ and $\beta = 160$ for which $s = 1$ is not a fixed point.

The functions where we examine stability of iterative method (36) are given as

$$\Omega_g'(1,\beta) = \min\left\{\Omega_g'(1,\beta), 1\right\}. \tag{49}$$

□

4.1. Analysis of Critical Points. The critical points of (36) satisfy $\Omega_g'(s,\beta) = 0$, i.e., $s = 0, -1$ and $s = \infty$ for $\beta \neq 0$, and

$$\begin{aligned}
\text{cr1} &= \frac{1}{16}(-16+\beta) - \frac{\sqrt{\beta(5\beta-288)}}{16\sqrt{5}} - \frac{\sqrt{\beta_1}}{8\sqrt{10}},\\
\text{cr2} &= \frac{1}{16}(-16+\beta) - \frac{\sqrt{\beta(5\beta-288)}}{16\sqrt{5}} + \frac{\sqrt{\beta_1}}{8\sqrt{10}},\\
\text{cr3} &= \frac{\beta-16}{16} - \frac{\sqrt{\beta(5\beta-288)}}{16\sqrt{5}} - \frac{\sqrt{\beta_2}}{2},\\
\text{cr4} &= \frac{\beta-16}{16} - \frac{\sqrt{\beta(5\beta-288)}}{16\sqrt{5}} + \frac{\sqrt{\beta_2}}{2},
\end{aligned} \tag{50}$$

where $\beta_1 = -224\beta + 5\beta^2 - (4608\sqrt{5\beta}/\sqrt{5\beta-288}) + (268\sqrt{5}\,\beta^{2/3}/\sqrt{5\beta-288}) - (5\sqrt{5}\,\beta^{5/2}/\sqrt{5\beta-288})$, and $\beta_2 = -2 + ((16-\beta)^2/32) - (30+2\beta)/5 + (\beta\bullet/\sqrt{5\beta-288})$, where $\beta\bullet = 2\sqrt{5}((\beta-32) - ((16-\beta)^2/64) + (2(16-\beta)(15+\beta)/5))$. We observe cr1 = 1/cr2 and cr3 = 1/cr4. Figure 2 presents the zones of stability of strange fixed points. Fixed points 1 and 0 are represented by black-dotted lines, while critical points $(-1, \text{cr1}, \text{cr2}, \text{cr3}, \text{cr4})$ are represented by black, red, blue, green and orange-dotted lines, respectively (see Figure 3).

Theorem 8. *The only member of the family of iterative methods whose operator is always conjugated to the rational map s^2 is the element corresponding to $\beta = 0$.*

Proof. From (51), we represent

$$\begin{aligned}
nn(s) &= s^4\left(-s^6 - 8s^5 - 27s^4 - 48s^3 + \beta s - 47s^2 - 24s - 5\right),\\
dd(s) &= -5s^6 + \beta s^5 - 24s^5 - 47s^4 - 48s^3 - 27s^2 - 8s - 1.
\end{aligned} \tag{51}$$

The unique value of β for which $nn(s) = dd(s)$ is 0. □

4.2. Parametric Planes. Parametric planes are obtained by taking β over a mesh of 500×500 values in the complex plane in $R_e(\beta) \times \text{Im}(\beta) \in [-4,4] \times [-4,4]$. A critical point is taken as the initial approximation. The method is then iterated until it reaches to the maximum iterations or converges to any fixed point. Taking 10^{-3} as a tolerance used for stopping criteria $|f(s_i)|$. The complex value of the parameter in the complex plane is paint in red if the method converges to any of the roots (0 and ∞), black in the other case.

In Figures 4(a)–4(d) and 5(a)–5(h), red color shows the convergence region, and all those values of parameter β show stable behavior, while those values of the

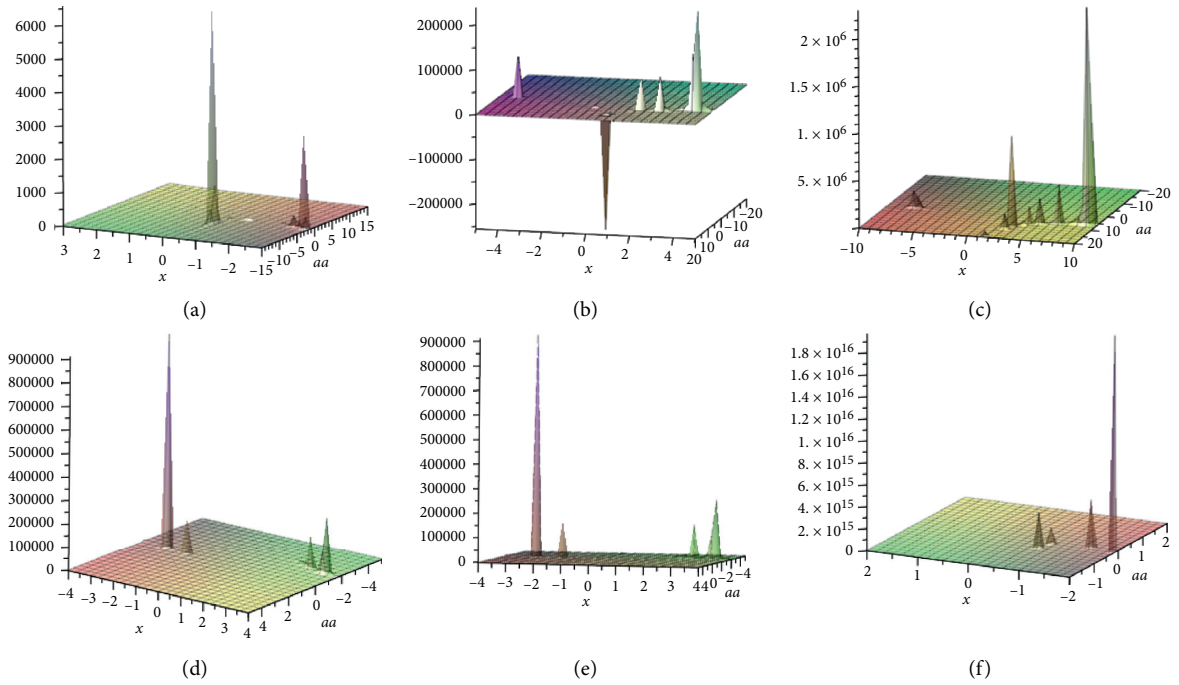

Figure 1: (a–f) Unions of the respective stability function of all strange fixed points of the rational map (BB1–BB5 and MM1), respectively.

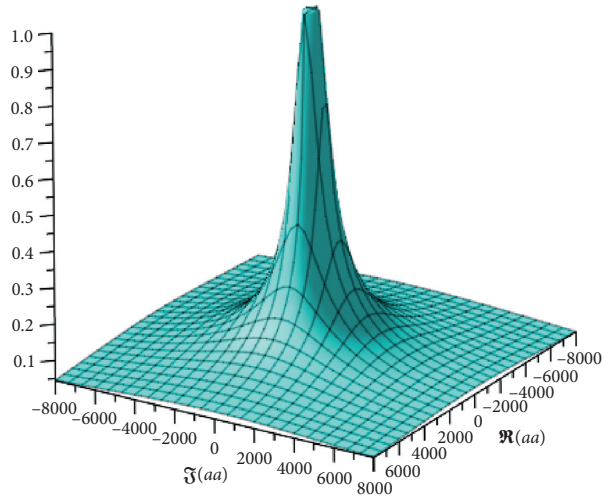

Figure 2: The zones of stability of strange fixed points of $\Omega'(1, \beta)$.

parameter which are taken from the black region (divergence region) show the unstable behavior of the iterative map (36). Stable and unstable behavior is shown in Figures 6 and 7, respectively.

4.3. Dynamical Planes. The generations of the dynamical planes are similar to the parametric plane. To execute them, the real and imaginary parts of the starting approximation are represented as two axes over a mesh of 250×250 in the complex plane. The stopping criteria are the same as in the parametric plane but assign different colors to indicate to which root the method converges and black in the other case.

Let us note that the iterative methods BB2–BB5, MM1–MM4, and BB1 satisfy Cayley's test [28–31] for all parameter values of β. It can be observed from Figure 8 that iterative methods BB1–BB5 and MM1–MM4 verifying Cayley's test [28] have the same dynamical properties as Newton's method. Figures 8(a)–8(e) clearly show quite stable behavior of iterative methods BB1–BB5 over MM1–MM4.

For unstable behavior, the value of parameter β is chosen from the black region. To generate basins of attractions, we take a square box of $[-1, 1] \times [-1, 1] \in \mathbb{C}$. To each root of (1), we assign color to which the corresponding orbit of iterative methods (BB1–BB5 and MM1–MM4) starts and convergences to a fixed point. Take color map as HSV. We use $|f(s_i)| < 10^{-3}$ as the stopping criteria, and the maximum iteration is taken as

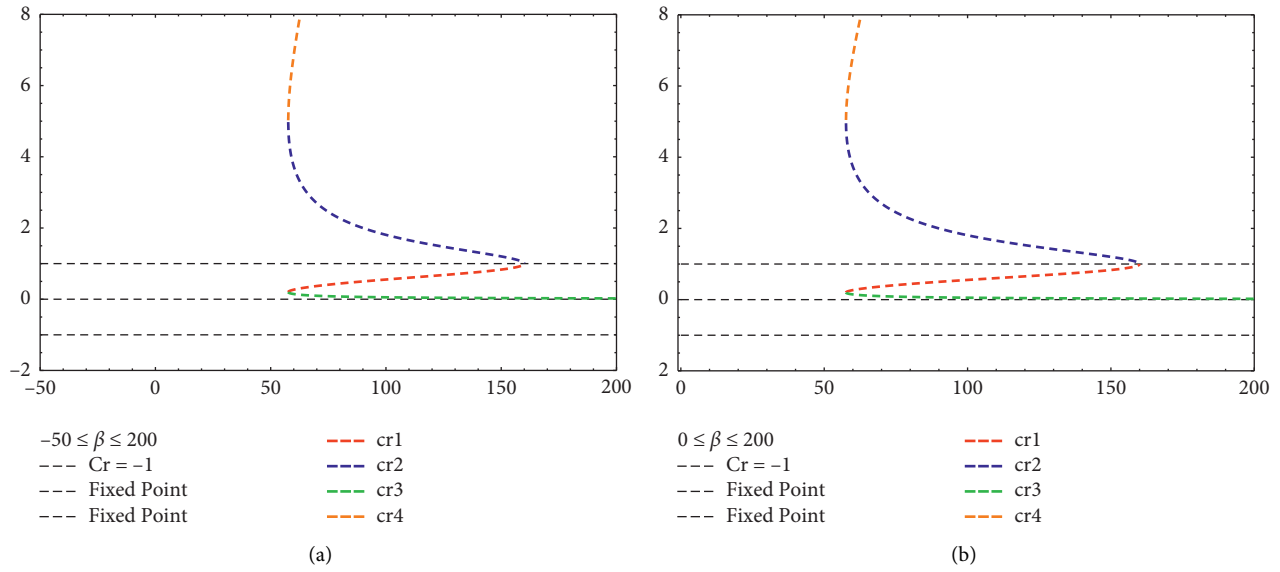

Figure 3: Dynamical behavior of strange fixed points and critical points for $0 \leq \beta \leq 20$ (a) and $-20 \leq \beta \leq 0$ (b).

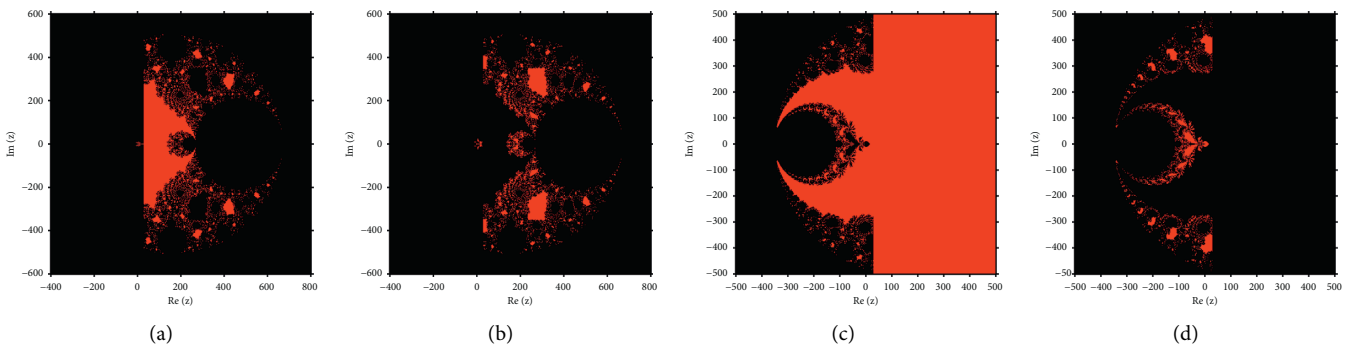

Figure 4: (a–d) Parametric planes associated with cr1, cr2, cr3, and cr4, respectively.

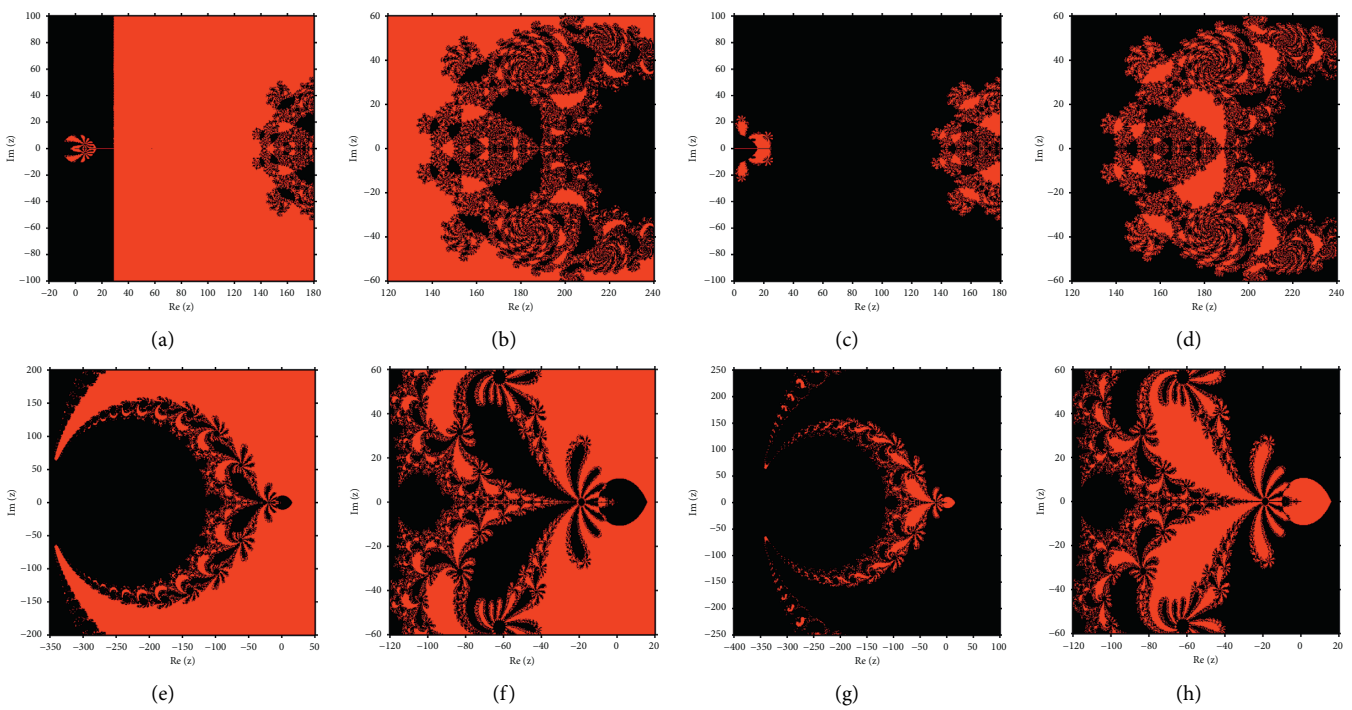

Figure 5: (a–h) Details of the parametric planes for cr1, cr2, cr3, and cr4.

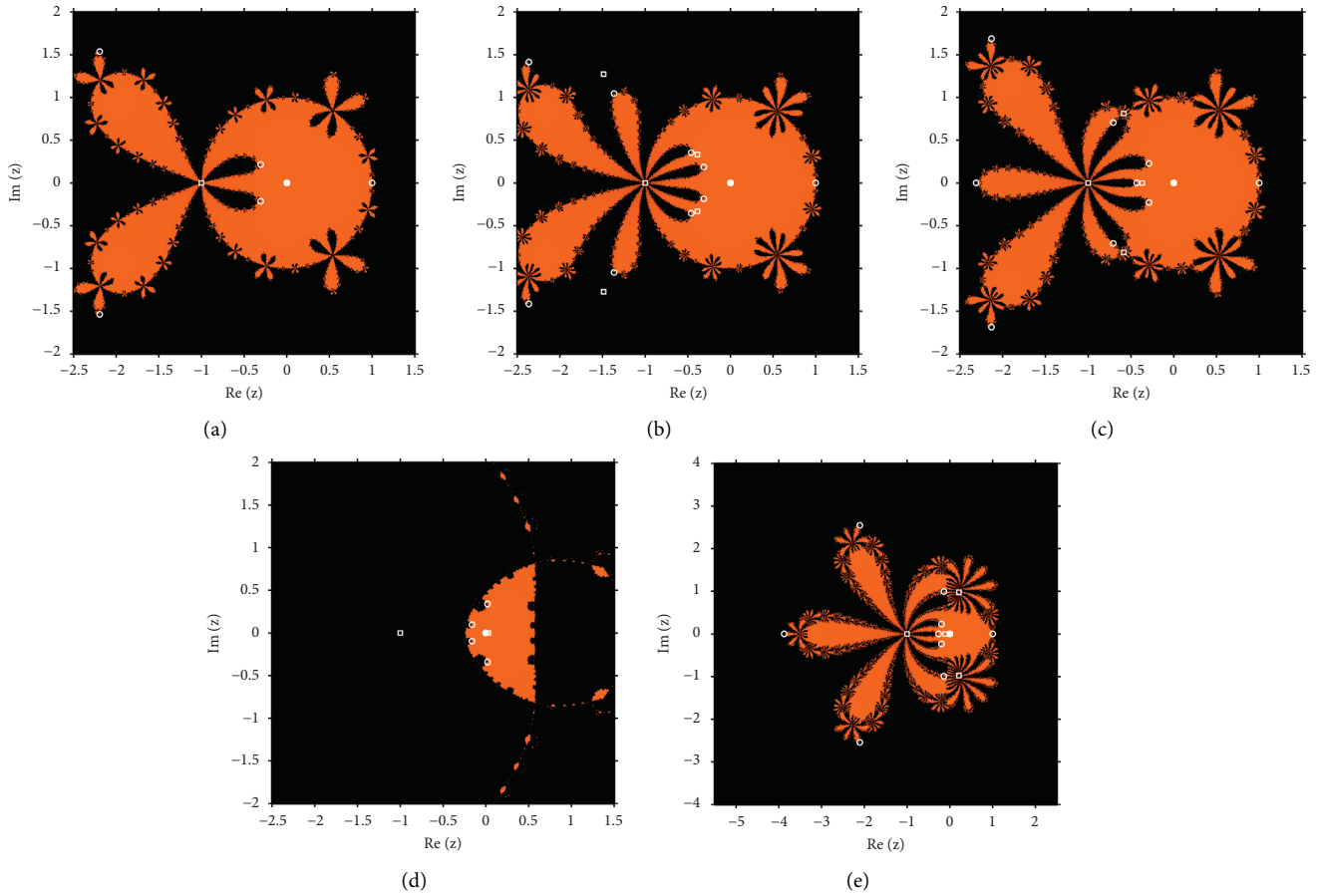

FIGURE 6: (a–e): Dynamical planes for stable behavior of iterative method (18) for different values of β, i.e., $\beta = 0, 1, -1, 160$, and -20, respectively.

30. We make dark red points if the orbit of the iterative methods does not converge to a root after 30 iterations. Different colors are used for different roots. Iterative methods have different basins of attractions distinguished by their color. In basins, brightness in color represents less number of iterations to achieve the roots of (1). Figures 9 and 10 show the basins of attractions of iterative methods (BB1–BB5 and MM1–MM4) for nonlinear function $f_1(s) = s^4 + 4s^3 - 24s^2 + 16s + 16$ and $f_2(s) = \sin((s-1)/2)\sin((s-2)/2)\sin((s-2.5)/2)$, respectively. From Figures 9 and 10, divergent regions and brightness in color present that BB1–BB5 are better than MM1–MM4, respectively.

5. Computational Aspect

Here, we compare the computational efficiency of M. S. Petkovi´c method [32] and the new methods (M1–M5) given by (28). As presented in [32], the efficiency of an iterative method can be estimated using the efficiency index given by

$$EL(m) = \frac{\log \mathbf{u}}{\mathbf{D}}, \tag{52}$$

where $\mathbf{D}$ is the computational cost and $\mathbf{u}$ is the order of convergence of the iterative method. Using arithmetic operation per iteration with certain weight depending on the execution time of operation, the computational cost $\mathbf{D}$ is evaluated. The weights used for division, multiplication, and addition plus subtraction are w_d, w_m, and w_{as}, respectively. For a given polynomial of degree m and n roots, the number of division, multiplication, addition and subtraction per iteration for all roots is denoted by D_m, M_m, and AS_m. The cost of computation can be calculated as

$$\mathbf{D} = \mathbf{D}(m) = w_{as}AS_m + w_mM_m + w_dD_m. \tag{53}$$

Thus, (52) becomes

$$EL(m) = \left(\frac{\log \mathbf{r}}{w_{as}AS_m + w_mM_m + w_dD_m}\right). \tag{54}$$

Reducing the number of operations of a complex polynomial of degree m with real and complex roots to operations of real arithmetic, as given in Table 2. Applying

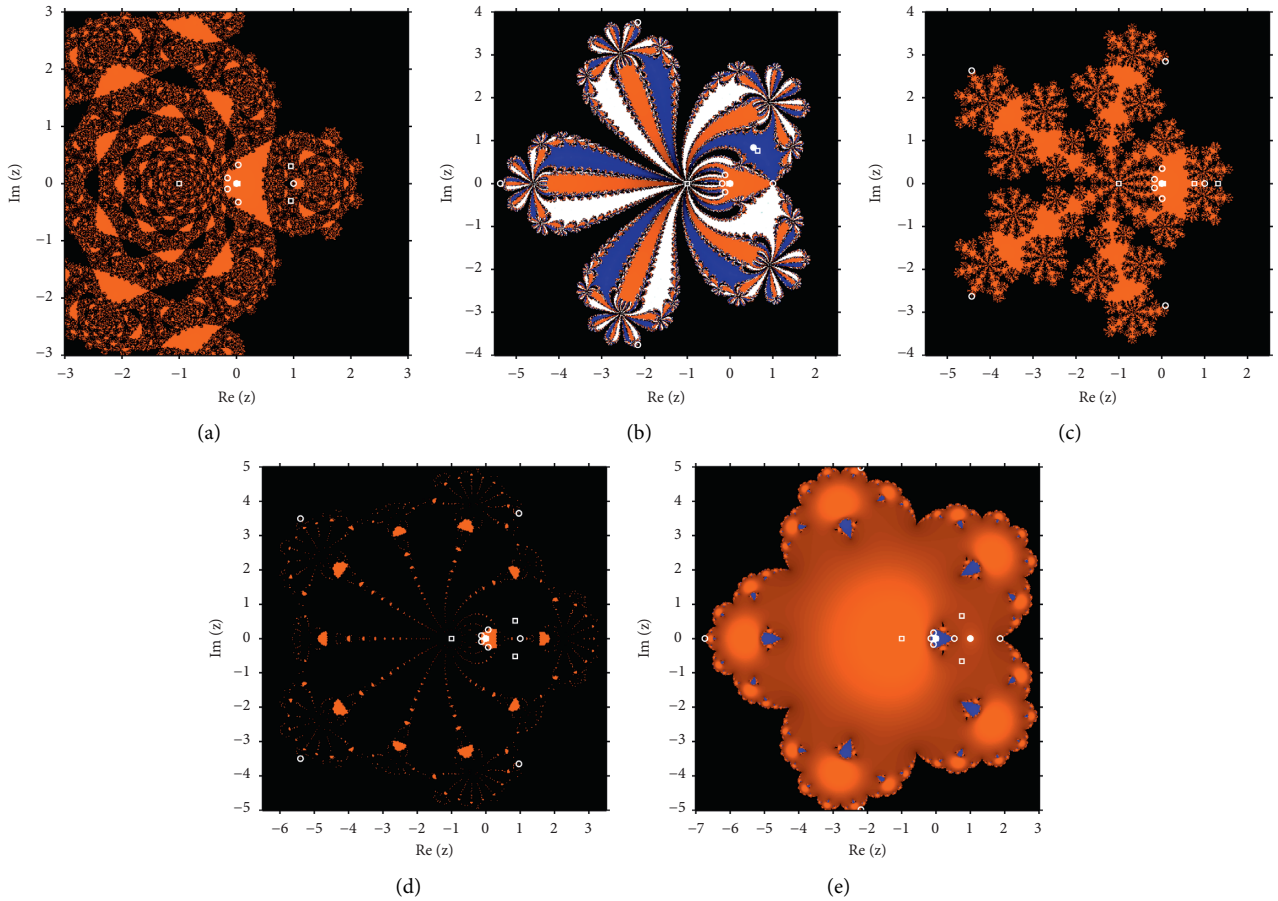

Figure 7: (a–e) Dynamical planes for unstable behavior of the iterative map (18) for different values of β, i.e., $\beta = 200, -140, 140, 500$, and -500, respectively.

(38) and data given in Table 2, we calculate the percentage ratio $\rho((M1 - M5), PJ6M)$ [3] which is given by

$$\rho(PJ6M, (M1 - M5)) = \left(\frac{EL(PJ6M)}{EL(M1 - M5)} - 1\right) \times 100,$$

$$\rho((M1 - M5), PJ6M) = \left(\frac{EL(M1 - M5)}{EL(PJ6M)} - 1\right) \times 100, \tag{55}$$

where $PJ6M$ is the Petkovic method [32] of order six. Figures 11(a)–11(d) graphically illustrate these percentage ratios. It is evident from Figures 11(a)–11(d) that the newly constructed simultaneous methods (M1–M5) are more efficient as compared with the Petkovic method [32].

We also calculate the CPU execution time as all the calculations are done using Maple 18 on Processor Intel(R) Core(TM) i3-3110m CPU@2.4 GHz with 64 bit Operating System. We observe that CPU time of the methods M1–M5 is less than M. S. Petkovi´c methods [32], showing the dominance efficiency of our methods (M1–M5) as compared to them.

6. Numerical Results

Here, some numerical examples are considered in order to demonstrate the performance of our family of two-step fourth-order single-root finding methods (BB1–BB5) and six-order simultaneous methods (M1–M5), respectively. We compare our family of optimal fourth-order single-root finding methods (BB1–BB5) with MM1–MM4 methods. Family of simultaneous methods (M1–M5) of convergence order six is compared with M. S. Petkovi´c method [32] of the same order (abbreviated as the PJ6M method). All the computations are performed using CAS Maple 18 with 2500 (64-digit floating-point arithmetic in case of simultaneous methods) significant digits. For single-root finding methods, the stopping criteria are as follows:

(i) $|f(s_i)| < \epsilon$,

(ii) $|s_i^{(k)} - \alpha| < \epsilon$,

whereas $e_i = \|f(s_i)\|_2$ for simultaneous methods.

We take $\epsilon = 10^{-600}$ for the single-root finding method and $\epsilon = 10^{-30}$ for simultaneous determination of all roots of nonlinear equation (1).

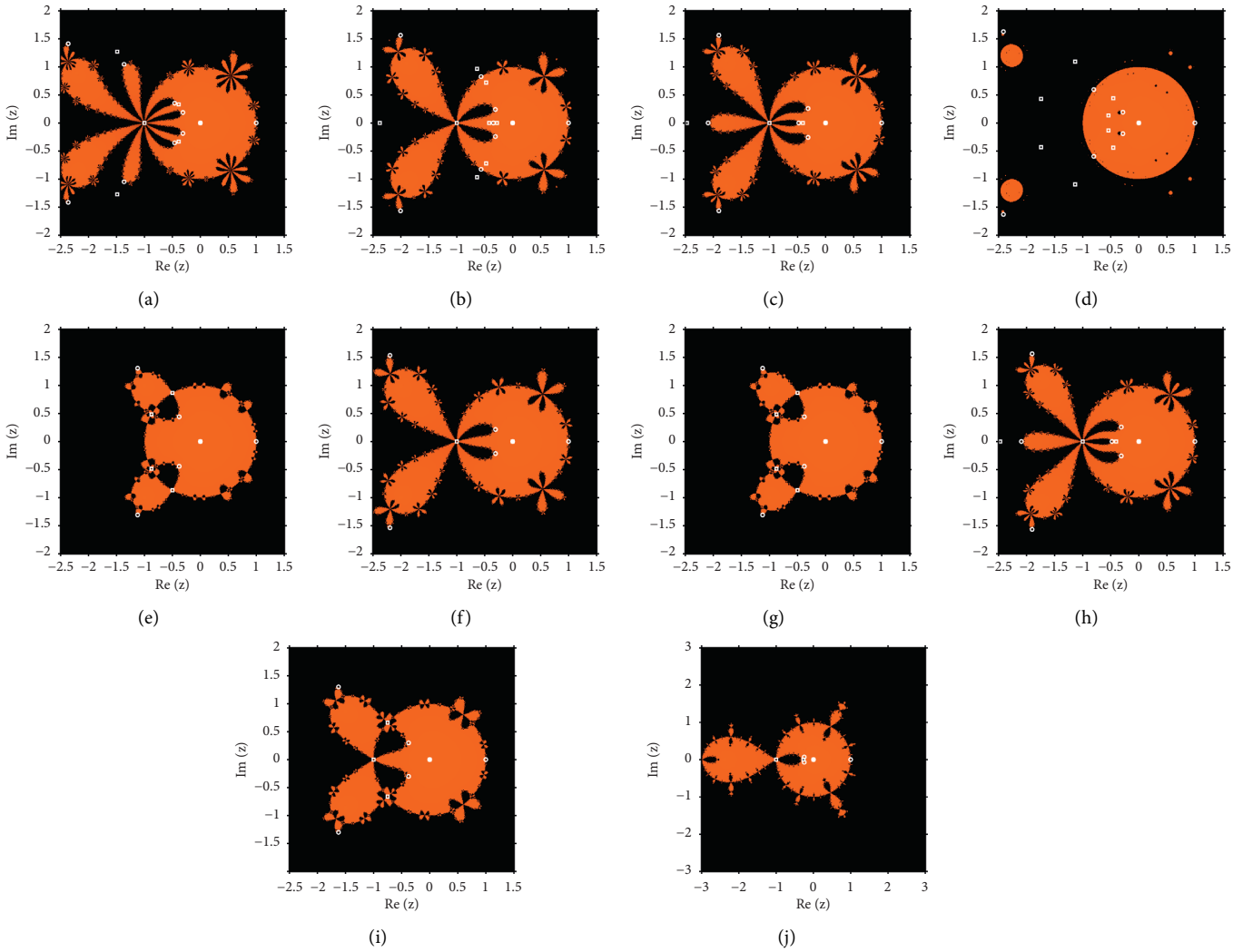

FIGURE 8: (a–i) Dynamical planes of iterative methods (BB1–BB5, MM2, MM4, MM3, and MM1) for $\beta = 1$. (j) Dynamical plane of method MM1 for $\beta = 3$.

Numerical test examples from [22, 33–35] are provided in Tables 3–12. In Tables 3, 4, 6, 7, 9, and 11, we present the numerical results for simultaneous determination of all roots while Tables 5, 8, 10, and 12 for single-root finding methods. In all tables, CO represents the convergence order, n, the number of iterations, ρ, the computational order of convergence, and CPU, the computational time in seconds. The value of arbitrary parameter β used in iterative methods BB1–BB5 and M1–M5 is 1.5 for test Examples 1–4. We observe that numerical results of single-root finding methods (BB1–BB5) as well as all-root finding methods (M1–M5) are better than MM1–MM4 and PJ6M, respectively, on the same number of iterations. Figures 12(a)–12(d) represent the residual fall for the iterative methods (M1–M5 and PJ6M), and Figures 12(e)–12(h) represent the residual fall of iterative methods (BB1–BB5 and MM1–MM4) for given nonlinear functions. Tables 3–12 show that the family of methods BB1–BB5 and M1–M5 are more efficient as compared with MM1–MM4 and PJ6M, respectively.

We also calculate the CPU execution time as all the calculations are done using Maple 18 on Processor Intel (R) Core (TM) i3-3110m CPU@2.4GHz with 64 bit Operating System. We observe from the tables that CPU time of the methods M1–M5 is better than the PJ6M method, showing the efficiency of our methods (M1–M5) as compared to them.

6.1. Applications in Engineering. In this section, we discuss the applications in engineering.

Example 1 (for the beam designing model, see [33]).

Consider a problem of beam positioning, resulting in a nonlinear function which is given as

$$f_1(s) = s^4 + 4s^3 - 24s^2 + 16s + 16 = (s-2)^2\left(s^2 + 8s + 4\right). \tag{56}$$

The exact roots of $f_1(s)$ are $\zeta_{1,2} = 2, \zeta_3 = -4 - 2\sqrt{3}, \zeta_3 = 2, \zeta_4 = -4 + 2\sqrt{3}$ as shown in Figure 13.

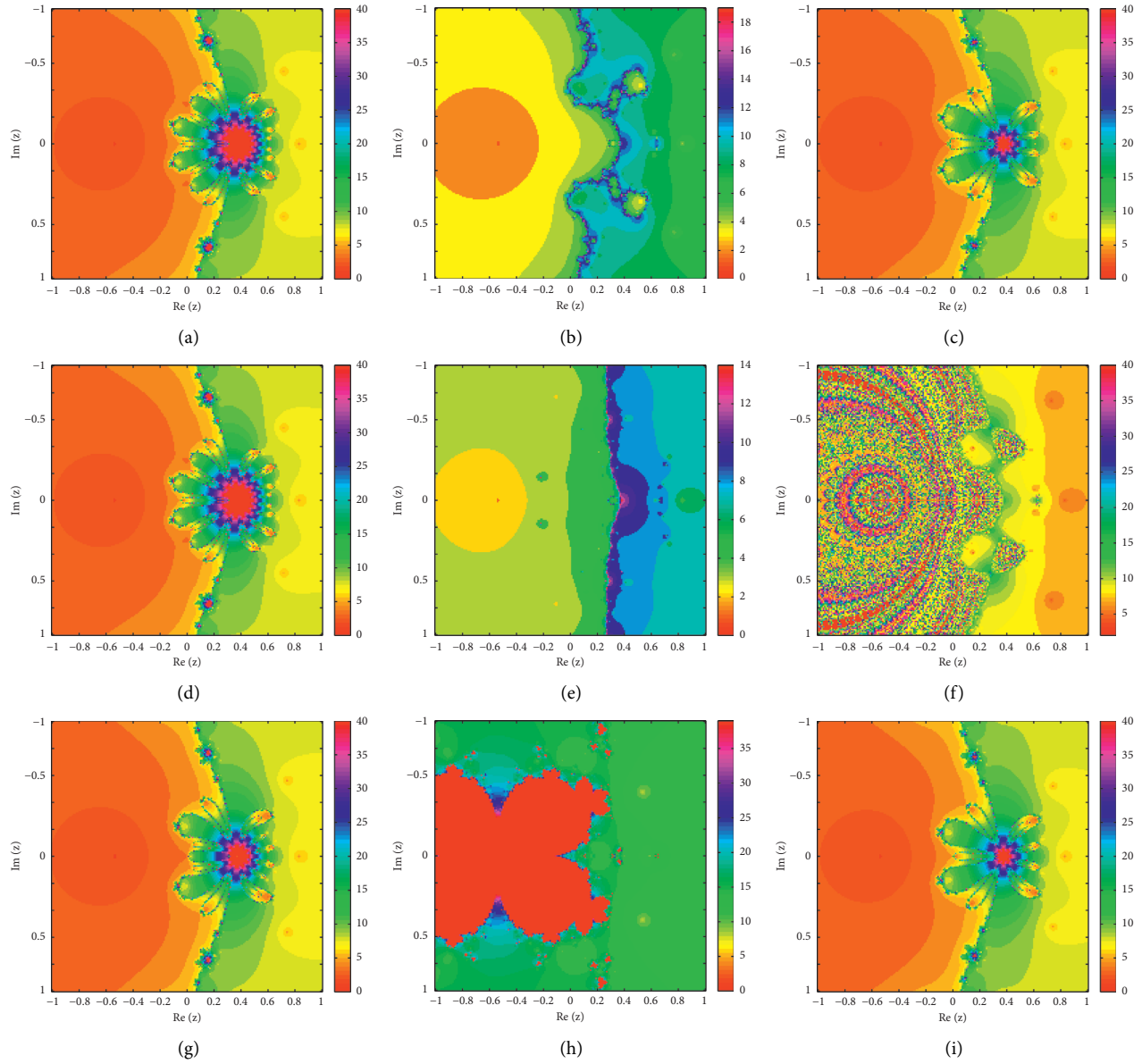

Figure 9: (a–i) Basins of attraction of methods BB1–BB5 and MM1–MM4 for the nonlinear equation $f_1(s)$, respectively.

The initial estimates for $f_3(s)$ are taken as

$$\begin{aligned} s_{1,2}^{(0)} &= 1.17, \\ s_3^{(0)} &= -7.4641, \\ s_4^{(0)} &= -0.5359. \end{aligned} \tag{57}$$

For distinct roots, we used method (28), and for multiple roots, we used method (29).

$$f_{1-1}(s) = s^4 + 4s^3 - 24s^2 + 16s + 16. \tag{58}$$

Example 2 (for determination of all distinct and multiple roots, see [22]). Here, we consider another standard test function for the demonstration of convergence behavior of newly constructed methods.

Consider

$$f_2(s) = \sin^3\left(\frac{s-1}{2}\right)\sin^3\left(\frac{s-2}{2}\right)\sin^3\left(\frac{s-2.5}{2}\right), \tag{59}$$

with exact roots $\zeta_1 = 1, \zeta_2 = 2, \zeta_3 = 2.5$ as shown in Figure 14.

The initial guessed values have been taken as $s_1^{(0)} = -0.2$, $s_2^{(0)} = 1.7$, $s_3^{(0)} = 3$.

For distinct roots, we used method (28), and for multiple roots, we used method (29).

$$f_{2-1}(s) = \sin\left(\frac{s-1}{2}\right)\sin\left(\frac{s-2}{2}\right)\sin\left(\frac{s-2.5}{2}\right). \tag{60}$$

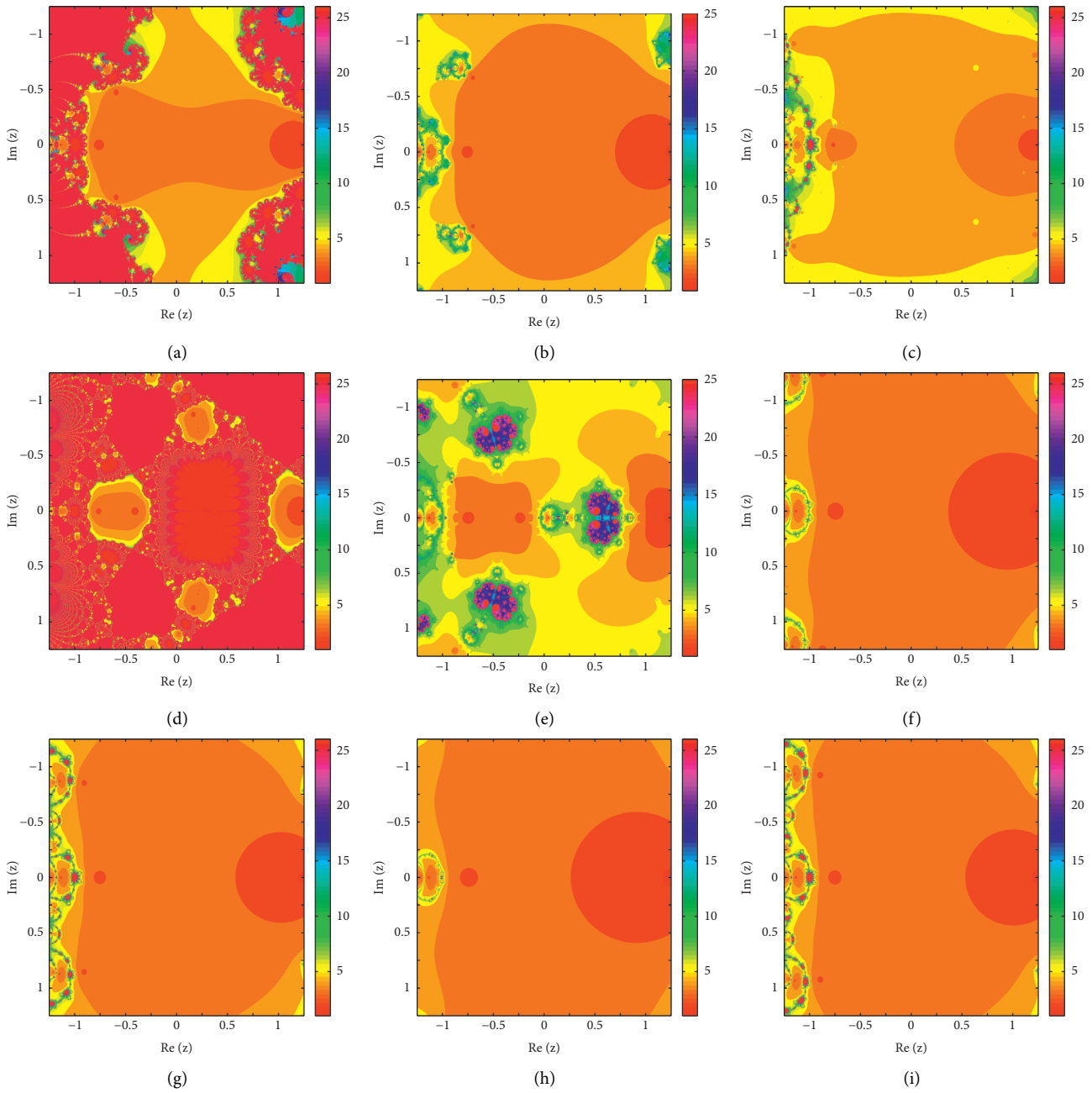

Figure 10: (a–i) Basins of attraction of methods BB1–BB5 and MM1–MM4 for the nonlinear equation $f_2(s)$, respectively.

Table 2: The number of basic operations.

Methods	CO	AS_m	M_m	D_m
M1	6	$7m^2 + O(m)$	$5m^2 + O(m)$	$2m^2 + O(m)$
M2	6	$7m^2 + O(m)$	$2m^2 + O(m)$	$2m^2 + O(m)$
M3	6	$7m^2 + O(m)$	$4m^2 + O(m)$	$2m^2 + O(m)$
M4	6	$7m^2 + O(m)$	$4m^2 + O(m)$	$2m^2 + O(m)$
M5	6	$7m^2 + O(m)$	$4m^2 + O(m)$	$2m^2 + O(m)$
PJ6M	6	$8m^2 + O(m)$	$6m^2 + O(m)$	$2m^2 + O(m)$

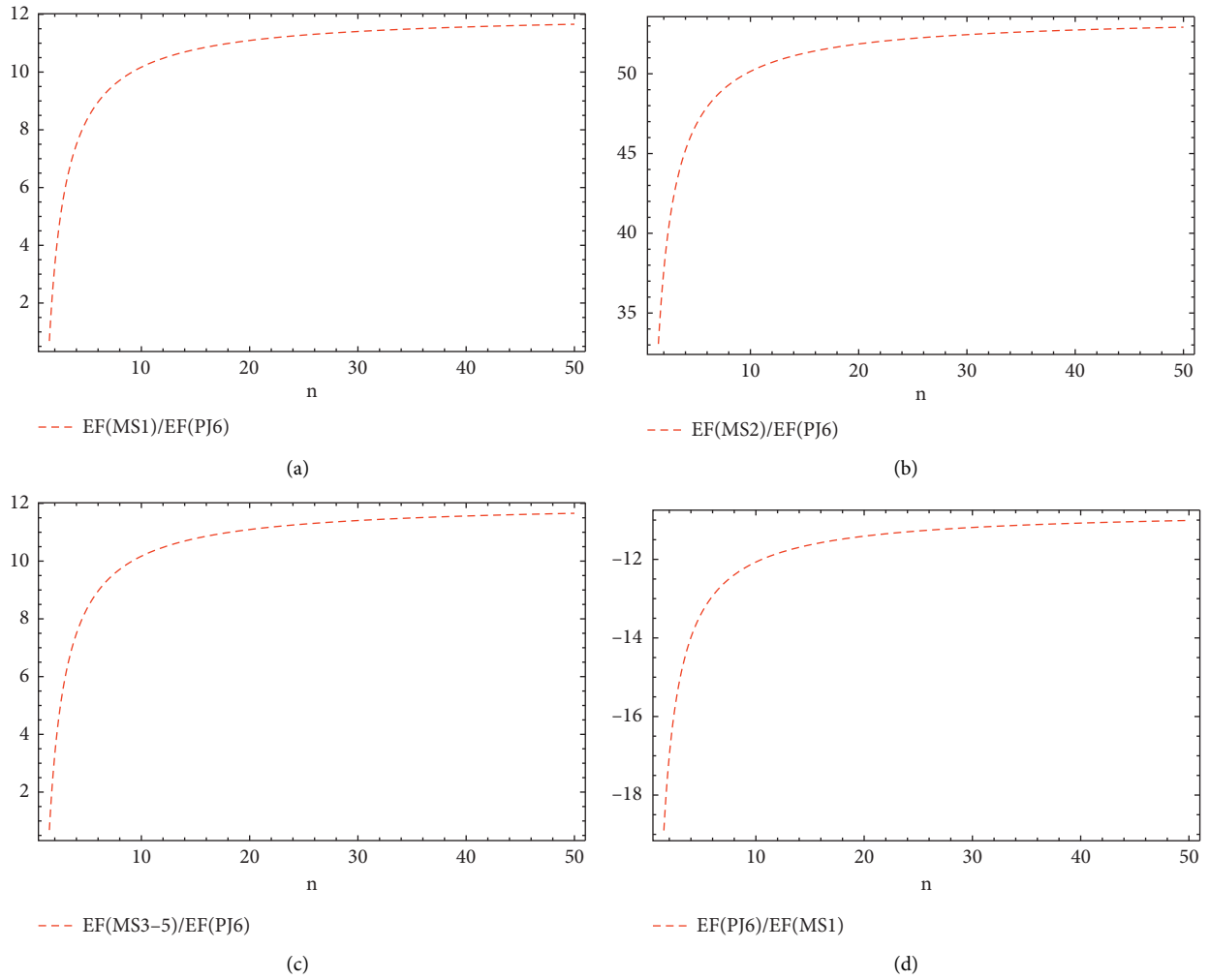

Figure 11: (a–c) Computational efficiency of methods M1, M3–M5, and M2 w.r.t method PJ6M. (d) Computational efficiency of method PJ6M w.r.t M1–M5.

Table 3: Simultaneous finding of all district roots.

$f_{1-1}(s) = s^4 + 4s^3 - 24s^2 + 16s + 16$

$\zeta_{1,2} = 2, \zeta_3 = -4 - 2\sqrt{3}, \zeta_3 = 2, \zeta_4 = -4 + 2\sqrt{3}.$

$s_{1,2}^{(0)} = 1.17, s_3^{(0)} = -7.4641, s_4^{(0)} = -0.5359$

Method	CO	CPU	n	$e1$	$e2$	$e3$	$e4$
PJ6M	6	0.031	4	$3.5e-2$	$3.5e-2$	$6.0e-11$	0.0
M1–M5	6	0.015	4	$5.0e-13$	$5.0e-13$	0.0	0.0

Table 4: Simultaneous findings of all multiple roots.

$f_1(s) = s^4 + 4s^3 - 24s^2 + 16s + 16 = (s-2)^2(s^2 + 8s + 4)$

$\zeta_{1,2} = 2, \zeta_3 = -4 - 2\sqrt{3}, \zeta_3 = 2, \zeta_4 = -4 + 2\sqrt{3}.$

$s_{1,2}^{(0)} = 1.17, s_3^{(0)} = -7.4641, s_4^{(0)} = -0.5359$

Methods	CO	CPU	n	$e1$	$e2$	$e3$	$e4$
PJ6M	6	0.032	4	$3.1e-10$	$3.1e-10$	$3.2e-6$	0.0
M1–M5	6	0.016	4	0.0	0.0	0.0	0.0

Table 5: Comparison of optimal fourth-order methods.

Method	$\lvert s_i^{(6)} - \alpha \rvert$	$\lvert f_i^{(6)}(s_i) \rvert$	CPU	p
$f_{1-1}(s) = s^4 + 4s^3 - 24s^2 + 16s + 16, s_0 = -0.5$				
BB1	$1.4e-1399$	$3.5e-5594$	0.032	4.00
BB2	$1.9e-1825$	$1.7e-7299$	0.063	4.00
BB3	$1.8e-1829$	$1.5e-7315$	0.047	4.01
BB4	$6.4e-1420$	$9.9e-5676$	0.031	4.00
BB5	$8.9e-1420$	$3.7e-5675$	0.047	4.01
MM1	$4.4e-1295$	$5.3e-5176$	0.016	4.00
MM2	$4.9e-1569$	$1.3e-6272$	0.031	4.00
MM3	$1.7e-1458$	$4.7e-5830$	0.063	4.01
MM4	$1.0e-1458$	$4.0e-5830$	0.032	4.00

Example 3. (for uniform beam design, see [34]).

Figure 15 shows a uniform beam subject to a linearly increasing distributed load. The nonlinear equation for the resulting elastic curve is

$$f(s) = \frac{w_0}{120EIL}\left(-s^5 + 2L^2s^3 - L^4s\right). \tag{61}$$

Table 6: Simultaneous finding of all distinct roots.

$f_{2-1}(s) = \sin(s-1/2)\sin(s-2/2)\sin(s-2.5/2)$

$\zeta_1 = 1, \zeta_2 = 2, \zeta_3 = 2.5,$
$s_1^{(0)} = -0.2, s_2^{(0)} = 1.7, s_3^{(0)} = 3$

Method	CO	CPU	n	$e1$	$e2$	$e3$
PJ6M	6	0.047	4	0.019	$4.6e-3$	$9.9e-10$
M1–M5	6	0.042	4	$6.9e-13$	$3.0e-5$	0.0

Table 7: Simultaneous finding of all multiple roots.

$f_2(s) = \sin^3(s-1/2)\sin^3(s-2/2)\sin^3(s-2.5/2)$

$\zeta_1 = 1$, $\zeta_2 = 2$, $\zeta_3 = 2.5$, each with multiplicity 3
$s_1^{(0)} = -0.2, s_2^{(0)} = 1.7, s_3^{(0)} = 3$

Method	CO	CPU	n	$e1$	$e2$	$e3$
PJ6M	6	0.094	4	$5.0e-5$	$1.7e-2$	$3.3e-5$
M1–M5	6	0.062	4	0.0	0.0	0.0

Table 8: Comparison of optimal fourth-order methods.

Methods	$\lvert s_i^{(6)} - \alpha \rvert$	$\lvert f_i^{(6)}(s_i) \rvert$	CPU	ρ
$f_{2-1}(s) = \sin^3(s-1/2)\sin^3(s-2/2)\sin^3(s-2.5/2), s_0 0.5$				
BB1	$5.0e-573$	$4.8e-2003$	1.516	4.00
BB2	$3.0e-501$	$8.4e-2291$	1.125	4.00
BB3	$1.5e-572$	$6.8e-2289$	1.125	4.00
BB4	$8.8e-506$	$2.9e-2021$	1.531	4.00
BB5	$1.2e-505$	$1.3e-2020$	1.409	4.00
MM1	$4.9e-462$	$5.1e-1846$	1.406	4.00
MM2	$1.0e-563$	$2.7e-2145$	1.657	4.00
MM3	$9.7e-516$	$3.5e-2061$	1.421	4.00
MM4	$9.7e-516$	$3.5e-060$	1.546	4.00

Table 9: Simultaneous finding of all distinct roots.

$f_3(s) = (2.5/120 * 50000 * 30000 * 600)(-5s^4 + 2160000s^2 - (600)^4)$

$\zeta_1 = -600, \zeta_2 = -268.328, \zeta_3 = 268.328, \zeta_4 = 600.$
$s_1^{(0)} = -590, s_2^{(0)} = -250, s_3^{(0)} = 255 s_4^{(0)} = 590$

Method	CO	CPU	n	$e1$	$e2$	$e3$	$e4$
PJ6M	6	0.031	4	$2.8e-3$	$2.3e-6$	$1.2e-6$	$7.2e-8$
M1–M5	6	0.015	4	0.0	$1.1e-30$	$1.1e-30$	0.0

We have to determine the point of maximum deflection which is the value of s where $f'(s) = 0$, i.e.,

$$\frac{w_0}{120EIL}\left(-5s^4 + 6L^2s^3 - L^4\right) = 0 \quad \text{or} \quad \frac{w_0}{120EIL}\left(-5s^4 + 6L^2s^3 - L^4\right) \tag{62}$$

where $L = 600$ cm, $E = 50,000$KN/cm^3, $I = 30,000$ cm^4, and $w_0 = 2.5$ KN/cm.

Table 10: Comparison of optimal fourth-order methods.

Method	$\lvert s_i^{(6)} - \alpha \rvert$	$\lvert f_i^{(6)}(s_i) \rvert$	CPU	ρ
$f_3(s) = (2.5/120*50000*30000*600)(-5s^4 + 2160000s^2 - (600)^4), s = -590$				
BB1	$5.7e-1050$	$3.1e-4208$	0.031	4.00
BB2	$3.1e-1332$	$1.4e-5338$	0.040	4.00
BB3	$3.6e-1342$	$2.5e-5378$	0.062	4.00
BB4	$1.5e-1060$	$1.3e-4250$	0.045	4.00
BB5	$1.0e-1059$	$2.6e-4247$	0.031	4.00
MM1	$5.4e-924$	$4.7e-3704$	0.047	4.00
MM2	$1.8e-1229$	$6.1e-4926$	1.734	4.00
MM3	$9.3e-1107$	$1.4e-4435$	0.046	4.00
MM4	$9.3e-1107$	$1.4e-4435$	0.032	4.00

Table 11: Simultaneous finding of all distinct roots.

$f_4(s) = s^4 + 11.50s^3 + 47.49s^2 + 83.06325s + 51.23266875$

$\zeta_1 = -1.45, \zeta_2 = -2.85, \zeta_3 = -2.85, \zeta_4 = -4.45$
$s_1^{(0)} = -1.0, s_2^{(0)} = -2.1, s_3^{(0)} = -1.8, s_4^{(0)} = -3.9$

Method	CO	CPU	n	e 1	e 2	e 3	e 4
PJ6M	6	0.047	3	$9.0e-3$	$2.0e-3$	0.4	$2.0e-3$
M1–M5	6	0.032	3	0.0	$3.6e-7$	$2.4e-7$	0.0

Table 12: Comparison of optimal fourth-order methods.

Method	$\lvert s_i^{(6)} - \alpha \rvert$	$\lvert f_i^{(6)}(s_i) \rvert$	CPU	ρ
$f_3(s) = s^4 + 11.50s^3 + 47.49s^2 + 83.06325s + 51.23266875, s_0 = -4.33$				
BB1	$65e-1239$	$3.2e-4951$	0.031	4.00
BB2	$1.5e-1551$	$5.1e-6203$	0.062	4.13
BB3	$1.3e-1556$	$3.4e-6223$	0.047	4.49
BB4	$2.6e-1256$	$1.4e-5021$	0.045	4.49
BB5	5.6e–1257	$1.4e-5019$	0.031	4.01
MM1	$7.2e-1125$	$9.1e-5679$	0.047	4.00
MM2	$8.9e-1299$	$3.3e-5192$	0.035	4.01
MM3	$5.2e-1299$	$8.3e-5192$	0.046	4.11
MM4	$5.7e-1299$	$3.3e-5192$	0.047	4.00

The exact roots of (62) are $\zeta_1 = -600$, $\zeta_2 = -268.328$, $\zeta_3 = 268.328$, and $\zeta_4 = 600$ as shown in Figure 16. The initial estimates for $f_3(s)$ have been taken as

$$s_{1-}^{(0)} = -590, \quad s_2^{(0)} = -250, \quad s_3^{(0)} = 255, \quad s_4^{(0)} = 590. \tag{63}$$

Example 4 (for continuous stirred tank reactor (CSTR), see [35]). We consider the isothermal stirred tank reactor (CSTR). Items A and R are fed to the reactor at rates of Q and

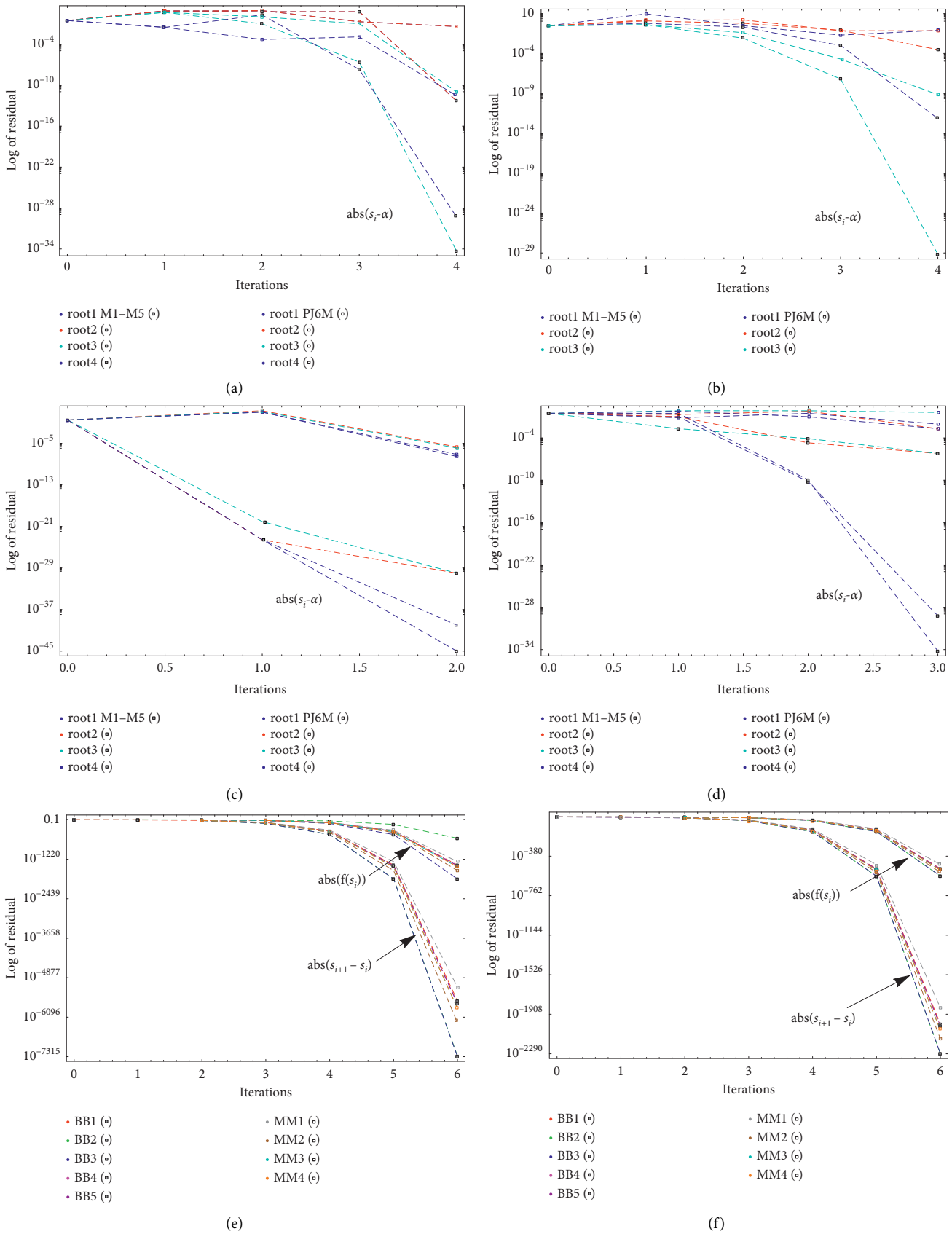

Figure 12: Continued.

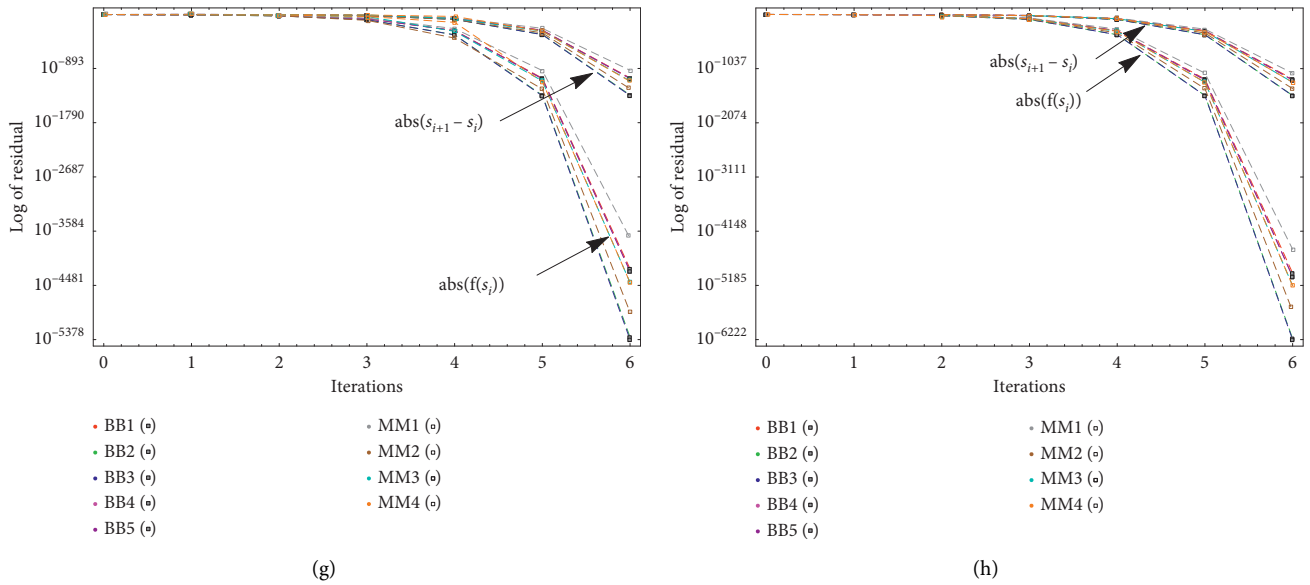

Figure 12: (a–d) Residual fall of simultaneous iterative methods (M1–M5 and PJ6M). (e–h) Residual fall of iterative methods (BB1–BB5 and MM1–MM4) for nonlinear function $f_1(s)$, $f_2(s)$, $f_3(s)$, and $f_4(s)$, respectively.

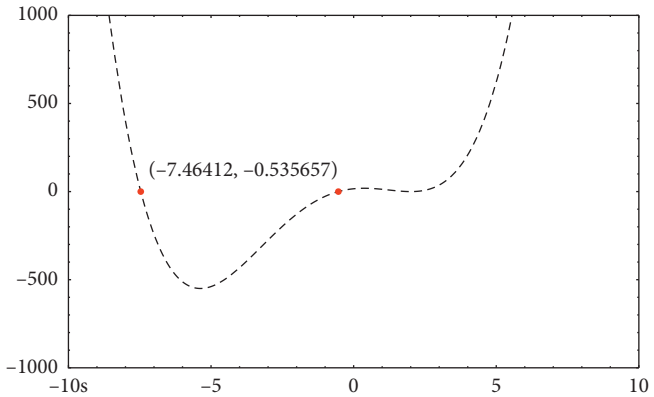

Figure 13: Exact roots of nonlinear equation $f_1(s)$.

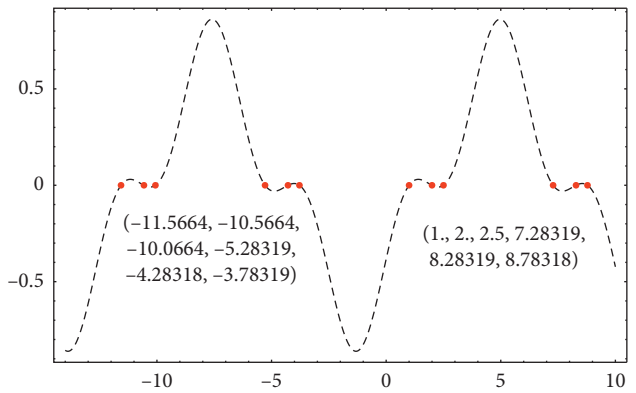

Figure 14: Exact roots of nonlinear equation $f_2(s)$.

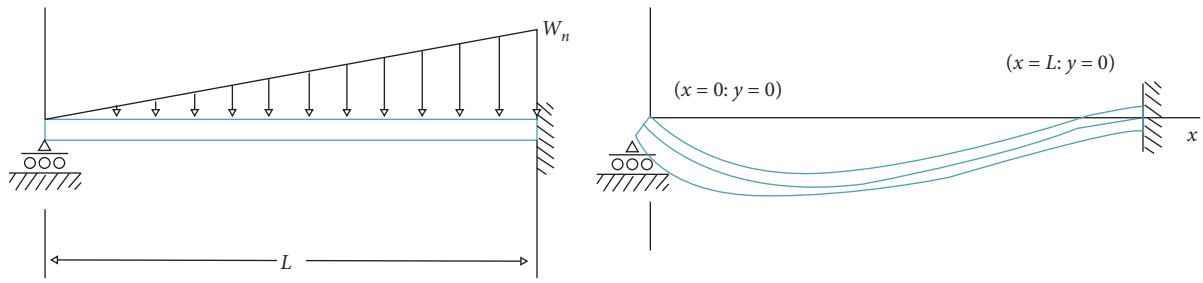

Figure 15 (a) Uniform beam design subject to a linearly increasing distributed load. (b) Resulting elastic curve.

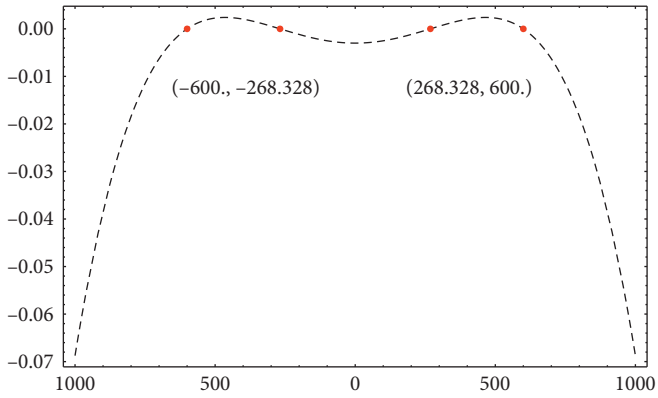

Figure 16: Exact roots of nonlinear equation $f(s)$.

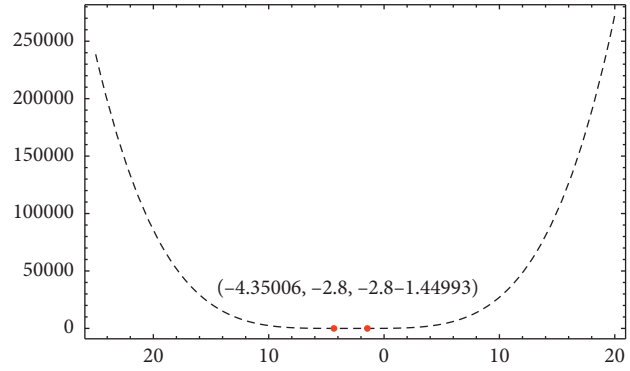

Figure 17: Exact roots of nonlinear equation $f_4(s)$.

q-Q, respectively. The following complex reaction system develops in the reactor (see [35]):

$$\begin{aligned} A + R &\longrightarrow B, \\ B + R &\longrightarrow C, \\ C + R &\longrightarrow D, \\ C + R &\longrightarrow E. \end{aligned} \tag{64}$$

This problem was tested by Douglas (see [36]) in order to construct the simple feedback control systems. In his searching of this system, he designed the following equation for the transfer function of the reactor with a proportional control system:

$$H_c \frac{2.98(s + 2.25)}{(s + 1.45)(s + 2.85)^2(s + 4.35)} = -1, \tag{65}$$

with H_c being the gain of the proportional controller. This control system is stable for values of H_c that yield roots of the transfer function having negative real parts. If we take $H_c = 0$, we get the nonlinear equation:

$$f_4(s) = s^4 + 11.50s^3 + 47.49s^2 + 83.06325s + 51.23266875 = 0. \tag{66}$$

The transfer function has four negative real roots as shown in Figure 17 that are

$$\begin{aligned} s_1 &= -1.45, \\ s_2 &= -2.85, \\ s_3 &= -2.85, \\ s_4 &= -4.45. \end{aligned} \tag{67}$$

These roots are known as poles of the open-loop transfer function.

The initial guessed values have been taken as

$$\begin{aligned} s_1^{(0)} &= -1.0, \\ s_2^{(0)} &= -2.1, \\ s_3^{(0)} &= -1.8, \\ s_3^{(0)} &= -3.9. \end{aligned} \tag{68}$$

7. Conclusion

Here, we have developed five families of two-step single-root finding methods of optimal convergence order four and five simultaneous iterative methods of order six, respectively. From Tables 3–10 and Figures 9, 10, and 12, we observe that our methods (BB1–BB5 and M1–M5) are superior in terms of efficiency, stability, CPU time, and residual error as compared with the methods MM1–MM4 and PJ6M, respectively [37].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally to the preparation of this manuscript.

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