Research Article

Novel Solution Method for Inventory Models with Stochastic Demand and Defective Units

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This paper is a response to two papers. We improve the lengthy proof for the first paper by an elegant verification. For the second paper, we point out the three-sequence approach will result in different convergent rates such that when the other two sequences are converged, the ordering quantity sequence may still not converge to the optimal solution. We construct a novel iterative method to simplify the previous approach proposed by the three-sequence approach for the optimal solution. By the same numerical examples of three published papers, we demonstrate that we can control our findings to converge more accurately than previous results. Moreover, we show that there are three distinct features of our proposed approach. (i) It converges to the desired solution within the preassigned threshold value. (ii) We estimate the convergent ratio. (iii) We find the dominant factors for our proposed convergent sequence.

1. Introduction

For the past several decades, many important and interesting inventory models had been developed by researchers. For example, Rosenblatt and Lee [1] examined the relationship between imperfect quality of goods and lot size. Paknejad et al. [2] studied the connection between defective items and lead time for inventory models. Huang [3], Ouyang et al. [4], and Wu et al. [5] worked on inventory systems with the imperfect quality of goods or stochastic property of lead time. Wee et al. [6] focused on the interaction between the imperfect quality and shortages or nonshortages in his model. Chung [7] derived the necessary and sufficient constraints for the existence of the optimal solution for a single-vendor single-buyer integrated production-inventory problem under the condition of an unreliable process. Chung and Wee [8] constructed an integrated inventory model with warranty policy, tactical price, imperfect manufacture process, and inspection procedure. There are several researchers to explore the shortages during the lead

time, for example, Das [9], Deng et al. [10], Ben-Daya and Raouf [11], Ouyang and Wu [12], and Wadhwa et al. [13]. Recently, there is a trend to improve previously published papers, for example, Deng [14], Deng et al. [15, 16], Lan et al. [17], Chang et al. [18], Jung et al. [19], and Tang et al. [20], that had provided useful analytical works to revise some questionable results in previous papers.

We will follow this trend to examine inventory models with stochastic demand, crashable lead time, and defective items that were developed by Wu and Ouyang [21], but they did not show the uniqueness of the optimal solution for their minimum cost problem. There are thirty-five papers that had cited Wu and Ouyang [21] in their references. We list them in the following: Mostard et al. [22], Wu et al. [5], Ho [23], Annadurai and Uthayakumar [24], Tung et al. [25], Çapar et al. [26], Drezner et al. [27], Lin et al. [28], Wang [29], Zheng and Liu [30], Gholami-Qadikolaei et al. [31], Ma and Qiu [32], Ouyang et al. [33], Lin [34–37], Parvathi et al. [37], Lin [38], Jindal and Solanki [39], Lin [40], Lin et al. [41], Vishkaei et al. [42], Wang et al. [43], Madhusoodhanan et al.

[44, 45], Chang et al. [46], Jindal and Solanki [47], Madhusoodhanan et al. [48], Chang et al. [49], Sarkar and Mahapatra [50], Chuang et al. [51], Hu et al. [52], Maleki Vishkaei et al. [53], Noh et al. [54], and Singha et al. [55]. Most of them only mentioned Wu and Ouyang [21] in their introduction but did not offer further study for Wu and Ouyang [21].

On the contrary, Tung et al. [25] and Lin [38] provided a detailed examination for Wu and Ouyang [21] concerning the solution procedure of the iterative sequence whether or not they will converge to the optimal solution. Three conditions were obtained by Tung et al. [25] which are two upper bounds and one lower bound. They used a numerical examination to compare two upper bounds to decide the smaller one. Under their conditions, they merged the system of two first partial derivatives into a function of order quantity and verify the uniqueness of the optimal solution. Moreover, Tung et al. [25] criticized the iterative approach in Wu and Ouyang [21] to point out that the original development suggested by Wu and Ouyang [21] cannot be executed. And then Tung et al. [25] provided an example to illustrate that the results presented in Wu and Ouyang [21] did not match the optimal solution derived by Tung et al. [25] through a revised iterative approach proposed by Tung et al. [25]. Three papers had cited Tung et al. [25] in their references: Yang [56], Lin [38], and Hu et al. [50]. Two of them, Yang [56] and Hu et al. [50], mentioned Tung et al. [25] in their introduction, but did not provide any discussion for the solution approach proposed by Tung et al. [25]. Only one paper, Lin [38], constructed a new system that contains three convergent sequences to generate the desired sequence that converges to the optimal ordering quantity. Lin [38] further considered the iterative approach of Tung et al. [25] to point out that they did not verify the convergence of the iterative approach in their paper. She constructed three sequences to prove the convergence of them. Only one paper, Hu et al. [50], had cited Lin [38] in their reference. However, Hu et al. [50] did not provide any comments for the three-sequence approach proposed by Lin [38]. In this paper, firstly we will provide a simplified verification to replace the complicated proof proposed by Tung et al. [25]. Secondly, we develop a novel approach to generate a new sequence and then prove its convergence. Moreover, by the same numerical example of Tung et al. [25] and Lin [38], we show that our sequence converges better than that of Lin [38] so our findings are simple and yield better convergent result.

Several related papers concerning inventory models with defective items are worthy to mention: Khanna et al. [57], Gautam et al. [58], Gautam and Khanna [59], and Khanna et al. [60]. The organization of this paper is explained as follows. Section 2 provides notation and assumptions for the examined inventory model. Section 3 describes the results of Wu and Ouyang [21]. Section 4 reviews Tung et al. [25]. Section 5 discusses the findings of Lin [38]. Section 6 provides our new proof to verify that the solution for the first partial system is unique and is the optimal solution. Section 7 presents a numerical example to illustrate that our approach can obtain an optimal solution. Section 8 shows the

three distinct features of our approach which is the key contribution of this paper. Section 9 concludes our paper.

2. Notation and Assumptions

We adopt the same notation and assumptions as Wu and Ouyang [21], Tung et al. [25], and Lin [38] and list them as follows:

- D : expected demand per year
- A : setup cost per setup
- h : nondefective holding cost per unit per year
- h' : defective holding cost per unit per year, $h' < h$
- π : shortage cost per unit short
- π_0 : marginal profit per unit
- v : unit inspection cost
- β : the fraction of the demand during the stock out period will be backordered, $0 \leq \beta \leq 1$
- p : the defective rate in an order lot (independent of lot size), $0 \leq p < 1$, and it is a random variable
- $g(p)$: the probability density function (p.d.f.) of p
- f : the proportion of order quantity inspected
- Q : lot size (order quantity), a decision variable
- L : length of the lead time, a decision variable
- X : the lead time demand with finite mean μL , and standard deviation $\sigma\sqrt{L}$ for lead time L
- r : reorder point, $r = \mu L + k\sigma\sqrt{L}$, where k is the safety factor that is a decision variable

An arrival order may contain some defective items. We assume that the number of defective items in an arriving order of size Q is a binomial random variable with parameters Q and p , where p ($0 \leq p \leq 1$) represents the defective rate in an order lot. Upon the arrival of an order, all the items are inspected and defective items in each lot will be returned to the vendor at the time of delivery of the next lot.

The inventory is continuously reviewed. Replenishments are made whenever the inventory level (based on the number of nondefective items) falls to the reorder point r .

The lead time L has n mutually independent components. The i th component has a minimum duration a_i and normal duration b_i , and a crashing cost per unit time c_i . Furthermore, for convenience, we rearrange c_i such that $c_1 \leq c_2 \leq \dots \leq c_n$.

The components of lead time are crashed one at a time starting with component 1 (because it has the minimum unit crashing cost) and then component 2, etc.

If we let $L_0 = \sum_{j=1}^n b_j$ and $L_i = \sum_{j=1}^i b_j + \sum_{j=i+1}^n a_j$, the lead time crashing cost $\bar{R}(L)$ per cycle for a given $L \in [L_i, L_{i-1}]$ is given by $R(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$.

3. Recap of Wu and Ouyang [21]

Wu and Ouyang [21] constructed inventory models with stochastic demand, crashable lead time, and defective items. They applied the minimax distribution-free approach of

Moon and Gallego [61] to develop the next minimum problem for variables: order quantity, Q , safety factor, k , and lead time L :

$$\begin{aligned} \text{EAC}^u(Q, k, L) = & \frac{AD}{Q(1-E(p))} + \frac{h}{2} \left\{ Q(1-E(p)) + Q \frac{E(p^2) - E^2(p)}{1-E(p)} + \frac{E(p(1-p))}{1-E(p)} \right\} \\ & + h \left\{ k\sigma\sqrt{L} + \frac{1-\beta}{2} \sigma\sqrt{L} \left(\sqrt{1+k^2} - k \right) \right\} + \frac{D(\pi + \pi_0(1-\beta))}{2Q(1-E(p))} \sigma\sqrt{L} \left(\sqrt{1+k^2} - k \right) \\ & + (Q-1)h' \frac{E(p(1-p))}{1-E(p)} + \frac{Dv}{1-E(p)} + \frac{D}{Q(1-E(p))} \left(c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j) \right), \end{aligned} \quad (1)$$

for $L \in [L_i, L_{i-1}]$, with $i = 1, \dots, n$.

Wu and Ouyang [21] proved that $\text{EAC}^u(Q, k, L)$ concaves down for $L \in [L_i, L_{i-1}]$ such that the minimum will happen on the two boundary points L_i or L_{i-1} . Consequently, under the restriction of $L = L_i$ or $L = L_{i-1}$, they derived the first partial derivatives for Q and k .

To simplify the expression, Wu and Ouyang [21] used L to represent $L = L_i$ or $L = L_{i-1}$ in the following derivations.

Using $(\partial/\partial Q)\text{EAC}^u(Q, k, L) = 0$ and $(\partial/\partial k)\text{EAC}^u(Q, k, L) = 0$, Wu and Ouyang [21] obtained that

$$\begin{aligned} Q = & \left[\frac{2D}{h\delta} \left\{ A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j) \right. \right. \\ & \left. \left. + \frac{\pi + \pi_0(1-\beta)}{2} \sigma\sqrt{L} \left(\sqrt{1+k^2} - k \right) \right\} \right]^{(1/2)}, \end{aligned} \quad (2)$$

$$\frac{2\sqrt{1+k^2}}{\sqrt{1+k^2} - k} = 1 - \beta + \frac{D(\pi + \pi_0(1-\beta))}{hQ(1-E(p))}, \quad (3)$$

where

$$\delta = 1 - 2E(p) + E(p^2) + 2\frac{h'}{h}E(p(1-p)). \quad (4)$$

Nevertheless, the expressions in equations (2) and (3) and the variables Q and k are mixed. Hence, Wu and Ouyang [21] cannot verify the uniqueness of the optimal solution. On the contrary, Wu and Ouyang [21] mentioned that they will apply equations (2) and (3) to generate two sequences which will converge to the optimal solution as follows:

$$\begin{aligned} Q_{n+1} = & \left[\frac{2D}{h\delta} \left\{ A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j) \right. \right. \\ & \left. \left. + \frac{\pi + \pi_0(1-\beta)}{2} \sigma\sqrt{L} \left(\sqrt{1+k_n^2} - k_n \right) \right\} \right]^{(1/2)}, \end{aligned} \quad (5)$$

$$\frac{2\sqrt{1+k_{n+1}^2}}{\sqrt{1+k_{n+1}^2} - k_{n+1}} = 1 - \beta + \frac{D(\pi + \pi_0(1-\beta))}{hQ_n(1-E(p))}. \quad (6)$$

When $\beta = 0$ and $L_3 = 3$, Wu and Ouyang [21] informed us that $Q^* = 183$ without offering a detailed description for their solution procedure.

However, Tung et al. [25] pointed out that the results of Wu and Ouyang [21] did not match the reproduction examined by Tung et al. [25] which will be explained after Tables 1 and 2 of this paper.

4. Review of Tung et al. [25]

Based on the results of equations (2) and (3), Tung et al. [25] obtained that

$$Q^2 = \alpha_1 + \alpha_2 \left(\sqrt{1+k^2} - k \right), \quad (7)$$

$$\frac{\sqrt{1+k^2} - k}{\sqrt{1+k^2}} = \frac{2\alpha_3 Q}{(1-\beta)\alpha_3 Q + \alpha_4}, \quad (8)$$

where

TABLE 1: Reproduction Table 2 of Tung et al. [25]. Comparisons between Wu and Ouyang [21] and Tung et al. [25].

Wu and Ouyang [21]			Tung et al. [25]	
β	Q	$EAC^u(Q, k, L)$	Q	$EAC^u(Q, k, L)$
0	183	5697.95	179.74	5697.41

TABLE 2: Reproduction Table 4 of Tung et al. [25]. The revised iterative method of Wu and Ouyang [21] and executed by Tung et al. [25].

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
Q_n	315.62	194.45	181.25	179.90	179.75	179.74
k_n	2.2916	3.0248	3.1386	3.1510	3.1523	3.1524

$$\alpha_1 = \left(\frac{2D}{h\delta} \right) \left(A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j) \right), \quad (9)$$

$$\alpha_2 = \left(\frac{D}{h\delta} \right) (\pi + \pi_0(1 - \beta))\sigma\sqrt{L}, \quad (10)$$

$$\alpha_3 = h(1 - E(p)), \quad (11)$$

$$\alpha_4 = D(\pi + \pi_0(1 - \beta)), \quad (12)$$

that satisfy $\alpha_j > 0$, for $j = 1, \dots, 4$.

Based on equation (8), Tung et al. [25] obtained that

$$1 - \frac{\sqrt{1+k^2} - k}{\sqrt{1+k^2}} = 1 - \frac{2\alpha_3 Q}{(1 - \beta)\alpha_3 Q + \alpha_4}, \quad (13)$$

and then Tung et al. [25] simplified equation (13) to find that

$$\frac{k}{\sqrt{1+k^2}} = \frac{\alpha_4 - (1 + \beta)\alpha_3 Q}{(1 - \beta)\alpha_3 Q + \alpha_4}. \quad (14)$$

Tung et al. [25] observed equation (14), under the condition that k is the safety factor, with the condition $k \geq 0$. The solution for the first partial derivative system is an interior solution and then the restriction is modified from $k \geq 0$ to $k > 0$.

Hence, Tung et al. [25] derived an upper bound for the ordering quantity, Q , as

$$\alpha_4 > (1 + \beta)\alpha_3 Q. \quad (15)$$

We follow the solution approach of Tung et al. [25] to flip over equation (14) to derive that

$$\frac{\sqrt{1+k^2}}{k} = \frac{(1 - \beta)\alpha_3 Q + \alpha_4}{\alpha_4 - (1 + \beta)\alpha_3 Q}, \quad (16)$$

and then we take square on both sides of equation (16) to yield that

$$\frac{1+k^2}{k^2} = \frac{[(1 - \beta)\alpha_3 Q + \alpha_4]^2}{[\alpha_4 - (1 + \beta)\alpha_3 Q]^2}. \quad (17)$$

We take “minutes one” from both sides of equation (17) to imply that

$$\frac{1+k^2}{k^2} - 1 = \frac{[(1 - \beta)\alpha_3 Q + \alpha_4]^2}{[\alpha_4 - (1 + \beta)\alpha_3 Q]^2} - 1, \quad (18)$$

and then we simplify equation (18) to find that

$$\frac{1}{k^2} = \frac{4\alpha_3 Q(\alpha_4 - \beta\alpha_3 Q)}{[\alpha_4 - (1 + \beta)\alpha_3 Q]^2}. \quad (19)$$

After lengthy derivation, Tung et al. [25] obtained a relation to express k as a function in the variable Q as

$$k = \frac{\alpha_4 - (1 + \beta)\alpha_3 Q}{2\sqrt{\alpha_3 Q(\alpha_4 - \beta\alpha_3 Q)}}. \quad (20)$$

Based on equation (20), Tung et al. [25] further derived that

$$\sqrt{1+k^2} = \left(1 + \frac{[\alpha_4 - (1 + \beta)\alpha_3 Q]^2}{4\alpha_3 Q(\alpha_4 - \beta\alpha_3 Q)} \right)^{(1/2)} \quad (21)$$

$$= \frac{\alpha_4 + (1 - \beta)\alpha_3 Q}{2\sqrt{\alpha_3 Q(\alpha_4 - \beta\alpha_3 Q)}}.$$

Plugging equations (20) and (21) into equation (7), then

$$Q^2 = \alpha_1 + \alpha_2 \sqrt{\frac{\alpha_3 Q}{\alpha_4 - \beta\alpha_3 Q}}. \quad (22)$$

Based on equation (22), Tung et al. [25] obtained a lower bound of Q as

$$Q > \sqrt{\alpha_1}. \quad (23)$$

If we recall the inequality of equation (15), then

$$\alpha_4 - \beta\alpha_3 Q > \alpha_3 Q. \quad (24)$$

Then, it follows that

$$1 > \frac{\alpha_3 Q}{\alpha_4 - \beta\alpha_3 Q}, \quad (25)$$

that is,

$$1 > \sqrt{\frac{\alpha_3 Q}{\alpha_4 - \beta\alpha_3 Q}}. \quad (26)$$

Based on equation (26), Tung et al. [25] found another upper bound of Q as

$$Q < \sqrt{\alpha_1 + \alpha_2}. \quad (27)$$

Tung et al. [25] applied a numerical method to use data from Wu and Ouyang [21] to compare two upper bounds: $\alpha_4/(1 + \beta)\alpha_3$ by equation (15) and $\sqrt{\alpha_1 + \alpha_2}$ by equation (27). We cite Table 3 of Tung et al. [25] in the following.

Based on the results of Table 3, Tung et al. [25] observed that

TABLE 3: Reproduction Table 3 of Tung et al. [25]. The ratio of $(\alpha_4/(1+\beta)\alpha_3n\sqrt{\alpha_1+\alpha_2})$.

	$L = L_0$	$L = L_1$	$L = L_2$	$L = L_3$
$\beta = 0$	18.855	20.090	21.856	22.962
$\beta = 0.5$	9.675	10.264	11.073	11.513
$\beta = 0.8$	6.211	6.552	6.994	7.180
$\beta = 1$	4.174	4.369	4.600	4.648

$$\frac{\alpha_4}{(1+\beta)\alpha_3} > \sqrt{\alpha_1 + \alpha_2}. \quad (28)$$

Tung et al. [25] rewrote the rational function of equation (22) as follows:

$$f'(Q) = (2Q)\sqrt{\alpha_4 - \beta\alpha_3Q} + (Q^2 - \alpha_1)\frac{-\beta\alpha_3}{2\sqrt{\alpha_4 - \beta\alpha_3Q}} - \frac{\alpha_2\sqrt{\alpha_3}}{2\sqrt{Q}},$$

$$f''(Q) = \frac{2}{\sqrt{\alpha_4 - \beta\alpha_3Q}}(\alpha_4 - 2\beta\alpha_3Q) + \frac{\alpha_2\sqrt{\alpha_3}\left(\sqrt{\alpha_4 - \beta\alpha_3Q}\right)^3 - (\sqrt{Q})^3(\beta\alpha_3)^2(Q^2 - \alpha_1)}{4\left(\sqrt{(\alpha_4 - \beta\alpha_3Q)Q}\right)^3}. \quad (32)$$

Tung et al. [25] obtained excellent derivations to claim that

$$\alpha_4 \geq (1+\beta)\alpha_3Q \geq 2\beta\alpha_3Q, \quad (33)$$

$$\begin{aligned} \sqrt{\alpha_3}\left(\sqrt{\alpha_4 - \beta\alpha_3Q}\right)^3 &\geq \sqrt{\alpha_3}(\sqrt{\alpha_3Q})^3 = (\sqrt{Q})^3(\alpha_3)^2 \\ &\geq (\sqrt{Q})^3(\beta\alpha_3)^2. \end{aligned} \quad (34)$$

Applying equations (33) and (34) with the inequality of equation (27), Tung et al. [25] proved the convexity property of $f(Q)$, that is,

$$f''(Q) > 0. \quad (35)$$

They further checked that

$$\begin{aligned} f(\sqrt{\alpha_1}) &= -\alpha_2\sqrt{\alpha_1\alpha_3} < 0, \\ f(\sqrt{\alpha_1 + \alpha_2}) &= \alpha_2[\alpha_4 - (1+\beta)\alpha_3\sqrt{\alpha_1 + \alpha_2}] \\ &\quad \cdot \left[\sqrt{\alpha_4 - \beta\alpha_3\sqrt{\alpha_1 + \alpha_2}} + \sqrt{\alpha_3}(\alpha_1 + \alpha_2)^{(1/4)}\right]. \end{aligned} \quad (36)$$

Based on the results of Table 3, Tung et al. [25] obtained that

$$f(\sqrt{\alpha_1 + \alpha_2}) > 0. \quad (37)$$

Hence, $f(Q)$ has a unique root for $\sqrt{\alpha_1 + \alpha_2} > Q > \sqrt{\alpha_1}$ which satisfies the first-order partial derivatives of $(\partial/\partial Q)EAC''(Q, k, L) = 0$ and $(\partial/\partial k)EAC''(Q, k, L) = 0$, so it is the optimal solution of $EAC''(Q, k, L)$. On the contrary, Tung et al. [25] pointed out that the iterative approach

$$(Q^2 - \alpha_1)\sqrt{\alpha_4 - \beta\alpha_3Q} - \alpha_2\sqrt{\alpha_3Q} = 0, \quad (29)$$

under the conditions

$$\frac{\alpha_4}{(1+\beta)\alpha_3} > \sqrt{\alpha_1 + \alpha_2} > Q > \sqrt{\alpha_1}, \quad (30)$$

with conditions of equations (9)–(12).

Based on equation (29), Tung et al. [25] assumed an auxiliary function:

$$f(Q) = (Q^2 - \alpha_1)\sqrt{\alpha_4 - \beta\alpha_3Q} - \alpha_2\sqrt{\alpha_3Q}. \quad (31)$$

Tung et al. [25] found that

proposed by Wu and Ouyang [21] was not consistent with their findings in the numerical examples.

Tung et al. [25] mentioned that given $k_0 = 0$ in equation (5), then Q_1 is obtained by the following formula:

$$\begin{aligned} Q_1 &= \left[\frac{2D}{h\delta} \left\{ A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j) \right. \right. \\ &\quad \left. \left. + \frac{\pi + \pi_0(1-\beta)}{2} \sigma\sqrt{L} \left(\sqrt{1+k_0^2} - k_0 \right) \right\} \right]^{(1/2)}. \end{aligned} \quad (38)$$

However, plugging Q_1 into equation (8), researchers only find that

$$\frac{2\sqrt{1+k_1^2}}{\sqrt{1+k_1^2} - k_1} = 1 - \beta + \frac{D(\pi + \pi_0(1-\beta))}{hQ_1(1-E(p))}, \quad (39)$$

which is a relation containing k_1 . However, researchers cannot directly derive k_1 .

Therefore, Tung et al. [25] applied their finding of equation (20), together with equation (7) to generate the following approach:

$$\begin{aligned} Q_{n+1} &= \sqrt{\alpha_1 + \alpha_2 \left(\sqrt{1+k_n^2} - k_n \right)}, \\ k_{n+1} &= \frac{\alpha_4 - (1+\beta)\alpha_3Q_n}{2\sqrt{\alpha_3Q_n(\alpha_4 - \beta\alpha_3Q_n)}} \end{aligned} \quad (40)$$

For completeness, we reproduce Tables 2 and 4 of Tung et al. [25] to show that the findings of Wu and Ouyang [21] did not satisfy their claim.

From Table 2, Tung et al. [25] mentioned that if researchers execute the iterative method proposed by Wu and Ouyang [21], then the optimal ordering quantity, $Q^* = 179.74$. However, Wu and Ouyang [21] claimed that $Q^* = 183$. Hence, Tung et al. [25] asserted that the iterative method proposed by Wu and Ouyang [21], which is too complicated, cannot execute their method.

In Section 6, we will provide one simplification and another alternative approach to replace the lengthy solution procedure proposed by Tung et al. [25].

5. Discussion of Lin [38]

Lin [38] further examined the sequence convergent problem discussed by Wu and Ouyang [21] and Tung et al. [25]. In the following, we provide our comments for Lin [38]. Lin [38] admitted that Tung et al. [25] improved the iterative approach of Wu and Ouyang [21], but Tung et al. [25] did not prove the convergence of their new approach. Hence, Lin [38] developed a new derivation that consists of three sequences as follows:

$$Q_{n+1} = \left(B_1 + B_2 \left(\sqrt{1 + k_n^2} - k_n \right) \right), \quad (41)$$

with

$$\begin{aligned} B_1 &= \frac{2D}{h\delta} \left[A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j) \right], \\ B_2 &= \frac{2D}{h\delta} \left[\frac{\pi + \pi_0(1 - \beta)}{2} \right] \sigma \sqrt{L}, \\ d_{n+1} &= \frac{1}{2} \left(1 - \beta + \frac{D[\pi + \pi_0(1 - \beta)]}{hQ_{n+1}(1 - E(p))} \right), \\ k_{n+1} &= \frac{d_{n+1} - 1}{\sqrt{2d_{n+1} - 1}}. \end{aligned} \quad (42)$$

Moreover, Lin [38] proved the convergence of her approach. For interested readers, please refer to Lin [38] for her detailed proof. In the next section, we will develop our method which will be shown superior to that of Lin [38] which will be demonstrated by the example in Section 7. Therefore, to save the precious space of this journal, we did not further discuss the detailed proof proposed by Lin [38] concerning the convergence of her three convergent sequence approach.

For later discussion, we cite results derived by Lin [38] in the following.

6. Our Improved Approach

Now, we begin to develop our verification for the uniqueness of the optimal ordering quantity to simplify the lengthy approach proposed by Tung et al. [25]. We observe (8) to imply that

TABLE 4: Reproduction from Lin [38]. For $\beta = 0.5$ and $L_3 = 3$, the three sequences generated by Lin [38].

i	k_i	Q_i	d_i
0	0	—	—
1	1.925382	271.444019	8.884377
2	2.433960	179.201796	13.328831
3	2.490503	171.872873	13.886532
4	2.495817	171.206597	13.939601
5	2.496308	171.145237	13.944509
6	2.496353	171.139579	13.944962
7	2.496357	171.139060	13.945003
8	2.496357	171.139014	13.945007
9	2.496357	171.139014	13.945007

$$\frac{\sqrt{1 + k^2} - k}{2\sqrt{1 + k^2}} = \frac{\alpha_3 Q}{(1 - \beta)\alpha_3 Q + \alpha_4}. \quad (43)$$

If $(A/B) = (C/D)$, then $(A/(-A + B)) = (C/(-C + D))$. We apply this rule to equation (43) to derive that

$$\frac{\sqrt{1 + k^2} - k}{\sqrt{1 + k^2} + k} = \frac{\alpha_3 Q}{\alpha_4 - \beta\alpha_3 Q}. \quad (44)$$

We know that

$$\sqrt{1 + k^2} - k = \frac{1}{\sqrt{1 + k^2} + k}. \quad (45)$$

Using equation (54), we find that

$$(\sqrt{1 + k^2} - k)^2 = \frac{\alpha_3 Q}{\alpha_4 - \beta\alpha_3 Q}. \quad (46)$$

If we plug our findings of equation (55) into equation (7), then we derive the results of equation (22) as proposed by Tung et al. [25]. Therefore, we demonstrate a simplified approach to find the same result as Tung et al. [25] without deriving $\sqrt{1 + k^2}$ of equation (21) or finding k of equation (20).

After simplifying the approach of Tung et al. [25], we will show our new solution method. Based on equation (8), we express Q as a function in the variable k ; then,

$$Q = \frac{(\sqrt{1 + k^2} - k)\alpha_4}{\alpha_3[(1 + \beta)\sqrt{1 + k^2} + (1 - \beta)k]}. \quad (47)$$

We plug the findings of equation (47) into (7) to derive that

$$\left(\frac{(\sqrt{1 + k^2} - k)\alpha_4}{\alpha_3[(1 + \beta)\sqrt{1 + k^2} + (1 - \beta)k]} \right)^2 = \alpha_1 + \alpha_2 \left(\sqrt{1 + k^2} - k \right). \quad (48)$$

We rewrite equation (48) as

$$\left(\frac{(\sqrt{1+k^2} - k)\alpha_4}{\alpha_3[\beta(\sqrt{1+k^2} - k) + (1/\sqrt{1+k^2} - k)]} \right)^2 = \alpha_1 + \alpha_2 \left(\sqrt{1+k^2} - k \right). \quad (49)$$

Therefore, our goal becomes to prove that there is a unique point that satisfies equation (49).

We observe equation (49) to find if we assume a new variable, denoted as s , where

$$s = \sqrt{1+k^2} - k. \quad (50)$$

With the new variable, we rewrite equation (49) as follows:

$$\alpha_3^2(\beta s^2 + 1)^2(\alpha_1 + \alpha_2 s) = \alpha_4^2 s^4. \quad (51)$$

Owing to equation (51), we assume an auxiliary function, denoted as $g(s)$, where

$$g(s) = \alpha_3^2(\beta s^2 + 1)^2(\alpha_1 + \alpha_2 s) - \alpha_4^2 s^4. \quad (52)$$

We derive that

$$g'(s) = 4\beta\alpha_3^2 s(\beta s^2 + 1)(\alpha_1 + \alpha_2 s) + \alpha_2\alpha_3^2(\beta s^2 + 1)^2 - 4\alpha_4^2 s^3, \quad (53)$$

$$g''(s) = 8\beta^2\alpha_3^2 s^2(\alpha_1 + \alpha_2 s) + 4\beta\alpha_3^2(\beta s^2 + 1)(\alpha_1 + 3\alpha_2 s) - 12\alpha_4^2 s^2, \quad (54)$$

$$g'''(s) = 24\beta^2\alpha_3^2 s(\alpha_1 + 2\alpha_2 s) + 12\beta\alpha_2\alpha_3^2(\beta s^2 + 1) - 24\alpha_4^2 s, \quad (55)$$

$$g^{(4)}(s) = 24\beta^2\alpha_3^2(\alpha_1 + 5\alpha_2 s) - 24\alpha_4^2. \quad (56)$$

For the later discussion, we will begin to show that

$$3\alpha_1 > \alpha_2, \quad (57)$$

$$\alpha_1 > 0, \quad (58)$$

$$\alpha_2 > 0. \quad (59)$$

We recall $\alpha_1 = (2D/h\delta)(A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j))$ of equation (9), and from equation (10), we know that $\alpha_2 = (D/h\delta)(\pi + \pi_0(1 - \beta))\sigma\sqrt{L}$ such that two inequalities of equations (58) and (59) are valid.

We will recall examples from Wu and Ouyang [23], Tung et al. [20], and Lin [13]. Hence, we adopt the same data as them with $D = 600$ units/year, $A = 200$ per order, $h = 20$, $h' = 12$, $v = 1.6$, $\pi = 50$, $\pi_0 = 150$, and $\sigma = 7$; the lead time has three components such that $L_0 = 8$, $R(L_0) = 0$, $L_1 = 6$, $R(L_1) = 5.6$, $L_2 = 4$, $R(L_2) = 22.4$, $L_3 = 3$, and $R(L_2) = 57.4$; the defective rate p has a Beta distribution, with the probability density function $g(p) = 4(1 - p)^3$ to imply that $E(p) = (1/5)$; and $E(p^2) = (1/15)$, the fraction of the backordered demand, β , has four values, 0, 0.5, 0.8, and 1.

From the above data, we know that the lower bound $\sqrt{\alpha_1} = 136.683$ and the upper bound $\sqrt{\alpha_1 + \alpha_2} = 271.444$.

Based on $\alpha_1 = 18.68 \times 10^3$ and $\alpha_2 = 54.99 \times 10^3$, we derive that

$$3\alpha_1 - \alpha_2 = 1.05 \times 10^3 > 0. \quad (60)$$

Hence, the inequality of equation (57) is valid under data from Wu and Ouyang [21], Tung et al. [25], and Lin [38].

We recall that $s = \sqrt{1+k^2} - k$ of equation (50), with $k > 0$, and then we derive that

$$0 < s < 1. \quad (61)$$

Based on equations (58), (59), and (61), we find that

$$\alpha_2 s < \alpha_1 + \alpha_2. \quad (62)$$

Based on equation (28), we imply that

$$\alpha_4^2 > (1 + \beta)^2 \alpha_3^2 (\alpha_1 + \alpha_2), \quad (63)$$

and then owing to $0 \leq \beta \leq 1$, we know that

$$2\beta \leq 1 + \beta. \quad (64)$$

We derive that

$$g(0) = \alpha_1 \alpha_3^2 > 0, \quad (65)$$

$$g(1) = \alpha_3^2 (1 + \beta)^2 (\alpha_1 + \alpha_2) - \alpha_4^2 < 0, \quad (66)$$

owing to equation (28). Next, we find that

$$g'(0) = \alpha_2 \alpha_3^2 > 0, \quad (67)$$

$$\begin{aligned} g'(1) &= 4\beta\alpha_3^2(1 + \beta)(\alpha_1 + \alpha_2) + \alpha_2\alpha_3^2(1 + \beta)^2 - 4\alpha_4^2 \\ &= 2[(2\beta)(1 + \beta)\alpha_3^2(\alpha_1 + \alpha_2) - \alpha_4^2] \\ &\quad + [\alpha_2\alpha_3^2(1 + \beta)^2 - \alpha_4^2] - \alpha_4^2 < 0. \end{aligned} \quad (68)$$

And then, we compute that

$$g''(0) = 4\beta\alpha_1\alpha_3^2 > 0, \quad (69)$$

$$\begin{aligned} g'''(1) &= 8\beta^2\alpha_3^2(\alpha_1 + \alpha_2) + 4\beta\alpha_3^2(1 + \beta)(\alpha_1 + 3\alpha_2) - 12\alpha_4^2 \\ &= 2[(2\beta)^2\alpha_3^2(\alpha_1 + \alpha_2) - \alpha_4^2] \\ &\quad + 2[(2\beta)(1 + \beta)\alpha_3^2(\alpha_1 + 3\alpha_2) - 3\alpha_4^2] - 4\alpha_4^2 < 0. \end{aligned} \quad (70)$$

Finally, we find that

$$g'''(0) = 12\beta\alpha_2\alpha_3^2 > 0, \quad (71)$$

$$\begin{aligned} g'''(1) &= 24\beta^2\alpha_3^2(\alpha_1 + 2\alpha_2) + 12\beta\alpha_2\alpha_3^2(1 + \beta) - 24\alpha_4^2 \\ &= 6[(2\beta)^2\alpha_3^2(\alpha_1 + 2\alpha_2) - 2\alpha_4^2] \\ &\quad + 6[(2\beta)(1 + \beta)\alpha_2\alpha_3^2 - \alpha_4^2] - 6\alpha_4^2 < 0. \end{aligned} \quad (72)$$

From the inequality $3\alpha_1 > \alpha_2$ of equation (57), we know that

$$4(\alpha_1 + \alpha_2) > \alpha_1 + 5\alpha_2. \quad (73)$$

Hence, we derive that

$$\alpha_4^2 > 4\beta^2 \alpha_3^2 (\alpha_1 + \alpha_2) > \beta^2 \alpha_3^2 (\alpha_1 + 5\alpha_2). \quad (74)$$

Based on equation (74), we rewrite $g^{(4)}(s)$ of equation (56) to show that

$$g^{(4)}(s) = 24[\beta^2 \alpha_3^2 (\alpha_1 + 5\alpha_2 s) - \alpha_4^2] < 0. \quad (75)$$

From equation (75), we know that $g^{(4)}(s) < 0$, for $0 < s < 1$ and then $g^{(3)}(s)$ is a strictly increasing function. We recall that $g^{(3)}(0) > 0$ of equation (71) and $g^{(3)}(1) < 0$ of equation (72) such that we know that there is a point, denoted as $s^\#$, that satisfying

$$g'''(s^\#) = 0, \quad (76)$$

with

$$g'''(s) > 0, \quad (77)$$

for $0 < s < s^\#$, and

$$g'''(s) < 0, \quad (78)$$

for $s^\# < s < 1$.

Based on our findings of equations (77) and (78), we derive that $g''(s)$ is an increasing function for $0 < s < s^\#$, and $g''(s)$ is a decreasing function for $s^\# < s < 1$.

Together with $g''(0) > 0$ of equation (69) and $g''(1) < 0$ of equation (70), we know that there is a point, denoted as s^Δ that satisfies $g''(s^\Delta) = 0$ such that

$$g''(s) > 0, \quad (79)$$

for $0 < s < s^\Delta$, and

$$g''(s) < 0, \quad (80)$$

for $s^\Delta < s < 1$.

Based on our findings of equations (79) and (80), we derive that $g'(s)$ is an increasing function for $0 < s < s^\Delta$, and $g'(s)$ is a decreasing function for $s^\Delta < s < 1$.

Together with $g'(0) > 0$ of equation (67) and $g'(1) < 0$ of equation (68), we know that there is a point, denoted as s^Ω that satisfies $g'(s^\Omega) = 0$ such that

$$g'(s) > 0, \quad (81)$$

for $0 < s < s^\Omega$, and

$$g'(s) < 0, \quad (82)$$

for $s^\Omega < s < 1$.

Based on our findings of equations (81) and (82), we derive that $g(s)$ is an increasing function for $0 < s < s^\Omega$, and $g(s)$ is a decreasing function for $s^\Omega < s < 1$.

Together with $g(0) > 0$ of equation (65) and $g(1) < 0$ of equation (66), we know that there is a point, denoted as s^* that satisfies $g(s^*) = 0$ such that

$$g(s) > 0, \quad (83)$$

for $0 < s < s^*$, and

$$g(s) < 0, \quad (84)$$

for $s^* < s < 1$.

Based on the above discussion, we prove that there is a unique point that satisfies $0 < s^* < 1$, and

$$g(s^*) = 0. \quad (85)$$

Based on equation (50), we find that

$$k^* = \frac{1 - (s^*)^2}{2s^*} \quad (86)$$

satisfies equation (48).

We recall equation (47), and we find that

$$Q^* = \frac{\left(\sqrt{1 + (k^*)^2} - k^*\right)\alpha_4}{\alpha_3 \left[(1 + \beta)\sqrt{1 + (k^*)^2} + (1 - \beta)k^*\right]}. \quad (87)$$

From the above discussion, we provide an alternative solution approach to show the uniqueness of the optimal solution.

Our proof needs an extra condition: $3\alpha_1 > \alpha_2$, which was supported by numerical examples in Wu and Ouyang [21], Tung et al. [25], and Lin [38]. Hence, we provide a simplified proof to replace the complicated verification in Tung et al. [25] with an extra condition of $\alpha_4 > (1 + \beta)\alpha_3\sqrt{\alpha_1 + \alpha_2}$ to find an upper bound and a lower bound of the order quantity.

From equation (31), we rewrite $f(Q) = 0$ to yield the following fifth-degree polynomial:

$$\beta\alpha_3 Q^5 - \alpha_4 Q^4 - 2\beta\alpha_1\alpha_3 Q^3 + 2\alpha_1\alpha_4 Q^2 + (\beta\alpha_1^2 + \alpha_2^2)\alpha_3 Q - \alpha_1^2\alpha_4 = 0. \quad (88)$$

Based on equation (31), we derive that

$$Q^3 = \frac{2\alpha_1\alpha_4 Q^2 + (\beta\alpha_1^2 + \alpha_2^2)\alpha_3 Q - \alpha_1^2\alpha_4}{Q(\alpha_4 - \beta\alpha_3 Q) + 2\beta\alpha_1\alpha_3}. \quad (89)$$

Owing to the upper bound in equation (30), we know that $\alpha_4 - \beta\alpha_3 Q > 0$, and the lower bound in equation (30) yields that $\alpha_1\alpha_4 Q^2 - \alpha_1^2\alpha_4 > 0$ such that the numerator and the denominator of equation (89) are both positive.

Motivated by equation (89), we will use the following relation:

$$Q_{n+1} = \sqrt[3]{\frac{2\alpha_1\alpha_4 Q_n^2 + (\beta\alpha_1^2 + \alpha_2^2)\alpha_3 Q_n - \alpha_1^2\alpha_4}{Q_n(\alpha_4 - \beta\alpha_3 Q_n) + 2\beta\alpha_1\alpha_3}}, \quad (90)$$

to construct a sequence, (Q_n) , with an initial point Q_1 . We will show that (Q_n) is a convergent sequence. We evaluate $Q_{n+2}^3 - Q_{n+1}^3$ and express the result as

$$Q_{n+2}^3 - Q_{n+1}^3 = (Q_{n+1} - Q_n) \frac{N}{M}, \quad (91)$$

where

$$N = y_1(Q_{n+1} + Q_n) + y_3Q_nQ_{n+1} + y_4 > 0, \quad (92)$$

$$M = \Delta_n\Delta_{n+1} > 0, \quad (93)$$

with

$$\begin{aligned} y_1 &= 3\beta\alpha_1^2\alpha_3\alpha_4, \\ y_2 &= \beta\alpha_3^2(\beta\alpha_1^2 + \alpha_2^2), \\ y_3 &= 2\alpha_1\alpha_4^2 + y_2, \\ y_4 &= \alpha_1^2\alpha_4^2 + 2\alpha_1y_2, \\ \Delta_n &= Q_n(\alpha_4 - \beta\alpha_3Q_n) + 2\beta\alpha_1\alpha_3. \end{aligned} \quad (94)$$

Depending on the relation of $Q_2 - Q_1$, our solution results are divided into three cases. Case (a) $Q_2 - Q_1 < 0$; Case (b) $Q_2 - Q_1 = 0$; Case (c) $Q_2 - Q_1 > 0$.

For Case (a), owing to $Q_2 - Q_1 < 0$ and equations (91)–(93), we know that $Q_3 - Q_2 < 0$, and then $Q_{n+1} - Q_n < 0$ for $n = 1, 2, \dots$, such that (Q_n) is an increasing sequence which is bounded above by $\sqrt{\alpha_1 + \alpha_2}$, and then the increasing sequence will converge to its least upper bound.

For Case (b), from $Q_2 = Q_1$, it yields that $Q_3 = Q_2$ and $Q_{n+1} = Q_n$ for $n = 1, 2, \dots$, such that (Q_n) is a constant sequence that converges to Q_1 .

For Case (c), from $Q_2 - Q_1 > 0$ and equations (91)–(93), it yields that $Q_3 - Q_2 > 0$, and then $Q_{n+1} - Q_n > 0$ for $n = 1, 2, \dots$, such that (Q_n) is a decreasing sequence which is bounded below by $\sqrt{\alpha_1}$, and then the decreasing sequence will converge to its greatest lower bound.

Now, we summarize our results for three different cases to derive that our proposed sequence (Q_n) will converge.

From our findings, we can claim that the optimal solution is between the limit of Case (a) and the limit of Case (c) such that depending on the accuracy of the optimal solution, we can obtain the optimal solution as accuracy as desired which will be demonstrated in the next section.

7. Numerical Examples

We will compare our findings with that of Wu and Ouyang [21], Tung et al. [25], and Lin [38]. Hence, we adopt the same data as them with $D = 600$ units/year, $A = 200$ per order, $h = 20$, $h' = 12$, $\nu = 1.6$, $\pi = 50$, $\pi_0 = 150$, and $\sigma = 7$; the lead time has three components such that $L_0 = 8$, $R(L_0) = 0$, $L_1 = 6$, $R(L_1) = 5.6$, $L_2 = 4$, $R(L_2) = 22.4$, $L_3 = 3$, and $R(L_3) = 57.4$; the defective rate p has a Beta distribution, with probability density function $g(p) = 4(1-p)^3$ to imply that $E(p) = (1/5)$; and $E(p^2) = (1/15)$, the fraction of the backordered demand, β , has four values, 0, 0.5, 0.8, and 1. From the above data, we know that the lower bound $\sqrt{\alpha_1} = 136.683$ and the upper bound $\sqrt{\alpha_1 + \alpha_2} = 271.444$. Therefore, for Case (a), we select $Q_1 = \sqrt{\alpha_1 + \alpha_2}$ with $L = L_3 = 3$ and $\beta = 0.5$ to derive our first numerical example. To save the precious space of the journal, we use different decimal places without influencing the tendency of the limit for our proposed decreasing sequence. We derive that $Q_1 = 271.4$, $Q_2 = 231.1$, $Q_3 = 191.6$, $Q_4 = 181.9$, $Q_5 = 177.0$, $Q_6 = 174.4$,

$$\begin{aligned} Q_7 &= 173.0, & Q_8 &= 172.2, & Q_9 &= 171.7, & Q_{10} &= 171.5, \\ Q_{11} &= 171.33, & Q_{12} &= 171.25, & Q_{13} &= 171.20, & Q_{14} &= 171.18, \\ Q_{15} &= 171.16, & Q_{16} &= 171.151, & Q_{17} &= 171.146, & Q_{18} &= 171.143, \\ Q_{19} &= 171.1412, & Q_{20} &= 171.1403, & Q_{21} &= 171.1397, & Q_{22} &= 171.1394, \\ Q_{23} &= 171.1392, & Q_{24} &= 171.1391, & Q_{25} &= 171.139085, & Q_{26} &= 171.139052, \\ Q_{27} &= 171.139033, & Q_{28} &= 171.139022, & Q_{29} &= 171.139016, & Q_{30} &= 171.139013, \\ Q_{31} &= 171.139010, & Q_{32} &= 171.1390093, & Q_{33} &= 171.1390087, & Q_{34} &= 171.1390083, \\ Q_{35} &= 171.1390081, & Q_{36} &= 171.1390080, & Q_{37} &= 171.13900793, & Q_{38} &= 171.13900789, \\ Q_{39} &= 171.13900786, & Q_{40} &= 171.13900785, & Q_{41} &= 171.139007844, & \text{and } Q_{42} &= 171.139007840. \end{aligned}$$

From the above sequence, if we decide to stop up to the eighth decimal place, we can claim that the optimal solution

$$Q^* \leq 171.13900784. \quad (95)$$

For our second numerical example, we select $Q_1 = \sqrt{\alpha_1}$ with $L = L_3 = 3$ and $\beta = 0.5$ and then list our results for the increasing sequence as follows: $Q_1 = 136.7$, $Q_2 = 147.2$, $Q_3 = 155.6$, $Q_4 = 161.6$, $Q_5 = 165.4$, $Q_6 = 167.8$, $Q_7 = 169.2$, $Q_8 = 170.0$, $Q_9 = 170.5$, $Q_{10} = 170.8$, $Q_{11} = 170.9$, $Q_{12} = 171.02$, $Q_{13} = 171.07$, $Q_{14} = 171.10$, $Q_{15} = 171.116$, $Q_{16} = 171.126$, $Q_{17} = 171.132$, $Q_{18} = 171.135$, $Q_{19} = 171.1366$, $Q_{20} = 171.1376$, $Q_{21} = 171.1382$, $Q_{22} = 171.13855$, $Q_{23} = 171.138746$, $Q_{24} = 171.138858$, $Q_{25} = 171.138922$, $Q_{26} = 171.138959$, $Q_{27} = 171.138980$, $Q_{28} = 171.138992$, $Q_{29} = 171.138999$, $Q_{30} = 171.1390026$, $Q_{31} = 171.1390048$, $Q_{32} = 171.1390061$, $Q_{33} = 171.1390069$, $Q_{34} = 171.1390073$, $Q_{35} = 171.13900751$, $Q_{36} = 171.13900765$, $Q_{37} = 171.13900773$, $Q_{38} = 171.13900777$, $Q_{39} = 171.1390077998$, and $Q_{40} = 171.1390078145$.

From the above sequence, if we decide to stop up to the eighth decimal place, we can state that the optimal solution

$$Q^* \geq 171.13900781. \quad (96)$$

We have computed to the tenth decimal place to obtain that

$$171.1390078339 \leq Q^* \leq 171.1390078341. \quad (97)$$

Because we have constructed a decreasing sequence with $Q_1 = \sqrt{\alpha_1 + \alpha_2}$ and an increasing sequence with $Q_1 = \sqrt{\alpha_1}$, we can accurately estimate the optimal solution.

If we compare our findings with Tung et al. [25], then they derived that $Q^* = 171.139009$ which was obtained by a decreasing sequence proposed by Tung et al. [25]. The result of Tung et al. [25] is consistent with our results when we check our decreasing sequence with $Q_{32} = 171.1390093$ and $Q_{33} = 171.1390087$ to express the finding to the sixth decimal place; then, we will stop at $Q^* \leq 171.139009$.

Furthermore, we compare our results with Lin [38]. She derived that $Q^* = 171.139014$ which is found by a decreasing sequence proposed by Lin [38]. It indicates that the decreasing sequence (Q_i) of Lin [38] did not converge to the desired result. The implicit reason for her unsuccessful converge may result from that there are three sequences in the iterative process of Lin [38]. We can predict that (k_i) or (d_i) converge faster than (Q_i) so that the equal value to the sixth decimal place of (k_i) or (d_i) mislead the convergence

for (Q_i) . The above observation supports that our derivation is better than the three-sequence approach of Lin [38]. Hence, our approach implies one-sequence is a good method. In the next section, we will explore three distinct characters proposed by our approach.

8. Three Distinct Features of Our New Approach

In this section, we show that there are three distinct features of our new approach.

The first feature is that we can derive the optimal solution within the preassigned threshold value. If we obtained a decreasing sequence, say (x_n) , with $x_1 = 271.4$ and an increasing sequence, say (z_n) , with $z_1 = 136.7$, the exact optimal solution is denoted as Q^* .

We know that

$$z_1 < z_2 < \dots < z_n < \dots < Q^* < \dots < x_n < \dots < x_2 < x_1. \quad (98)$$

For a given threshold value, say ε , after we find a number, say m that satisfies

$$x_m - z_m < \varepsilon, \quad (99)$$

then we assume our optimal solution, denoted as Q^Δ with

$$Q^\Delta = \frac{x_m + z_m}{2}, \quad (100)$$

such that we obtain

$$z_m < Q^* < x_m. \quad (101)$$

We combine our findings of equations (99)–(101) to derive that

$$|Q^\Delta - Q^*| < \varepsilon, \quad (102)$$

to indicate our selected solution, Q^Δ , is as close to the exact optimal solution, Q^* , within the preassigned threshold value.

The second feature is that we can estimate the converge ratio of our increasing sequence. Based on equation (91), we find that

$$\frac{Q_{n+2} - Q_{n+1}}{Q_{n+1} - Q_n} \approx \frac{2y_1Q^* + y_3(Q^*)^2 + y_4}{3(Q^*)^2[Q^*(\alpha_4 - \beta\alpha_3Q^*) + 2\beta\alpha_1\alpha_3]^2}. \quad (103)$$

From our results, we obtain that

$$\frac{2y_1Q^* + y_3(Q^*)^2 + y_4}{3(Q^*)^2[Q^*(\alpha_4 - \beta\alpha_3Q^*) + 2\beta\alpha_1\alpha_3]^2} = 0.571840729. \quad (104)$$

We compute several related difference ratios and then list them in the following:

$$\frac{Q_{29} - Q_{28}}{Q_{28} - Q_{27}} = 0.571840853, \quad (105)$$

$$\frac{Q_{30} - Q_{29}}{Q_{29} - Q_{28}} = 0.571840779, \quad (106)$$

$$\frac{Q_{31} - Q_{30}}{Q_{30} - Q_{29}} = 0.571840770. \quad (107)$$

We compare results from equations (105)–(107) with our estimation of equation (104) to show that our estimation of equation (103) is very accurate when n is big enough.

The third feature is that we can estimate the dominant factors to influence the convergence ratio proposed by our approach.

To estimate which factor influences the convergent ratio, we treat

$$\alpha_4 - \beta\alpha_3Q^* \approx \alpha_4, \quad (108)$$

such that we rewrite equation (104) as follows:

$$\frac{y_3(Q^*)^2 + 2y_1Q^* + y_4}{3(Q^*)^2[\alpha_4Q^* + 2\beta\alpha_1\alpha_3]^2}. \quad (109)$$

Based on

$$y_3(Q^*)^2 = 6.16773 \times 10^{18}, \quad (110)$$

$$2y_1Q^* = 2.15036 \times 10^{17}, \quad (111)$$

$$y_4 = 1.97858 \times 10^{18}, \quad (112)$$

$$\alpha_4Q^* = 12835425.59, \quad (113)$$

$$2\beta\alpha_1\alpha_3 = 298916.129, \quad (114)$$

we compare magnitudes of equations (110)–(114) to decide the dominant factors are (i) Q^* , (ii) α_4 , (iii) y_3 , and (iv) y_4 . Hence, we improve our findings of equation (103) to claim that

$$\frac{Q_{n+2} - Q_{n+1}}{Q_{n+1} - Q_n} \approx \frac{y_3(Q^*)^2 + y_4}{3\alpha_4^2(Q^*)^4}. \quad (115)$$

We compute that

$$\begin{aligned} & \frac{(y_3(Q^*)^2 + y_4/3\alpha_4^2(Q^*)^4)}{(2y_1Q^* + y_3(Q^*)^2 + y_4/3(Q^*)^2[Q^*(\alpha_4 - \beta\alpha_3Q^*) + 2\beta\alpha_1\alpha_3]^2)} \\ &= 98.412\%. \end{aligned} \quad (116)$$

According to equation (116), we claim that our dominant factors can explain 98% of our convergent ratio.

We may provide a hypothetical example as follows: $Q_1 = 5.327$, $Q_2 = 2.049$, $Q_3 = 2.043$, and $Q_4 = 2.041$ with $\lim_{n \rightarrow \infty} Q_n = 1$.

The researcher assumed that he will adopt the second decimal place to imply that $Q_1 = 5.33$, $Q_2 = 2.05$, $Q_3 = 2.04$, and $Q_4 = 2.04$.

Because the decimal expressions of Q_3 and Q_4 are the same, the researcher will accept that $Q^* = 2.04$.

However, the convergent result at the second decimal place is not the optimal solution. The above example points out that depending on the convergence of the truncated decimal place without considering the convergent accuracy and ratio may not derive the optimal solution.

Our proposed approach can control the convergent accuracy and ratio that will be useful for future researchers to develop their convergent algorithms.

9. Conclusion

We follow the research trend to revise previously published papers of Wu and Ouyang [21], Tung et al. [25], and Lin [38]. Our approach simplifies the complex derivation in Tung et al. [25], and our generated sequence converges better than that of Lin [38]. At last, we show that our proposed method contained three distinct features. Our findings will provide a complete method for future papers in their proof for the existence and uniqueness of the optimal solution and explanation for their convergent environment.

Data Availability

The data of this inventory model are cited from Wu and Ouyang [21] which was published in Computers & Industrial Engineering; Tung et al. [25] which was published in Abstract and Applied Analysis; and Lin [35] which was published in Abstract and Applied Analysis.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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