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Research Article

Asymptotic Behaviors for Delay Lotka–Volterra Model Disturbed by *G*-Brownian Motion

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In this paper, we propose the stochastic Lotka–Volterra model with delay disturbed by G-Brownian motion $\mathrm{d}x = \mathrm{diag}(x_1, x_2, \ldots, x_n)[(Ax(t-\tau) + b)d\langle B\rangle(t) + \sigma x \mathrm{d}B(t)]$. Under a natural assumption on noise, we study existence and uniqueness of the global positive solution for the system and its asymptotic pathwise moment behavior and prove that the solution does not explode to infinity in a finite time.

1. Introduction

Since the Lotka–Volterra model (LVM in short) was provided by Lotka [1] and Volterra [2], there were extensive works concerned with the dynamics of this system and global stability and the stochastic Lotka–Volterra population model, and in here, we only mention [3, 4] (for deterministic situation) and [5–8] (for stochastic situation). The well-known two-dimensional delay Lotka–Volterra ecological population model driven by Brownian motion is

$$\begin{cases} \mathrm{d}x_1 = x_1\left[(a_1 + b_{11}x_1(t-\tau) + b_{12}x_2(t-\tau))\mathrm{d}t \right. \\ \left. +(c_{11}x_1 + c_{12}x_2)\mathrm{d}W(t)\right], \\ \mathrm{d}x_2 = x_2\left[(a_2 + b_{21}x_1(t-\tau) + b_{22}x_2(t-\tau))\mathrm{d}t \right. \\ \left. +(c_{21}x_1 + c_{22}x_2)\mathrm{d}W(t)\right]. \end{cases} \tag{1}$$

Bahar and Mao in [9] proved that the solution of (1) is almost surely nonnegative and finite. Wu and Xu in [10] investigated stochastic LVM with infinite delay. Global asymptotic stability for a stochastic delay LVM was obtained in [11].

Peng first established the stochastic analysis theory under the G-expectation framework in references [12–14]. Peng's G-expectation space is an essential extension for probability measure space. Since then, many important theoretical results in this field are obtained, for example, SLL for sublinear expectations are obtained in [15], capacity theory results are discussed in [16] and [17–19], and other related technologies in [20–22]. Inspired by these results, we investigate a stochastic delay Lotka–Volterra model disturbed by G-Brownian motion:

$$\mathrm{d}x = \mathrm{diag}(x_1, x_2, \ldots, x_n)[(b + Ax(t-\tau))\mathrm{d}\langle B\rangle(t) + \sigma x \mathrm{d}B(t)], \tag{2}$$

with $\{x(s)\colon -\tau \le s \le 0\} \in C([-\tau, 0]; R_+^n)$, where $x = (x_1, \ldots, x_n)^T$ is a n-dimensional vector, $x_i(t)$ is the population size of species i at time $t(t \ge 0)$, $b = (b_1, b_2, \ldots, b_n)^T$, b_i is the species i's growth rate, $A = (a_{ij})_{n\times n}$ is a $n \times n$ community matrix, $a_{ij}(I \ne j)$ is the interspecific interaction effect, and a_{ii} is the intraspecific interaction effect. We assume that the interaction effect in this system was disturbed by a G-Brownian motion with $\hat{\mathbb{E}}[B(t)^2] = \overline{\sigma}^2 t$ and $\hat{\mathbb{E}}[-B(t)^2] = -\underline{\sigma}^2 t$, where $\sigma = (\sigma_{ij})_{n\times n}$ is a constant matrix, representing the total interference intensity matrix for the system; $B(t)$ has a variance-uncertainty but not mean-uncertainty; $\langle B\rangle(t)$ has a mean-uncertainty property. Therefore, $(\langle B\rangle, B)$ is used to characterise the disturbed growth rate, disturbed interspecific, or intraspecific interactions and interference intensity at the same time. We think the model (2) considers the stochastic interference from both mean-uncertainty and variance-uncertainty, but the traditional stochastic model cannot describe this property. Indeed, we prove the solution of (2) is quasi-surely nonnegative and finite. Some asymptotic pathwise moment estimations for the solutions of this system are presented.

2. Stochastic Delay Lotka–Volterra Model Driven by G-Brownian Motion

Definitions about sublinear expectations, G-Brownian motions, and quadratic variation process $\langle B\rangle(t)$ and notations, as well as more details can also be found in [12–14]. For a matrix A, we denote $|A| = \sqrt{(A^T A)}$ and $\|A\| = \sup\{|Ax|: |x| = 1\}$. $C([-\tau, 0]; R_+^n)$ denotes the family of continuous functions from $[-\tau, 0]$ to R_+^n. We assume the matrix σ satisfies the following assumption:

$$(A)\quad \begin{cases} \sigma_{ii} > 0, & i \in [1, n], \\ \sigma_{ij} \geq 0, & i \neq j \in [1, n]. \end{cases} \tag{3}$$

The assumption (A) was first assumed by Mao et al. in [5], and it is also necessary in our framework.

Theorem 1. *If the matrix σ in system (2) satisfies assumption (A), then $\forall A \in R^{n\times n}$, $b \in R^n$ and $\{x(s): s \in [-\tau, 0]\}$, then there exists a unique solution x of equation (2). Furthermore, $x(s) \in R_+^n$ for all $s \geq -\tau$ quasi-surely, namely, $\nu(\omega: x(s) \in R_+^n, s \in [-\tau, \infty)) = 1$.*

Proof. Because the coefficients of equation (2) are locally Lipschitz continuous, there exist a unique local solution $x(s)$ on $s \in [-\tau, \tau_e)$, where τ_e is called explosion time. To see it is also global, we must show $\tau_\infty = \infty$ q.s. Suppose $k_0 (k_0 > 0)$ is large enough s.t. $x(t) (t \in [-\tau, 0])$ satisfies $1/k_0 < \min |x(t)|$, $\max |x(t)| < k_0$. For any $k (k \geq k_0)$, set $\tau_k = \inf\{s \in [0, \tau_e): x_i(s) \notin (1/k, k), 1 < i \leq n\}$, where $\inf \varnothing = \infty$. Noting that τ_k is increasing when $k \longrightarrow \infty$, let $\tau_\infty = \lim_{k\longrightarrow\infty} \tau_k$, then $\tau_\infty \leq \tau_e$ q.s. If we can prove $\tau_\infty = \infty$ q.s., then $\tau_e = \infty$ q.s. and $x(t) \in R_n^+$ q.s., $t \geq 0$. If $\tau_\infty \neq \infty$ q.s., then $\exists$ a constant $T > 0$ s.t. $V(\omega: \tau_\infty(\omega) \leq T) \geq \varepsilon$ for any $\varepsilon > 0$, namely, $\exists$ an integer $k_1 (k_1 \geq k_0)$ s.t. $V(A_k) \coloneqq V(\omega: \tau_k(\omega) \leq T) \geq \varepsilon$ for all $k \geq k_1$. Let $U: R_n^+ \longrightarrow R^+$ be $U(x) = \sum_{i=1}^n (\sqrt{x_i} - 0.5\log(x_i) - 1)$. Set $k \geq k_0$ and $T > 0$. Using the G-Itô lemma for $\check{V}(t, x) =: U(x) + \int_{t-\tau}^t |x(s)|^2 d\langle B\rangle(s)$, $t \in [0, \tau_k \wedge T]$, we get

$$\begin{aligned} d\check{V}(x, t) = {}& \sum_{i=1}^n \left\{ \left(0.5x_i^{0.5} - 0.5\right)\left(\sum_{j=1}^n a_{ij} x_j(t-\tau) + b_i\right) d\langle B\rangle(t) + \left(-0.125x_i^{0.5} + 0.25\right)\left(\sum_{j=1}^n \sigma_{ij} x_j\right)^2 d\langle B\rangle(t) \right\} \\ & + \sum_{i=1}^n \left(0.5x_i^{0.5} - 0.5\right) \sum_{j=1}^n \sigma_{ij} x_j dB(t) + \left(|x|^2 - |x(t-\tau)|^2\right) d\langle B\rangle(t), \end{aligned} \tag{4}$$

and noting that

$$\sum_{i=1}^n \left[\left(0.5x_i^{0.5} - 0.5\right)\left(b_i + \sum_{j=1}^n a_{ij} x_j(t-\tau)\right)\right] \leq \frac{1}{2}\sum_{i=1}^n \left(x_i^{0.5} - 1\right) b_i + |x(t-\tau)|^2 + \sum_{i=1}^n \sum_{j=1}^n \frac{n a_{ij}^2}{16}\left(x_i^{0.5} - 1\right)^2, \tag{5}$$

and $\sum_{i=1}^n (\sum_{j=1}^n \sigma_{ij} x_j)^2 \leq \sum_{i=1}^n |\sigma|^2 x_i^2 = |\sigma|^2 |x|^2$, as well as $\sum_{i=1}^n x_i^{0.5} (\sum_{j=1}^n \sigma_{ij} x_j)^2 \geq \sum_{i=1}^n \sigma_{ii}^2 x_i^{2.5}$ by the assumption (A). Thus,

$$\begin{aligned} d\check{V}(t, x(t)) \leq {}& \left[\sum_{i=1}^n \frac{1}{2}\left(x_i^{0.5} - 1\right) b_i + |x(t)|^2 + \sum_{i=1}^n \sum_{j=1}^n \frac{n a_{ij}^2}{16}\left(x_i^{0.5} - 1\right)^2\right] d\langle B\rangle(t) + \left[-0.125 \sum_{i=1}^n \sigma_{ii}^2 x_i^{2.5} + 0.25|\sigma|^2 |x|^2\right] d\langle B\rangle(t) \\ & + \sum_{i=1}^n \left(0.5x_i^{0.5} - 0.5\right) \sum_{j=1}^n \sigma_{ij} x_j dB(t). \end{aligned} \tag{6}$$

Denote

$$f(x,a,b,\sigma) = \sum_{i=1}^{n}\frac{1}{2}\left(x_i^{0.5}-1\right)b_i + |x|^2\left(1+\frac{1}{4}|\sigma|^2\right) + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{na_{ij}^2}{16}\left(x_i^{0.5}-1\right)^2 - \frac{1}{8}\sum_{i=1}^{n}\sigma_{ii}^2 x_i^{2.5}, \tag{7}$$

since we note that there is K s.t. $f(x, a, b, \sigma)$ is bounded, namely, $f(x, a, b, \sigma) < K$, then

$$\hat{\mathbb{E}}\left[\int_{\tau_k\wedge T-\tau}^{T\wedge\tau_k}|x|^2\mathrm{d}\langle B\rangle(s) + U(x(T\wedge\tau_k))\right] \le \hat{\mathbb{E}}\left[\int_{-\tau}^{0}|x|^2\mathrm{d}\langle B\rangle(s)\right] + U(x_0) + K\overline{\sigma}^2\hat{\mathbb{E}}[T\wedge\tau_k] \le \overline{\sigma}^2\int_{-\tau}^{0}\hat{\mathbb{E}}\left[|x|^2\right]\mathrm{d}s + U(x_0) + K\overline{\sigma}^2\hat{\mathbb{E}}[T\wedge\tau_k], \tag{8}$$

then $\hat{\mathbb{E}}[U(x(\tau_k\wedge T))] \le U(x_0) + \overline{\sigma}^2\int_{-\tau}^{0}\hat{\mathbb{E}}[|x(s)|^2]\mathrm{d}s + K\overline{\sigma}^2 T < \infty$. From the definition of τ_k, we know $\forall\omega\in A_k$, $\exists$ some i s.t. $x_i(\tau_k, \omega)\notin(1/k, k)$, namely, $x_i(\tau_k)\le 1/k$, or $x_i(\tau_k)\ge k<\infty$. Noting that the function $U(x_i)$ is decreasing when $0<x_i\le 1$ and is increasing when $x_i>1$, hence $U(x(\tau_k))\ge U(1/k, \ldots, 1/k)$ and $U(x(\tau_k))\ge U(k, \ldots, k)$, namely, $U(x(\tau_k))\ge \max\left\{\sqrt{(1/k)} - 0.5\log(1/k) - 1, \sqrt{k} - 0.5\log(k) - 1\right\}$. Therefore, we have

$$\varepsilon C_k \le C_k V(A_k) \le \hat{\mathbb{E}}\left[I_{A_k}U(x(\tau_k))\right] \le \hat{\mathbb{E}}[U(x(T\wedge\tau_k))] \le U(x_0) + \overline{\sigma}^2\int_{-\tau}^{0}\hat{\mathbb{E}}\left[|x|^2\right]\mathrm{d}t + K\overline{\sigma}^2 T. \tag{9}$$

Setting $k\longrightarrow\infty$, we have the contradiction $\infty\le\hat{\mathbb{E}}[U(x(\tau_k\wedge T))]<\infty$; therefore, we have $\tau_\infty=\infty$ q.s., namely, $\tau_e=\infty$ q.s., so $\nu(\omega: x(t)\in R_+^n, t\ge 0) = 1$.

3. Asymptotic Behaviors of the Solution

Theorem 2. *Under the assumption (A), if* $\hat{\mathbb{E}}[B(1)^2] = \overline{\sigma}^2 \le 1$, *for any* $\beta\in(0, 1)$ *and* $\delta\in(0, 1)$, $\exists C_0 = C(\delta) > 0$ *s.t. the solution* $x(t)$ *of equation (2) is as follows:*

$$\limsup_{t\longrightarrow\infty} V\left(e^{(\langle B\rangle(t)-\overline{\sigma}^2 t)/\beta}|x(t)|\le C_0\right)\ge 1-\delta. \tag{10}$$

Proof. Let

$$U(x) = \sum_{i=1}^{n}x_i^{\beta}. \tag{11}$$

Using the G-Itô lemma for $U(x)$ and noting

$$\mathrm{d}x_i(t) = x_i\left[\left(\sum_{j=1}^{n}a_{ij}x_j(t-\tau)+b_i\right)\mathrm{d}\langle B\rangle(t) + \sum_{j=1}^{n}\sigma_{ij}x_j\mathrm{d}B(t)\right], \tag{12}$$

we have

$$\mathrm{d}U(x) = \sum_{i=1}^{n}\beta x_i^{\beta}\left[\left(b_i+\sum_{j=1}^{n}a_{ij}x_j(t-\tau)\right)\mathrm{d}\langle B\rangle(t) + \sum_{j=1}^{n}\sigma_{ij}x_j\mathrm{d}B(t)\right] + \frac{1}{2}\beta(\beta-1)\sum_{i=1}^{n}x_i^{\beta}\left[\sum_{j=1}^{n}\sigma_{ij}x_j\right]^2\mathrm{d}\langle B\rangle(t). \tag{13}$$

Since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\beta x_i^{\beta}a_{ij}x_j(t-\tau) \le \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{n}{4}\left(\beta a_{ij}x_i^{\beta}\right)^2 + \frac{x_j(t-\tau)^2}{n}\right], \tag{14}$$

and from equation (13), we get

$$\mathrm{d}U(x) \le \left[\sum_{i=1}^{n}\beta b_i x_i^{\beta} + \frac{n}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\beta a_{ij}x_i^{\beta}\right)^2\right]\mathrm{d}\langle B\rangle(t) + \left[\frac{1}{2}\beta(\beta-1)\sum_{i=1}^{n}x_i^{\beta+2}\sigma_{ii}^2 + |x(t-\tau)|^2\right]\mathrm{d}\langle B\rangle(t) + \left(\sum_{i=1}^{n}\beta x_i^{\beta}\sum_{j=1}^{n}\sigma_{ij}x_j\right)\mathrm{d}B(t), \tag{15}$$

then

$$d\left(e^{\langle B\rangle(t)}U(x(t))\right) = e^{\langle B\rangle(t)}U(x(t))d\langle B\rangle(t) + e^{\langle B\rangle(t)}\mathrm{d}U(x(t))$$
$$\leq e^{\langle B\rangle(t)}\left[\sum_{i=1}^{n}(\beta b_i+1)x_i^{\beta} + \frac{n}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\beta a_{ij}x_i^{\beta}\right)^2\right]\mathrm{d}\langle B\rangle(t) + \exp(\langle B\rangle(t))\left[|x(t-\tau)|^2 + 0.5\beta(\beta-1)\sum_{i=1}^{n}x_i^{\beta+2}\sigma_{ii}^2\right]\mathrm{d}\langle B\rangle(t)$$
$$+ e^{\langle B\rangle(t)}\left(\sum_{i=1}^{n}\beta x_i^{\beta}\sum_{j=1}^{n}\sigma_{ij}x_j\right)\mathrm{d}B(t). \tag{16}$$

We set

$$F_1(x) = \sum_{i=1}^{n}(1+\beta b_i)x_i^{\beta} + \frac{n}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\beta a_{ij}x_i^{\beta}\right)^2 + e^{\tau}|x|^2 - \frac{1}{2}\beta(1-\beta)\sum_{i=1}^{n}x_i^{\beta+2}\sigma_{ii}^2, \tag{17}$$

then $F_1(x)$ is bounded in R_+^n , say K_1, from (16),

$$d\left(e^{\langle B\rangle(t)}U(x(t))\right) \leq e^{\langle B(t)\rangle}\left[K_1 - e^{\tau}|x(t)|^2 + |x(t-\tau)|^2\right]d\langle B\rangle(t) + e^{\langle B\rangle(t)}\left(\sum_{i=1}^{n}\beta x_i^{\beta}\sum_{j=1}^{n}\sigma_{ij}x_j\right)\mathrm{d}B(t), \tag{18}$$

namely,

$$\begin{aligned}\widehat{\mathbb{E}}\left[e^{\langle B\rangle(t)}U(x(t))\right] &\leq U_0 + \widehat{\mathbb{E}}\left[\int_0^t e^{\langle B\rangle(s)}\left(K_1 - e^{\tau}|x(s)|^2 + |x(s-\tau)|^2\right)\mathrm{d}\langle B\rangle(s)\right]\\ &\leq U_0 + \widehat{\mathbb{E}}\left[\int_0^t e^{\overline{\sigma}^2 s}\left(K_1 - e^{\tau}|x(s)|^2 + |x(s-\tau)|^2\right)\mathrm{d}\langle B\rangle(s)\right]\\ &\leq U_0 + \overline{\sigma}^2\widehat{\mathbb{E}}\left[\int_0^t e^{\overline{\sigma}^2 s}\left(K_1 - e^{\tau}|x(s)|^2 + |x(s-\tau)|^2\right)\mathrm{d}s\right]\\ &\leq U_0 + K_1 e^{\overline{\sigma}^2 t} + \overline{\sigma}^2\widehat{\mathbb{E}}\left[\int_0^t e^{\overline{\sigma}^2 s}|x(s-\tau)|^2 - e^{\overline{\sigma}^2 s+\tau}|x(s)|^2\mathrm{d}s\right]\\ &= U_0 + K_1 e^{\overline{\sigma}^2 t} + \overline{\sigma}^2\widehat{\mathbb{E}}\left[\int_{-\tau}^{t-\tau} e^{\overline{\sigma}^2(s+\tau)}|x(s)|^2 - e^{\overline{\sigma}^2 s+\tau}|x(s)|^2\mathrm{d}s\right],\end{aligned} \tag{19}$$

where $U_0 = U(x(0))$, and noting that $\overline{\sigma}^2$ satisfies $\overline{\sigma}^2 \leq 1$, by (19), then

$$\widehat{\mathbb{E}}\left[e^{\langle B\rangle(t)}U(x(t))\right] \leq U_0 + K_1 e^{\overline{\sigma}^2 t} + \overline{\sigma}^2\widehat{\mathbb{E}}\left[\int_{-\tau}^{0} e^{\overline{\sigma}^2(s+\tau)}|x(s)|^2\mathrm{d}s\right], \tag{20}$$

therefore,

$$\limsup_{t\longrightarrow\infty}\widehat{\mathbb{E}}\left[e^{\langle B\rangle(t)-\overline{\sigma}^2 t}U(x(t))\right] \leq K_1. \tag{21}$$

In addition, we note

$$|x|^{\beta} = \left(\sum_{i=1}^{n}x_i^2\right)^{\beta/2} \leq n^{\beta/2}\max_{1\leq i\leq n}x_i^{\beta} \leq n^{\beta/2}U(x), \tag{22}$$

so

$$\limsup_{t\longrightarrow\infty}\widehat{\mathbb{E}}\left[e^{\langle B\rangle(t)-\overline{\sigma}^2 t}|x|^{\beta}\right] \leq K_0, \tag{23}$$

$\forall\delta > 0$, let $C_0 = (K_0/\delta)^{1/\beta}$, then

$$v\left(\exp\left(\frac{\langle B\rangle(t) - \overline{\sigma}^2 t}{\beta}\right)|x| > C_0\right) \leq \frac{\widehat{\mathbb{E}}\left[\exp(\langle B\rangle(t) - \overline{\sigma}^2 t)|x|^{\beta}\right]}{C_0^{\beta}} \leq \frac{K_0}{C_0^{\beta}} = \delta. \tag{24}$$

Hence,

$$\limsup_{t\longrightarrow\infty}V\left(e^{(\langle B\rangle(t)-\overline{\sigma}^2 t)/\beta}|x| \leq C_0\right) \geq 1-\delta. \tag{25}$$

Theorem 3. *Suppose the (A) is true, and there exists $K(K > 0)$ is independent of $\{x(s)\colon s\in[-\tau, 0]\}$, then*

$$\limsup_{T\longrightarrow\infty}\frac{-1}{T}\hat{\mathbb{E}}\left[\int_0^T -|x|^2 \mathrm{d}\langle B\rangle(s)\right]\le K\overline{\sigma}^2. \quad (26)$$

Proof. Write (7) as $g(x,a,b,\sigma)=g_1(x,a,b,\sigma)-|x|^2$, where

$$g_1(x,a,b,\sigma) := \sum_{i=1}^n 0.5\left(x^{0.5}-1\right)b_i+\sum_{i=1}^n\sum_{j=1}^n\frac{na_{ij}^2}{16}\left(x^{0.5}-1\right)^2 - 0.125\sum_{i=1}^n\sigma_{ii}^2x_i^{2.5}+|x|^2\left(2+0.25|\sigma|^2\right), \quad (27)$$

then $g(x,a,b,\sigma)\le K-|x|^2$, where $K:=\max_{x\in R_+^n}g_1(x)<\infty$. Taking expectation from 0 to $\tau_k\wedge T$ on both sides of equation (6), we have

$$\begin{aligned}0&\le\hat{\mathbb{E}}\left[\int_{-\tau}^0|x|^2\mathrm{d}\langle B\rangle(s)\right]+U(x_0)+\hat{\mathbb{E}}\left[\int_0^{\tau_k\wedge T}\left(K-|x|^2\right)\mathrm{d}\langle B\rangle(s)\right]\\&\le\overline{\sigma}^2\int_{-\tau}^0\hat{\mathbb{E}}\left[|x|^2\right]\mathrm{d}s+U(x_0)+K\overline{\sigma}^2\hat{\mathbb{E}}[T\wedge\tau_k]+\hat{\mathbb{E}}\left[\int_0^{T\wedge\tau_k}-|x|^2\mathrm{d}\langle B\rangle(s)\right].\end{aligned} \quad (28)$$

Letting $k\longrightarrow\infty$ yields

$$-\hat{\mathbb{E}}\left[\int_0^T-|x|^2\mathrm{d}\langle B\rangle(s)\right]\le\overline{\sigma}^2\int_{-\tau}^0\hat{\mathbb{E}}\left[|x|^2\right]\mathrm{d}s+U(x_0)+K\overline{\sigma}^2T. \quad (29)$$

Therefore, setting $T\longrightarrow\infty$,

$$\limsup_{T\longrightarrow\infty}\frac{-1}{T}\hat{\mathbb{E}}\left[\int_0^T -|x|^2 \mathrm{d}\langle B\rangle(s)\right]\le K\overline{\sigma}^2. \quad (30)$$

4. Asymptotic Moment Estimations

Theorem 4. *If condition (A) is true, then* $\forall\ \{x(s): s\in[-\tau,0]\}$, *x(t) in (2) satisfies*

$$\limsup_{t\longrightarrow\infty}\frac{1}{t}\hat{\mathbb{E}}\left[\log\left(\frac{|x(t)|}{\sqrt{n}}\right)+\frac{\widehat{\sigma}^2}{4n}\int_0^t|x(s)|^2\mathrm{d}\langle B\rangle(s)\right]\le\overline{\sigma}^2K, \quad (31)$$

where $\widehat{\sigma}=\min_{1\le i\le n}\sigma_{ii}$.

Proof. Let $\widetilde{V}(x)=\sum_{i=1}^n x_i(t)$ for $x\in R_+^n$, then by G-Itô's lemma of Reference [13], we have

$$\log(\widetilde{V}(x))=C_0+\int_0^t\frac{x^T(s)}{V(x(s))}(b+Ax(s-\tau))\mathrm{d}\langle B\rangle(s)+\int_0^t\frac{x^T(s)\sigma x(s)}{\widetilde{V}(x(s))}\mathrm{d}B(s)-\int_0^t\frac{\left|x^T(s)\sigma x(s)\right|^2}{2\widetilde{V}^2(x)}\mathrm{d}\langle B\rangle(s), \quad (32)$$

where $C_0=\log(\tilde{V}(x(0)))$. Noting that

$$\left\langle\int_0^t\frac{x^T\sigma x}{\widetilde{V}(x)}\mathrm{d}B(s),\left|\int_0^t\frac{\left|x^T\sigma x\right|^2}{\widetilde{V}^2(x)}\right|\mathrm{d}\langle B\rangle(s)\right\rangle=\int_0^t\frac{\left|x^T\sigma x\right|^2}{\widetilde{V}^2(x)}\mathrm{d}\langle B\rangle(s), \quad (33)$$

$\forall\varepsilon\in(0,1/2)$, from Lemma 3.1 in reference [19], for any integer $k\ge1$, we have

$$V\left(\sup_{0\le t\le k}\left[\int_0^t\frac{x^T\sigma x}{\widetilde{V}(x)}\mathrm{d}B(s)-\frac{\varepsilon}{2}\int_0^t\frac{\left|x^T\sigma x\right|^2}{\widetilde{V}^2(x)}\mathrm{d}\langle B\rangle(s)\right]>\frac{2}{\varepsilon}\ln k\right)\le\frac{1}{k^2}, \quad (34)$$

so

$$\sum_{k=1}^\infty V\left(\sup_{0\le t\le k}\left[\int_0^t\frac{x^T\sigma x}{\widetilde{V}(x)}\mathrm{d}B(s)-\frac{\varepsilon}{2}\int_0^t\frac{\left|x^T\sigma x\right|^2}{\widetilde{V}^2(x)}\mathrm{d}\langle B\rangle(s)\right]>\frac{2}{\varepsilon}\ln k\right)<\infty, \quad (35)$$

applying Lemma 2 in [15], we know for all but finitely many k,

$$\sup_{0\le t\le k}\left[\int_0^t\frac{x^T\sigma x}{\widetilde{V}(x)}\mathrm{d}B(s)-\frac{\varepsilon}{2}\int_0^t\frac{\left|x^T\sigma x\right|^2}{\widetilde{V}^2(x)}\mathrm{d}\langle B\rangle(s)\right]\le\frac{\ln k^2}{\varepsilon}, \quad (36)$$

quasi-surely true, i.e., $\exists\Omega_i\subset\Omega$ $(v(\Omega i)=1)$ s.t. $\forall\omega\in\Omega_i$ and $k_i=k_i(\omega)$ s.t.

$$\int_0^t\frac{x^T(s)\sigma x}{\widetilde{V}(x)}\mathrm{d}B(s)-\frac{\varepsilon}{2}\int_0^t\frac{\left|x^T\sigma x\right|^2}{\widetilde{V}^2(x)}d\langle B\rangle(s)\le\frac{\ln k^2}{\varepsilon},\quad 0\le t\le k, \quad (37)$$

$k\ge k_i(\omega)$. From equation (32) and inequality (37),

$$\log(\widetilde{V}(x))\le C_0+\frac{\ln k^2}{\varepsilon}+\int_0^t\left[\sqrt{n}\left(|x(s-\tau)|\|A\|+|b|\right)-\widehat{\sigma}^2|x|^2\frac{(1-\varepsilon)}{2n}\right]\mathrm{d}\langle B\rangle(s), \quad (38)$$

$t\in[0,k_i(\omega)]$, $k\ge k_i(\omega)$, in other words,

$$\log(\widetilde{V}(x))+\widehat{\sigma}^2\frac{(1-2\varepsilon)}{4n}\int_0^t|x|^2d\langle B\rangle(s)\le C_0+\frac{\ln k^2}{\varepsilon}+\int_0^t\left[\sqrt{n}\left(|x(s-\tau)|\|A\|+|b|\right)-|x|^2\frac{\widehat{\sigma}^2}{4n}\right]d\langle B\rangle(s), \quad (39)$$

where $\widehat{\sigma}=\min\sigma_{ii}\ (i\in[1,n])$. Taking G-expectation $\hat{\mathbb{E}}$ for (39), and then $\forall\omega\in\cap_{i=1}^n\Omega_i$, from (39), we get

$$\begin{aligned}
&\widehat{\mathbb{E}}\left[\widehat{\sigma}^2\frac{(1-2\varepsilon)}{4n}\int_0^t|x|^2\mathrm{d}\langle B\rangle(s)+\log(\widetilde{V}(x))\right]\\
&\le\frac{\ln k^2}{\varepsilon}+\overline{\sigma}^2\widehat{\mathbb{E}}\left[\int_0^t(|x(s-\tau)|\|A\|+|b|)\sqrt{n}-\frac{\widehat{\sigma}^2}{4n}|x|^2\mathrm{d}s\right]+C_0\\
&\le C_0+\frac{\ln k^2}{\varepsilon}+\sqrt{n}\|A\|\overline{\sigma}^2\widehat{\mathbb{E}}\left[\int_{-\tau}^0|x|\mathrm{d}s\right]\\
&\quad+\overline{\sigma}^2\widehat{\mathbb{E}}\left[\int_0^t\sqrt{n}(|b|+\|A\||x|)-\frac{\widehat{\sigma}^2}{4n}|x|^2\mathrm{d}s\right]\\
&\le C_0+\frac{2\ln k}{\varepsilon}+\sqrt{n}\|A\|\overline{\sigma}^2\widehat{\mathbb{E}}\left[\int_{-\tau}^0|x|\mathrm{d}s\right]+\overline{\sigma}^2Kt,
\end{aligned} \tag{40}$$

where $\sqrt{n}(\|A\||x|+|b|)-(\widehat{\sigma}^2/4n)|x(s)|^2\le K$. Set $\max\{k_i(\omega), i\in[1,n]\}=k_0(\omega)$, then $\forall\omega\in\cap_{i=1}^n\Omega_i$, $t\in[k-1,k]$, $k\ge k_0(\omega)$, it gets from (40):

$$\limsup_{t\longrightarrow\infty}\frac{1}{t}\widehat{\mathbb{E}}\left[\log(\widetilde{V}(x))+\frac{\widehat{\sigma}^2(1-2\varepsilon)}{4n}\int_0^t|x|^2d\langle B\rangle(s)\right]\le\overline{\sigma}^2K. \tag{41}$$

Letting ε tend to zero and noting that $|x|\le\sqrt{n}V(x)$ yield

$$\limsup_{t\longrightarrow\infty}\frac{1}{t}\widehat{\mathbb{E}}\left[\log\left(\frac{|x(t)|}{\sqrt{n}}\right)+\frac{\widehat{\sigma}^2}{4n}\int_0^t|x(s)|^2d\langle B\rangle(s)\right]\le\overline{\sigma}^2K. \tag{42}$$

The proof is complete.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

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References

[1] A. J. Lotka, *Elements of Physical Biology*, William and Wilkins, Baltimore, MD, USA, 1925.
[2] V. Volterra, "Variazioni e fluttuazioni del numero dindividui in specie d'animani conviventi," *Mem. Acad. Lincei*, vol. 2, pp. 31–113, 1926.
[3] Y. Lou, S. Martínez, and P. Poláčik, "Loops and branches of coexistence states in a Lotka-Volterra competition model," *Journal of Differential Equations*, vol. 230, no. 2, pp. 720–742, 2006.
[4] Y. Zhang and M. Wang, "Bifurcation of Lotka-Volterra competition model with nonlinear boundary conditions," *Applied Mathematics Letters*, vol. 38, pp. 52–56, 2014.
[5] X. Mao, G. Marion, and E. Renshaw, "Environmental Brownian noise suppresses explosions in population dynamics," *Stochastic Processes and Their Applications*, vol. 97, no. 1, pp. 95–110, 2002.
[6] X. Mao, S. Sabanis, and E. Renshaw, "Asymptotic behaviour of the stochastic Lotka-Volterra model," *Journal of Mathematical Analysis and Applications*, vol. 287, no. 1, pp. 141–156, 2003.
[7] A. Hastings, "Global stability in Lotka-Volterra systems with diffusion," *Journal of Mathematical Biology*, vol. 6, no. 2, pp. 163–168, 1978.
[8] N. H. Du and V. H. Sam, "Dynamics of a stochastic Lotka-Volterra model perturbed by white noise," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 1, pp. 82–97, 2006.
[9] A. Bahar and X. Mao, "Stochastic delay Lotka-Volterra model," *Journal of Mathematical Analysis and Applications*, vol. 292, no. 2, pp. 364–380, 2004.
[10] F. Wu and Y. Xu, "Stochastic Lotka-Volterra population dynamics with infinite delay," *SIAM Journal on Applied Mathematics*, vol. 70, no. 3, pp. 641–657, 2009.
[11] Y. Huang, Q. Liu, and Y. Liu, "Global asymptotic stability of a general stochastic Lotka-Volterra system with delays," *Applied Mathematics Letters*, vol. 26, no. 1, pp. 175–178, 2013.
[12] S. Peng, "*G*-expectation, *G*-brownian motion and related stochastic calculus of Itô type," in *Stochastic Analysis and Applications, Abel Symp. 2*, F. E. Benth, G. Di Nunno, T. Lindstrøm et al., Eds., pp. 541–567, Springer, Berlin, Germany, 2007.
[13] S. Peng, "Nonlinear expectations and stochastic calculus under uncertainty-with robust central limit theorem and *G*-Brownian motion," 2010, https://arxiv.org/abs/1002.4546.
[14] S. Peng, "Backward stochastic differential equation, nonlinear expectation and their applications," in *Proceedings of the International Congress of Mathematicians*, World Scientific, Singapore, pp. 393–432, 2010.
[15] Z. Chen, "Strong laws of large numbers for sub-linear expectations," *Science China Mathematics*, vol. 59, no. 5, pp. 945–954, 2016.
[16] L. Denis, M. Hu, and S. Peng, "Function spaces and capacity related to a sublinear expectation: application to *G*-brownian motion paths," *Potential Analysis*, vol. 34, no. 2, pp. 139–161, 2011.
[17] H. M. Soner, N. Touzi, and J. Zhang, "Martingale representation theorem for the *G*-expectation," *Stochastic Processes and Their Applications*, vol. 121, no. 2, pp. 265–287, 2011.
[18] F. Gao, "Pathwise properties and homeomorphic flows for stochastic differential equations driven by *G*-brownian motion," *Stochastic Processes and Their Applications*, vol. 119, pp. 3356–3382, 2009.
[19] D. Zhang and Z. Chen, "Exponential stability for stochastic differential equation driven by *G*-brownian motion," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1906–1910, 2012.
[20] W. Yu, Y. Yong, G. Guan, Y. Huang, W. Su, and C. Cui, "Valuing guaranteed minimum death benefits by cosine series expansion," *Mathematics*, vol. 7, no. 9, p. 835, 2019.
[21] Z. Zhang, Y. Yong, and W. Yu, "Valuing equity-linked death benefits in general exponential Lévy models," *Journal of Computational and Applied Mathematics*, vol. 365, Article ID 112377, 2020.
[22] W. Yu, F. Wang, Y. Huang, and H. Liu, "Social Optimal mean field control problem for population growth model," *Asian Journal of Control*, vol. 21, 2019.