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Review Article

A Summary of Dynamic Output Feedback Robust MPC for Linear Polytopic Uncertainty Model with Bounded Disturbance

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This paper is the summary and enhancement of the previous studies on dynamic output feedback robust model predictive control (MPC) for the linear parameter varying model (described in a polytope) with additive bounded disturbance. When the state is measurable and there is no bounded disturbance, the robust MPC has been developed with several paradigms and seems becoming mature. For the output feedback case for the LPV model with bounded disturbance, we have published a series of works. Anyway, it lacks a unification of these publications. This paper summarizes the existing results and exposes the ideas in a unified framework. Indeed there is a long way to go for the output feedback case for the LPV model with bounded disturbance. This paper can pave the way for further research on output feedback MPC.

1. Introduction

In the control community, it is widely recognized that linear parameter varying (LPV) model, whose system matrices lie in the polytope, is a good tool for representing the nonlinearity and uncertainty. The well-known Takagi–Sugeno (T-S) model (see, e.g., [1, 2]), often when the stability is considered, can be considered as the LPV model. Therefore, it is not surprising that there are a lot of research works on the LPV model-based and T-S model-based controls. Moreover, it is impossible that all the uncertainties can be included in the parametric polytopes. The additive bound disturbance, with its real-time value arbitrarily changing, without useful statistics, is another widely accepted uncertainty description. This paper considers the above LPV model (including T-S model) with additive bound disturbance.

The research on robust model predictive control (MPC) for LPV model has begun as early as in 1996 (see [3]). After researching for slightly longer than a decade, the robust MPC for LPV model (excluding T-S model and bounded disturbance), when the state is assumed measurable, seems becoming mature; there are four types in this robust MPC community, i.e., open-loop MPC, partial feedback MPC, feedback MPC, and parameter-dependent open-loop MPC (see the Introduction of [4]). In the partial feedback, the control move u is defined as $u = Fx + c$ (i.e., state feedback Fx plus perturbation c); when $c = 0$, the partial feedback becomes the feedback and when $F = 0$, open-loop. When the switching horizon $N = 0$ or $N = 1$, the four types are equivalent. When $N \geq 2$, u can be defined as parameter-dependent as in [4]; in this parameter-dependent case, open-loop is equivalent to partial feedback.

From 2006, we have begun research on robust MPC for LPV model (including T-S model and bounded disturbance), where the state can be unmeasurable. We have published several works, emphasizing on $N = 0$, i.e., a close generalization of [3]. For $N > 1$, we have not reached to a technique which is, to us, as satisfactory as that in the case when the state x is measurable. Therefore, this paper concentrates on the output feedback robust MPC with $N = 0$. $N = 0$ here indicates that there is no free control move, i.e., both u and c will not be the immediate decision variables.

Although we have published several works on output feedback MPC, there lacks a unified and updated framework. These works are given across more than 10 years. The results are scattered in different works; there are necessary overlaps due to problem statements and recalls; some of the results are improved which are not easy to trace back; some of the details are missed in all published results; the original thoughts may be overlooked. In this paper, we rearrange the results of output feedback MPC for the LPV model during these years, compromising the above demerits in the existing works. We think that this is useful for future research; it is not only a guideline, but also a summary for readers.

Notations: I is the unitary matrix with appropriate dimension; $x(k+i\,|\,k)$ is the prediction of $x(k+i)$ at time k. Moreover,

(i) u: in $\mathfrak{R}^{n_u}$, the control input signal
(ii) w: in $\mathfrak{R}^{n_w}$, the disturbance
(iii) x: in $\mathfrak{R}^{n_x}$, the true state
(iv) x_c: in $\mathfrak{R}^{n_{x_c}}$, the estimator state or controller state
(v) y: in $\mathfrak{R}^{n_y}$, the output
(vi) $|\xi|$: the component-wise absolute value of ξ
(vii) ε_M: the ellipsoid associated with the positive-definite matrix M, i.e., $\varepsilon_M = \left\{\xi\,|\,\xi^{\mathrm{T}}M\xi \le 1\right\}$
(viii) Co$\mathcal{S}$: an element belonging to Co$\mathcal{S}$ means that it is a convex combination of the elements in the polytope $\mathcal{S}$, with the scalar combing coefficients being nonnegative and summing as 1
(ix) ★: this symbol induces a symmetric structure in any square matrix
(x) $*$: a value with superscript $*$ means that it is the solution of the optimization problem

2. Dynamic Output Feedback Robust MPC Problem

Consider the following linear parameter varying (LPV) model:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) + D(k)w(k), \\ y(k) = C(k)x(k) + E(k)w(k), \\ z(k) = \mathscr{C}(k)x(k) + \mathscr{E}(k)w(k), \\ z'(k) = \mathscr{F}(k)x(k) + \mathscr{G}(k)w(k), \end{cases} \tag{1}$$

where $z(k) \in \mathfrak{R}^{n_z}$ (see [5, 6]) and $z'(k) \in \mathfrak{R}^{n_z'}$ (see [7, 8]) are the constrained signal and penalized signal, respectively, and w is unknown, norm-bounded, and persistent.

Assumption 1. $\|w(k)\| \le 1$ for all $k \ge 0$.

Assumption 2. $[A\,|\,B\,|\,C\,|\,D\,|\,E\,|\,\mathscr{C}\,|\,\mathscr{E}\,|\,\mathscr{F}\,|\,\mathscr{G}](k) \in \Omega := \mathrm{Co}\left\{[A_l\,|\,B_l\,|\,C_l\,|\,D_l\,|\,E_l\,|\,\mathscr{C}_l\,|\,\mathscr{E}_l\,|\,\mathscr{F}_l\,|\,\mathscr{G}_l\,]\,|\,l = 1,\ldots,L]\right\}$, i.e., there exist nonnegative coefficients $\lambda_l(k), l = 1,\ldots,L$ such that $\sum_{l=1}^{L}\lambda_l(k) = 1$ and $[A\,|\,B\,|\,C\,|\,D\,|\,E\,|\,\mathscr{C}\,|\,\mathscr{E}\,|\,\mathscr{F}\,|\,\mathscr{G}](k) = \sum_{l=1}^{L}\lambda_l(k)[A_l\,|\,B_l\,|\,C_l\,|\,D_l\,|\,E_l\,|\,\mathscr{C}_l\,|\,\mathscr{E}_l\,|\,\mathscr{F}_l\,|\,\mathscr{G}_l]$.

Since $D(k), E(k)$ are shaping matrices, Assumption 1 applies to any norm-bounded disturbance. If $\lambda_l(k)$'s are exactly known at the current time k, but $\lambda_l(k+i)$ for all $i > 0$ are unknown at the current time k, then we specially call (1) the quasi-LPV model.

The hard physical constraints are

$$\begin{aligned} |u(k)| &\le \overline{u}, \\ |\Psi z(k+1)| &\le \overline{\psi}, \\ k &\ge 0, \end{aligned} \tag{2}$$

where $\overline{u} = [\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{n_u}]^{\mathrm{T}}$; $\overline{\psi} = [\overline{\psi}_1, \overline{\psi}_2, \ldots, \overline{\psi}_q]^{\mathrm{T}}$; $\overline{u}_j > 0$, $j = 1,\ldots,n_u$; $\overline{\psi}_j > 0$, $j = 1,\ldots,q$; $\Psi \in \mathfrak{R}^{q\times n_z}$.

When x is fully measurable and $w(k) \equiv 0$, Kothare et al. [3] have developed a technique which, at each time k, solves a linear matrix inequality (LMI) optimization problem with four constraints (confinement of the current state, invariance/stability/optimality condition, input constraint, and state/output constraint). In the following, we will generalize the procedure of [3] to the cases when x can be unmeasurable and $w(k) \ne 0$.

Theorem 1 (see [9]). *Consider system (1), with Assumptions 1 and 2 being satisfied. Adopt the dynamic output feedback controller, i.e.,*

$$\begin{cases} x_c(k+1) = A_c(k)x_c(k) + B_c(k)y(k), \\ u(k) = C_c(k)x_c(k) + D_c(k)y(k), \end{cases} \tag{3}$$

where the controller parameters are defined as parameter-dependent, i.e.,

$$\begin{cases} A_c(k) = \sum_{l=1}^{L}\sum_{j=1}^{L}\lambda_l(k)\lambda_j(k)\overline{A}_c^{lj}(k), \\ B_c(k) = \sum_{l=1}^{L}\lambda_l(k)\overline{B}_c^{l}(k), \\ C_c(k) = \sum_{j=1}^{L}\lambda_j(k)\overline{C}_c^{j}(k), \\ D_c(k) = \overline{D}_c(k). \end{cases} \tag{4}$$

The controller parametric matrices $\left\{\overline{A}_c^{lj}, \overline{B}_c^{l}, \overline{C}_c^{j}, \overline{D}_c\right\}(k)$ *are taken as*

$$\begin{cases} \overline{D}_c = \widehat{D}_c, \\ \overline{C}_c^{j} = \left(\widehat{C}_c^{j} - \overline{D}_c C_j Q_1\right)Q_2^{-1}, \\ \overline{B}_c^{l} = M_2^{-T}\left(\widehat{B}_c^{l} - M_1 B_l \overline{D}_c\right), \\ \overline{A}_c^{lj} = M_2^{-T}\left(\widehat{A}_c^{lj} - M_1 A_l Q_1 - M_1 B_l \overline{D}_c C_j Q_1 - M_2^{T}\overline{B}_c^{l} C_j Q_1 \right. \\ \qquad \left. - M_1 B_l \overline{C}_c^{j} Q_2\right)Q_2^{-1}, \end{cases} \tag{5}$$

where "(k)" is omitted for brevity. Further, $\left\{\widehat{A}_c^{lj}, \widehat{B}_c^{l}, \widehat{C}_c^{j}, \widehat{D}_c\right\}(k)$ *are obtained by solving*

$$\min_{\left\{\gamma,\alpha_{lj},\varrho,Q_1,M_1,\widehat{A}_c^{lj},\widehat{B}_c^{l},\widehat{C}_c^{j},\widehat{D}_c\right\}(k)} \gamma(k), \tag{6}$$

$$\text{s.t.} \quad M_1(k) \le \varrho(k)M_e(k), \tag{7}$$

$$\begin{bmatrix} 1-\varrho(k) & \star & \star \\ U(k)x_c(k) & Q_1(k) & \star \\ 0 & I & M_1(k) \end{bmatrix} \geq 0, \tag{8}$$

$$\sum_{l=1}^{L} \mathscr{C}_l^{\ell}(d,2)\Upsilon_{ll}^{\mathrm{QB}}(k) + \sum_{l=1}^{L-1}\sum_{j=l+1}^{L} \mathscr{C}_{lj}^{\ell}(d,1,1) \cdot \left[\Upsilon_{lj}^{\mathrm{QB}}(k) + \Upsilon_{jl}^{\mathrm{QB}}(k)\right] \geq 0, \quad \ell = 1,\ldots,|\mathscr{K}(d+2)|, \tag{9}$$

$$\begin{bmatrix} M_1(k) & \star & \star & \star \\ I & Q_1(k) & \star & \star \\ 0 & 0 & I & \star \\ \frac{1}{\sqrt{1-\eta_{1s}}}\xi_s\widehat{D}_c(k)C_j & \frac{1}{\sqrt{1-\eta_{1s}}}\xi_s\widehat{C}_c^j(k) & \frac{1}{\sqrt{\eta_{1s}}}\xi_s\widehat{D}_c(k)E_j & \overline{u}_s^2 \end{bmatrix} \geq 0, \quad j = 1,\ldots,L,\ s = 1,\ldots,n_u, \tag{10}$$

$$\sum_{l=1}^{L} \mathscr{C}_l^{\ell}(d,2)\Upsilon_{hlls}^{z}(k) + \sum_{l=1}^{L-1}\sum_{j=l+1}^{L} \mathscr{C}_{lj}^{\ell}(d,1,1) \cdot \left[\Upsilon_{hljs}^{z}(k) + \Upsilon_{hjls}^{z}(k)\right] \geq 0, \quad \ell = 1,\ldots,|\mathscr{K}(d+2)|, \quad h = 1,\ldots,L,\ s = 1,\ldots,q, \tag{11}$$

where $U(k)$ is a transformation matrix being given before solving (6)–(11), d is a fixed nonnegative integer, $\eta_{1s} \in [0,1)$ are the fixed scalars, and ξ_s is *the* s-th row of n_u-ordered identity matrix. In (9) and (11), $\mathscr{K}(d+2)$ is the set of L-tuples obtained from all possible combinations of $d_1, d_2, \ldots, d_L$, $d_l \geq 0, l = 1,\ldots,L$ such that $d_1 + d_2 + \cdots + d_L = d+2$. The number of elements of $\mathscr{K}(d+2)$ is given by $|\mathscr{K}(d+2)| = ((L+d+1)!)/((d+2)!(L-1)!)$. The L-tuples of $\mathscr{K}(d+2)$ are lexically ordered as $\ell = 1,\ldots,|\mathscr{K}(d+2)|$. Moreover,

$$\mathscr{C}_l^{\ell}(d,2) = \begin{cases} \dfrac{d!}{d_1!\cdots d_{l-1}!(d_l-2)!d_{l+1}!\cdots d_L!}, & d_l \geq 2, \\ 0, & \text{otherwise}, \end{cases}$$
$$\mathscr{C}_{lj}^{\ell}(d,1,1) = \begin{cases} \dfrac{d!}{d_1!\cdots d_{l-1}!(d_l-1)!d_{l+1}!\cdots d_{j-1}!\left(d_j-1\right)!d_{j+1}!\cdots d_L!}, & d_l \geq 1, d_j \geq 1, \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

In (9),

$$\Upsilon_{lj}^{QB} = \begin{bmatrix} (1-\alpha)M_1 & \star & \star & \star & \star & \star & \star \\ (1-\alpha)I & (1-\alpha)Q_1 & \star & \star & \star & \star & \star \\ 0 & 0 & \alpha I & \star & \star & \star & \star \\ A_l + B_l\widehat{D}_cC_j & A_lQ_1 + B_l\widehat{C}_c^j & B_l\widehat{D}_cE_j + D_l & Q_1 & \star & \star & \star \\ M_1A_l + \widehat{B}_c^lC_j & \widehat{A}_c^{lj} & \widehat{B}_c^lE_j + M_1D_l & I & M_1 & \star & \star \\ \mathscr{Q}_1^{1/2}\mathscr{F}_j & \mathscr{Q}_1^{1/2}\mathscr{F}_jQ_1 & \mathscr{Q}_1^{1/2}\mathscr{G}_j & 0 & 0 & \gamma I & \star \\ \mathscr{R}^{1/2}\widehat{D}_cC_j & \mathscr{R}^{1/2}\widehat{C}_c^j & \mathscr{R}^{1/2}\widehat{D}_cE_j & 0 & 0 & 0 & \gamma I \end{bmatrix}, \tag{13}$$

where "(k)" is omitted, and $\{\mathcal{Q}_1, \mathcal{R}\}$ are the weighting matrices. In (11),

$$\Upsilon^z_{hljs} = \begin{bmatrix} M_1 & \star & \star & \star \\ I & Q_1 & \star & \star \\ 0 & 0 & I & \star \\ \spadesuit_1 & \spadesuit_2 & \frac{1}{\sqrt{1-\eta_{2s}}\sqrt{\eta_{3s}}}\Psi_s\mathcal{E}_h\left(B_l\widehat{D}_c E_j + D_l\right) & \overline{\psi}_s^2 - \frac{1}{\eta_{2s}}\Psi_s\mathcal{E}_h\mathcal{E}_h^T\Psi_s^T \end{bmatrix}, \quad (14)$$

$$\spadesuit_1 = \frac{1}{\sqrt{1-\eta_{2s}}\sqrt{1-\eta_{3s}}}\Psi_s\mathcal{C}_h\left(A_l + B_l\widehat{D}_c C_j\right),$$

$$\spadesuit_2 = \frac{1}{\sqrt{1-\eta_{2s}}\sqrt{1-\eta_{3s}}}\Psi_s\mathcal{C}_h\left(A_l Q_1 + B_l\widehat{C}_c^j\right),$$

where "(k)" is omitted, Ψ_s is the s-th row of Ψ, and $\{\eta_{2s}, \eta_{3s}\} \in [0,1)$ are the fixed scalars.

Take $U(0) = I$ *and an* $x_c(0)$, *and suppose* $x(0) - x_c(0) \in \varepsilon_{M_e(0)}$. *At each* $k \geq 0$,

(a) For $k > 0$, apply (4) and (5) to obtain $\{A_c, B_c\}(k-1)$, then calculate $x_c(k) = A_c(k-1)x_c(k-1) + B_c(k-1)y(k-1)$

(b) For $k > 0$, take

$$U(k) = U(k-1), \quad (15)$$

$$\varrho(k) = 1 - x_c(k)^{\mathrm{T}}[M_3^*(k-1) - U(k-1)^{\mathrm{T}} M_1^*(k-1) \cdot U(k-1)]x_c(k), \quad (16)$$

$$M_e(k) = M_1^*(k-1)\varrho(k)^{-1}, \quad (17)$$

where

$$M_3(k-1) = M_2(k-1)\left[M_1(k-1) - Q_1(k-1)^{-1}\right]^{-1} \cdot M_2(k-1)^{\mathrm{T}}, \quad (18)$$

with $M_2(k-1) = -U(k-1)^{\mathrm{T}} M_1(k-1)$

(c) For $k > 0$, find $\{M_e', U'\}(k)$ satisfying

$$\left\{x(k-1) - U(k-1)x_c(k-1) \in \varepsilon_{M_e(k-1)}, \|w(k-1)\| \leq 1\right\} \Longrightarrow x(k) - U'(k)x_c(k) \in \varepsilon_{M_e'(k)}, \quad (19)$$

$$M_e'(k) \geq M_e(k), \quad (20)$$

and if (19) and (20) are feasible, then change $M_e(k) = M_e'(k)$ and $U(k) = U'(k)$

(d) Solve (6)–(11) to find $\left\{Q_1, M_1, \widehat{A}_c^{lj}, \widehat{B}_c^l, \widehat{C}_c^j, \widehat{D}_c\right\}^*(k)$

(e) Take $\{Q_1, M_1\}(k) = \{Q_1, M_1\}^*(k)$, $Q_2(k) = U(k)^{-1}[Q_1(k) - M_1(k)^{-1}]$, and $M_2(k) = -U(k)^{\mathrm{T}} M_1(k)$

(f) Apply (4) and (5) to obtain $C_c(k)$ and $D_c(k)$, then implement $u(k) = C_c(k)x_c(k) + D_c(k)y(k)$

Suppose (6)–(11) is feasible at time $k = 0$. Then,

(i) (6)–(11) will be feasible at each $k > 0$

(ii) $\{y, z', u\}$ will converge to a neighborhood of 0, and the constraints in (2) are satisfied for all $k \geq 0$

In (6)–(11), the four constraints of [3] are generalized (i.e., the confinement of $x(k)$ being generalized to (7) and (8) which is the confinement of both $x(k)$ and $x_c(k)$, invariance/stability/optimality condition to (9) which is the combination of quadratic boundedness and optimality conditions, input constraint to (10), and state/output constraint to (11) which is the constraint on z).

In the following, let us show the details how the above generalizations happen, taking Theorem 1 as one of the examples.

3. Model and Controller Descriptions

The predictive form of (1) is

$$x(k+i+1|k) = A(k+i)x(k+i|k) + B(k+i)u(k+i|k) + D(k+i)w(k+i),$$
$$y(k+i|k) = C(k+i)x(k+i|k) + E(k+i)w(k+i), \quad (21)$$

for all $i \geq 0$. The predictive form of (2) is

$$|u(k+i|k)| \leq \overline{u},$$
$$|\Psi z(k+i+1|k)| \leq \overline{\psi}, \quad (22)$$
$$i \geq 0,$$

where $z(k+i|k) = \mathcal{C}(k+i)x(k+i|k) + \mathcal{E}(k+i)w(k+i)$. According to Assumption 2,

$$[A \mid B \mid C \mid D \mid E \mid \mathscr{C} \mid \mathscr{E} \mid \mathscr{F} \mid \mathscr{G}](k+i) = \sum_{l=1}^{L} \lambda_l(k+i)[A_l \mid B_l \mid C_l \mid D_l \mid E_l \mid \mathscr{C}_l \mid \mathscr{E}_l \mid \mathscr{F}_l \mid \mathscr{G}_l]. \tag{23}$$

3.1. Controller for LPV Model. For the LPV model (1), the dynamic output feedback controller is of the following form (see firstly [10, 11]):

$$\begin{cases} x_c(k+1) = A_c(k)x_c(k) + L_c(k)y(k), \\ u(k) = F_x(k)x_c(k) + F_y(k)y(k), \end{cases} \tag{24}$$

where $\{A_c, L_c\}$ are controller gain matrices and $\{F_x, F_y\}$ are feedback gain matrices. It is unnecessary that $n_x = n_{x_c}$. The predictive form of (24) is

$$\begin{cases} x_c(k+i+1 \mid k) = A_c(k)x_c(k+i \mid k) + L_c(k)y(k+i \mid k), \\ u(k+i \mid k) = F_x(k)x_c(k+i \mid k) + F_y(k)y(k+i \mid k). \end{cases} \tag{25}$$

Remark 1. There are 4 controller parameters $\{A_c, L_c, F_x, F_y\}$ in (24) and (25). In the literature, often there are only 2 controller parameters $\{L_c, F_x\}$ for output feedback. We found that with only 2 controller parameters $\{L_c, F_x\}$, for (1), it is more difficult to find the feasible solution to the optimization problem of output feedback MPC. With 4 parameters $\{A_c, L_c, F_x, F_y\}$, output feedback MPC can be applied to a much larger range of system models.

Define the augmented state $\tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}$. By applying (1) and (24), the augmented closed-loop system is

$$\tilde{x}(k+1) = \Phi(k)\tilde{x}(k) + \Gamma(k)w(k), \tag{26}$$

where

$$\Phi(k) = \begin{bmatrix} A(k) + B(k)F_y(k)C(k) & B(k)F_x(k) \\ L_c(k)C(k) & A_c(k) \end{bmatrix}, \quad \Gamma(k) = \begin{bmatrix} B(k)F_y(k)E(k) + D(k) \\ L_c(k)E(k) \end{bmatrix}. \tag{27}$$

The predictive form of (26) is

$$\tilde{x}(k+i+1 \mid k) = \Phi(i,k)\tilde{x}(k+i \mid k) + \Gamma(i,k)w(k+i), \tag{28}$$

where

$$\Phi(i,k) = \begin{bmatrix} A(k+i) + B(k+i)F_y(k)C(k+i) & B(k+i)F_x(k) \\ L_c(k)C(k+i) & A_c(k) \end{bmatrix}, \quad \Gamma(i,k) = \begin{bmatrix} B(k+i)F_y(k)E(k+i) + D(k+i) \\ L_c(k)E(k+i) \end{bmatrix}. \tag{29}$$

By applying (23), it is shown that

$$\begin{aligned} \Phi(i,k) &= \sum_{l=1}^{L} \lambda_l(k+i) \sum_{j=1}^{L} \lambda_j(k+i)\Phi_{lj}(k), \\ \Gamma(i,k) &= \sum_{l=1}^{L} \lambda_l(k+i) \sum_{j=1}^{L} \lambda_j(k+i)\Gamma_{lj}(k), \\ \Phi_{lj}(k) &= \begin{bmatrix} A_l + B_lF_y(k)C_j & B_lF_x(k) \\ L_c(k)C_j & A_c(k) \end{bmatrix}, \\ \Gamma_{lj}(k) &= \begin{bmatrix} B_lF_y(k)E_j + D_l \\ L_c(k)E_j \end{bmatrix}. \end{aligned} \tag{30}$$

3.2. Controller for Quasi-LPV Model. For the quasi-LPV model (1), the dynamic output feedback controller is (3) and (4) (see firstly [12, 13]), where $n_x = n_{x_c}$. The predictive form of (3) is

$$\begin{cases} x_c(k+i+1 \mid k) = A_c(k+i)x_c(k+i \mid k) + B_c(k+i)y(k+i \mid k), \\ u(k+i \mid k) = C_c(k+i)x_c(k+i \mid k) + D_c(k+i)y(k+i \mid k), \end{cases} \tag{31}$$

where

$$\begin{cases} A_c(k+i) = \sum_{l=1}^{L}\sum_{j=1}^{L} \lambda_l(k+i)\lambda_j(k+i)\overline{A}_c^{lj}(k), \\ B_c(k+i) = \sum_{l=1}^{L} \lambda_l(k+i)\overline{B}_c^{l}(k), \\ C_c(k+i) = \sum_{j=1}^{L} \lambda_j(k+i)\overline{C}_c^{j}(k), \\ D_c(k+i) = \overline{D}_c(k). \end{cases} \tag{32}$$

Remark 2. For the quasi-LPV, since $\lambda_l(k)$ are known, we can utilize $\{\overline{A}_c^{lj}, \overline{B}_c^{l}, \overline{C}_c^{j}, \overline{D}_c\}(k)$ to calculate the parameter-dependent $\{A_c, B_c, C_c\}(k)$. Such $\{A_c, B_c, C_c\}(k)$ allows to find convex optimization problem to simultaneously give $\{\overline{A}_c^{lj}, \overline{B}_c^{l}, \overline{C}_c^{j}, \overline{D}_c\}(k)$. Hence, the parameter-dependent $\{A_c, B_c, C_c\}(k)$ is considerably better than the non-parameter-dependent $\{A_c, L_c, F_x\}(k)$.

Define the augmented state $\tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}$. By applying (1) and (3), the augmented closed-loop system is

$$\tilde{x}(k+1) = \Phi(k)\tilde{x}(k) + \Gamma(k)w(k), \tag{33}$$

where

$$\Phi(k) = \begin{bmatrix} A(k) + B(k)D_c(k)C(k) & B(k)C_c(k) \\ B_c(k)C(k) & A_c(k) \end{bmatrix}, \quad \Gamma(k) = \begin{bmatrix} B(k)D_c(k)E(k) + D(k) \\ B_c(k)E(k) \end{bmatrix}. \tag{34}$$

The predictive form of (33) is

$$\tilde{x}(k+i+1 \mid k) = \Phi(i,k)\tilde{x}(k+i \mid k) + \Gamma(i,k)w(k+i), \quad (35)$$

where

$$\Phi(i,k) = \begin{bmatrix} A(k+i) + B(k+i)D_c(k+i)C(k+i) & B(k+i)C_c(k+i) \\ B_c(k+i)C(k+i) & A_c(k+i) \end{bmatrix},$$
$$\Gamma(i,k) = \begin{bmatrix} B(k+i)D_c(k+i)E(k+i) + D(k+i) \\ B_c(k+i)E(k+i) \end{bmatrix}. \quad (36)$$

By applying (32), it is shown that

$$\Phi(i,k) = \sum_{l=1}^{L} \lambda_l(k+i) \sum_{j=1}^{L} \lambda_j(k+i)\Phi_{lj}(k),$$
$$\Gamma(i,k) = \sum_{l=1}^{L} \lambda_l(k+i) \sum_{j=1}^{L} \lambda_j(k+i)\Gamma_{lj}(k),$$
$$\Phi_{lj}(k) = \begin{bmatrix} A_l + B_l\overline{D}_c(k)C_j & B_l\overline{C}_c^j(k) \\ \overline{B}_c^l(k)C_j & \overline{A}_c^{lj}(k) \end{bmatrix},$$
$$\Gamma_{lj}(k) = \begin{bmatrix} B_l\overline{D}_c(k)E_j + D_l \\ \overline{B}_c^l(k)E_j \end{bmatrix}. \quad (37)$$

In the sequel, we often use the notations for LPV, but the results can be simply transplanted to quasi-LPV.

4. Characterization of Stability and Optimality

Consider the closed-loop systems (28) and (35). They have the same form. Both (28) and (35) have uncertain system parametric matrices which are composed of double convex combinations (i.e., convex combinations by coefficients $\lambda_l(k+i)$ and $\lambda_j(k+i)$).

We will borrow the notion of quadratic boundedness (QB) in [14, 15] to characterize the closed-loop stability of (28) and (35).

4.1. Review of Quadratic Boundedness. In [14], the following model with nominal parametric matrices is considered:

$$x(k+1) = Ax(k) + Dv(k), \quad (38)$$

where A and D are time-invariant (fixed) matrix, $v \in \mathfrak{R}^{n_v}$. In [14], it is firstly assumed that $v \in \mathbb{V}$ where $\mathbb{V}$ is a compact (bounded and closed) set, and $\mathbb{V} \subset \mathfrak{R}^{n_v}$.

Definition 1 (see [14]). System (38) is said to be quadratically bounded with Lyapunov matrix $P > 0$ if

$$x^{\mathrm{T}}Px \geq 1 \Longrightarrow (Ax + Dv)^{\mathrm{T}}P(Ax + Dv) \leq x^{\mathrm{T}}Px, \quad \forall v \in \mathbb{V}. \quad (39)$$

System (38) is said to be strictly quadratically bounded with Lyapunov matrix $P > 0$ if

$$x^{\mathrm{T}}Px > 1 \Longrightarrow (Ax + Dv)^{\mathrm{T}}P(Ax + Dv) < x^{\mathrm{T}}Px, \quad \forall v \in \mathbb{V}. \quad (40)$$

Lemma 1 (see [14]). *Suppose there exists a $\xi \in \mathbb{V}$ such that $D\xi \neq 0$. If (38) is quadratically bounded with the Lyapunov matrix $P > 0$, then it is strictly quadratically bounded with the same Lyapunov matrix.*

Definition 2. The set $\mathbb{S}$ is a robust positively invariant set for (38), if

$$x \in \mathbb{S} \Longrightarrow (Ax + Dv) \in \mathbb{S}, \quad \forall v \in \mathbb{V}. \quad (41)$$

Theorem 2 (see [14]). *Suppose $v \in \varepsilon_{P_v}$ with $P_v > 0$. The following facts are equivalent:*

(i) (38) is quadratically bounded with Lyapunov matrix $P > 0$

(ii) (38) is strictly quadratically bounded with Lyapunov matrix $P > 0$

(iii) The ellipsoid ε_P is a robust positively invariant set for (38)

(iv) $x^{\mathrm{T}}Px \geq v^{\mathrm{T}}P_v v \Longrightarrow (Ax + Dv)^{\mathrm{T}}P(Ax + Dv) \leq x^{\mathrm{T}}Px$

(v) There exists $\alpha > 0$ such that

$$\begin{bmatrix} (1-\alpha)P - A^{\mathrm{T}}PA & \star \\ -D^{\mathrm{T}}PA & \alpha P_v - D^{\mathrm{T}}PD \end{bmatrix} \geq 0; \quad (42)$$

(vi) A is exponentially stable (i.e., there exists $\overline{P} > 0$ such that $\overline{P} - A^{\mathrm{T}}\overline{P}A > 0$)

In [15], the following model with uncertain parametric matrices is considered:

$$x(k+1) = A(k)x(k) + D(k)v(k), \quad (43)$$

where $[A(k) \mid D(k)]$ belongs to a known bounded set, i.e., $[A(k) \mid D(k)] \in \mathscr{P}$ for all $k \geq 0$, and $D \neq 0$ for at least one $[A \mid D] \in \mathscr{P}$.

Definition 3 (see [15]). Suppose $v(k) \in \varepsilon_{P_v}$ for all $k \geq 0$, in (43). System (43) is said to be strictly quadratically bounded with a common Lyapunov matrix $P > 0$, if

$$x^{\mathrm{T}}Px > 1 \Longrightarrow (Ax+Dv)^{\mathrm{T}}P(Ax+Dv) < x^{\mathrm{T}}Px, \quad \forall v \in \varepsilon_{P_v}, \forall [A \mid D] \in \mathscr{P}. \tag{44}$$

Since $D \neq 0$ for at least one $[A \mid D] \in \mathscr{P}$, and $v \in \varepsilon_{P_v}$, there exists a $Dv \neq 0$. Similarly to Lemma 1. if (43) is quadratically bounded with Lyapunov matrix $P > 0$, then it is strictly quadratically bounded with the same Lyapunov matrix. The definition of quadratic boundedness is similar to Definition 1.

Definition 4. Suppose $v(k) \in \varepsilon_{P_v}$ for all $k \geq 0$, in (43). The set $\mathbb{S}$ is a positively invariant set for (43), if

$$x \in \mathbb{S} \Longrightarrow (Ax+Dv) \in \mathbb{S}, \quad \forall v \in \varepsilon_{P_v}, \forall [A \mid D] \in \mathscr{P}. \tag{45}$$

Theorem 3 (see [15]). *Suppose $v(k) \in \varepsilon_{P_v}$ for all $k \geq 0$, in (43). The following facts are equivalent:*

(i) (43) is strictly quadratically bounded with a common Lyapunov matrix $P > 0$

(ii) The ellipsoid ε_P is a positively invariant set for (43)

(iii) There exists $\alpha(k) \in (0,1)$ such that

$$\begin{bmatrix} (1-\alpha(k))P - A(k)^{\mathrm{T}}PA(k) & \star \\ -D(k)^{\mathrm{T}}PA(k) & \alpha(k)P_v - D(k)^{\mathrm{T}}PD(k) \end{bmatrix} \geq 0. \tag{46}$$

Note that in the above theorem it is necessary to use a time-varying $\alpha(k)$.

4.2. Stability Condition. In the output feedback MPC, QB is equivalent to strict QB (see [16]). For the closed-loop systems (28) and (35), by generalizing the results in Section 4.1, we obtain the following results.

Definition 5 (see firstly [12, 13] for quasi-LPV and [11, 17] for LPV). Suppose (referring to Assumptions 1 and 2), at time k and for all $i \geq 0$:

$$\|w(k+i)\| \leq 1; \tag{47}$$

there exist nonnegative coefficients $\lambda_l(k+i), l = 1, \ldots, L$ such that $\sum_{l=1}^{L} \lambda_l(k+i) = 1$ and $[A \mid B \mid C \mid D \mid E](k) = \sum_{l=1}^{L} \lambda_l\ (k+i)[A_l \mid B_l \mid C_l \mid D_l \mid E_l]$.

System (28) or (35) is said to be quadratically bounded with a common Lyapunov matrix $M(k) > 0$, if

$$\|\tilde{x}(k+i \mid k)\|^2_{M(k)} \geq 1 \Longrightarrow \|\tilde{x}(k+i+1 \mid k)\|^2_{M(k)} \leq \|\tilde{x}(k+i \mid k)\|^2_{M(k)}, \quad \forall i \geq 0. \tag{48}$$

Definition 6. With the assumptions in Definition 5 satisfied, the set $\mathbb{S}$ is a positively invariant set for (28) or (35), if

$$\tilde{x}(k+i \mid k) \in \mathbb{S} \Longrightarrow \tilde{x}(k+i+1 \mid k) \in \mathbb{S}, \quad \forall i \geq 0. \tag{49}$$

Theorem 4 (see firstly [10, 11] for LPV and [12, 18] for quasi-LPV). With the assumptions in Definition 5 satisfied, the following facts are equivalent:

(i) (28) or (35) is quadratically bounded with a common Lyapunov matrix $M(k) > 0$

(ii) The ellipsoid $\varepsilon_{M(k)}$ is a positively invariant set for (28) or (35)

(iii) There exists $\alpha(i,k) \in (0,1)$ such that

$$\begin{bmatrix} (1-\alpha(i,k))M(k) - \Phi(i,k)^T M(k)\Phi(i,k) & \star \\ -\Gamma(i,k)^T M(k)\Phi(i,k) & \alpha(i,k)I - \Gamma(i,k)^T M(k)\Gamma(i,k) \end{bmatrix} \geq 0, \quad i \geq 0, \tag{50}$$

(iv) $\Phi(i,k)$ is exponentially stable for all $i > 0$ (i.e., there exists $\overline{M}(k) > 0$ such that $\overline{M}(k) - \Phi(i,k)^{\mathrm{T}}\overline{M}(k)\Phi(i,k) > 0$).

In [19], the single-valued α is firstly replaced by

$$\alpha(i,k) = \sum_{l=1}^{L}\sum_{j=1}^{L} \lambda_l(k+i)\lambda_j(k+i)\alpha_{lj}. \tag{51}$$

4.3. Optimality Condition. The disturbance-free form of (28) or (35) is

$$\tilde{x}_u(k+i+1 \mid k) = \Phi(i,k)\tilde{x}_u(k+i \mid k), \quad \forall i \geq 0, \tilde{x}_u(k \mid k) = \tilde{x}(k). \tag{52}$$

Correspondingly,

$$\begin{aligned} u_u(k+i \mid k) &= F_x(k)x_{c,u}(k+i \mid k) + F_y(k)y_u(k+i \mid k), \\ y_u(k+i \mid k) &= C(k+i)x_u(k+i \mid k), \\ z_u(k+i \mid k) &= \mathscr{C}(k+i)x_u(k+i \mid k), z'_u(k+i \mid k) \\ &= \mathscr{F}(k+i)x_u(k+i \mid k). \end{aligned} \tag{53}$$

Let us introduce the quadratic cost

$$J(k) = \sum_{i=0}^{\infty} J_i(k),$$
$$J_i(k) = \left\|z'_u(k+i \mid k)\right\|^2_{\mathscr{Q}_1} + \left\|x_{c,u}(k+i \mid k)\right\|^2_{\mathscr{Q}_2} + \left\|u_u(k+i \mid k)\right\|^2_{\mathscr{R}}, \tag{54}$$

where $\mathscr{Q}_1$, $\mathscr{Q}_2$, and $\mathscr{R}$ are positive-definite weighting matrices, and consider the condition

$$\left\|\tilde{x}_u(k+i+1\,|\,k)\right\|^2_{M(k)} - \left\|\tilde{x}_u(k+i\,|\,k)\right\|^2_{M(k)} \le -\frac{1}{\gamma(k)} J_i(k), \quad \forall i \ge 0. \tag{55}$$

In Theorem 1, it has taken $\mathcal{Q}_2 = 0$. For exponentially stable $\Phi(i,k)$, it will result in $\lim_{i\longrightarrow\infty} z'_u(k+i\,|\,k) = 0$, $\lim_{i\longrightarrow\infty} x_{c,u}(k+i\,|\,k) = 0$, and $\lim_{i\longrightarrow\infty} u_u(k+i\,|\,k) = 0$. Hence, summing (55) from $i=0$ to $i=\infty$ yields

$$J(k) \le \gamma(k)\left\|\tilde{x}_u(k\,|\,k)\right\|^2_{M(k)} = \gamma(k)\|\tilde{x}(k)\|^2_{M(k)}. \tag{56}$$

Further, let

$$\tilde{x}(k) \in \varepsilon_{M(k)}. \tag{57}$$

Then, applying (57) to (56) yields

$$J(k) \le \gamma(k), \tag{58}$$

that is, $\gamma(k)$ is an upper bound of $J(k)$. We will take $\gamma(k)$ as the cost function of the optimization problems which finds the controller parametric matrices.

The condition (55) can be rewritten as

$$\tilde{x}_u(k+i\,|\,k)^{\mathrm{T}} \Pi(i,k) \tilde{x}_u(k+i\,|\,k) \ge 0, \tag{59}$$

where

$$\begin{aligned}\Pi(i,k) = {} & M(k) - \Phi(i,k)^{\mathrm{T}} M(k)\Phi(i,k) \\ & - \frac{1}{\gamma(k)} \operatorname{diag}\left\{\mathcal{F}(k+i)^{\mathrm{T}}\mathcal{Q}_1\mathcal{F}(k+i), \mathcal{Q}_2\right\} \\ & - \frac{1}{\gamma(k)}\left[\, F_y(k)C(k+i) \;\; F_x(k)\,\right]^{\mathrm{T}} \\ & \cdot \mathcal{R}\left[\, F_y(k)C(k+i) \;\; F_x(k)\,\right].\end{aligned} \tag{60}$$

Hence, (55) is guaranteed by $\Pi(i,k) \ge 0$. By applying the Schur complement, it is shown that $\Pi(i,k) \ge 0$ can be transformed into

$$\begin{bmatrix} M(k) - \Phi(i,k)^{\mathrm{T}} M(k)\Phi(i,k) & \star & \star \\ \mathcal{Q}^{1/2}\operatorname{diag}\{\mathcal{F}(k+i), I\} & \gamma(k)I & \star \\ \mathcal{R}^{1/2}\left[\, F_y(k)C(k+i) \;\; F_x(k)\,\right] & 0 & \gamma(k)I \end{bmatrix} \ge 0, \quad i \ge 0, \tag{61}$$

where $\mathcal{Q} = \operatorname{diag}\{\mathcal{Q}_1, \mathcal{Q}_2\}$.

The condition (55) or (61) is for optimality, not primarily for stability. However, if

$$\begin{aligned}&\operatorname{diag}\left\{\mathcal{F}(k+i)^{\mathrm{T}}\mathcal{Q}_1\mathcal{F}(k+i), \mathcal{Q}_2\right\} + \left[\, F_y(k)C(k+i) \;\; F_x(k)\,\right]^{\mathrm{T}} \\ & \cdot \mathcal{R}\left[\, F_y(k)C(k+i) \;\; F_x(k)\,\right] > 0,\end{aligned} \tag{62}$$

then (61) means that $\overline{M}(k) - \Phi(i,k)^{\mathrm{T}}\overline{M}(k)\Phi(i,k) > 0$, i.e., $\Phi(i,k)$ is exponentially stable (referring to point (iv) of Theorem 4). We can indeed combine the optimality and stability conditions by imposing (see firstly [11, 17] for LPV and [12, 13] for quasi-LPV)

$$\begin{aligned}&\|\tilde{x}(k+i\,|\,k)\|^2_{M(k)} \ge 1 \\ &\Longrightarrow \|\tilde{x}(k+i+1\,|\,k)\|^2_{M(k)} - \|\tilde{x}(k+i\,|\,k)\|^2_{M(k)} \\ &\le -\frac{1}{\gamma(k)}\left[\left\|z'(k+i\,|\,k)\right\|^2_{\mathcal{Q}_1} + \left\|x_c(k+i\,|\,k)\right\|^2_{\mathcal{Q}_2} + \|u(k+i\,|\,k)\|^2_{\mathcal{R}}\right], \quad \forall i \ge 0.\end{aligned} \tag{63}$$

It is easy to show that (63) is equivalent to (in the sense for any $\tilde{x}(k+i\,|\,k)$ and $w(k+i)$)

$$\begin{bmatrix} (1-\alpha(i,k))M(k) - \Phi(i,k)^{\mathrm{T}}M(k)\Phi(i,k) & \star & \star & \star \\ -\Gamma(i,k)^{\mathrm{T}}M(k)\Phi(i,k) & \alpha(i,k)I - \Gamma(i,k)^{\mathrm{T}}M(k)\Gamma(i,k) & \star & \star \\ \mathcal{Q}^{1/2}\operatorname{diag}\{\mathcal{F}(k+i), I\} & \mathcal{Q}^{1/2}\begin{bmatrix}\mathcal{G}(k+i) \\ 0\end{bmatrix} & \gamma(k)I & \star \\ \mathcal{R}^{1/2}\left[\, F_y(k)C(k+i) \;\; F_x(k)\,\right] & \mathcal{R}^{1/2}F_y(k)E(k+i) & 0 & \gamma(k)I \end{bmatrix} \ge 0, \quad i \ge 0. \tag{64}$$

Remark 3. It is apparent that feasibility of (64) guarantees both (50) and (61). With $\gamma(k)$ free (i.e., as a decision variable), feasibility of (50) guarantees both (61) and (64). Therefore, on the feasibility aspect, (64) and (50) are equivalent.

4.4. A Paradox for State Convergence. Consider the condition group {(57), (64)} or {(57), (50)}. Condition (64) or (50) means that, if the augmented state $\tilde{x}(k)$ lies outside of the ellipsoid $\varepsilon_{M(k)}$, then $\tilde{x}(k+i\,|\,k)$ will converge to $\varepsilon_{M(k)}$ with the increase of $i \ge 0$. However, condition (57) requires that the initial augmented state lies within the ellipsoid $\varepsilon_{M(k)}$. With the satisfaction of (57), condition (64) or (50) cannot guarantee the convergence of $\tilde{x}(k+i\,|\,k)$; condition (64) or (50) only guarantees the invariance of $\tilde{x}(k+i\,|\,k)$ within $\varepsilon_{M(k)}$.

In the above, although there is no guarantee that $\tilde{x}(k+i\,|\,k)$ will converge, the convergence of $\tilde{x}(k+i\,|\,k)$ will happen when $\|\tilde{x}(k)\|$ is not small (see firstly [19] for LPV and [20] for quasi-LPV). The main reason lies in that (64) or (50) is a robust condition.

Let us impose that, if the augmented state $\tilde{x}(k)$ lies outside of the ellipsoid $\varepsilon_{\beta(k)^{-1}M(k)}$, then $\tilde{x}(k+i\mid k)$ converges to $\varepsilon_{\beta(k)^{-1}M(k)}$ with the increase of $i \geq 0$. Here, $\varepsilon_{\beta(k)^{-1}M(k)}$ is an ellipsoid not larger than $\varepsilon_{M(k)}$ since $0<\beta(k)\leq 1$ (see firstly [17] for LPV and [13, 20] for quasi-LPV). By applying such $\beta(k)$, we can change (48) as

$$\|\tilde{x}(k+i\mid k)\|^2_{M(k)} \geq \beta(k) \Longrightarrow \|\tilde{x}(k+i+1\mid k)\|^2_{M(k)} \leq \|\tilde{x}(k+i\mid k)\|^2_{M(k)}, \quad \forall i \geq 0, \tag{65}$$

which is equivalent to (in the sense for any $\tilde{x}(k+i\mid k)$ and $w(k+i)$)

$$\begin{bmatrix} \heartsuit & \star \\ -\Gamma(i,k)^{\mathrm{T}} M(k)\Phi(i,k) & \alpha(i,k)\beta(k)I - \Gamma(i,k)^{\mathrm{T}} M(k)\Gamma(i,k) \end{bmatrix} \geq 0,$$
$$\heartsuit = (1-\alpha(i,k))M(k) - \Phi(i,k)^{\mathrm{T}} M(k)\Phi(i,k), \quad i \geq 0. \tag{66}$$

We can also change (63) as

$$\|\tilde{x}(k+i|k)\|^2_{M(k)} \geq \beta(k) \Longrightarrow \|\tilde{x}(k+i+1|k)\|^2_{M(k)} - \|\tilde{x}(k+i\mid k)\|^2_{M(k)} \leq -\frac{1}{\gamma(k)}\left[\left\|z'(k+i|k)\right\|^2_{\mathcal{Q}_1} + \left\|x_c(k+i\mid k)\right\|^2_{\mathcal{Q}_2} + \|u(k+i\mid k)\|^2_{\mathcal{R}}\right], \quad \forall i \geq 0, \tag{67}$$

which is equivalent to (in the sense for any $\tilde{x}(k+i\mid k)$ and $w(k+i)$)

$$\begin{bmatrix} (1-\alpha(i,k))M(k) - \Phi(i,k)^{\mathrm{T}} M(k)\Phi(i,k) & \star & \star & \star \\ -\Gamma(i,k)^{\mathrm{T}} M(k)\Phi(i,k) & \alpha(i,k)\beta(k)I - \Gamma(i,k)^{\mathrm{T}} M(k)\Gamma(i,k) & \star & \star \\ \mathcal{Q}^{1/2}\operatorname{diag}\{\mathcal{F}(k+i), I\} & \mathcal{Q}^{1/2}\begin{bmatrix} \mathcal{G}(k+i) \\ 0 \end{bmatrix} & \gamma(k)I & \star \\ \mathcal{R}^{1/2}\left[F_y(k)C(k+i) \;\; F_x(k)\right] & \mathcal{R}^{1/2}F_y(k)E(k+i) & 0 & \gamma(k)I \end{bmatrix} \geq 0, \quad i \geq 0. \tag{68}$$

Adding $\beta(k) \in (0,1]$ as a free variable, due to the special position of $\beta(k)$ in either (66) or (68), does not affect the minimization of $\gamma(k)$ and feasibility. It is suggested to minimize $\beta(k)$ after the minimization of $\gamma(k)$ (see firstly [19] for LPV and [20] for quasi-LPV). If the controller parametric matrices are not reoptimized in minimizing $\beta(k)$, it is easy to know that we do not need $\beta(k)$, i.e., we can simply remove it.

5. General Optimization Problem

Define $\overrightarrow{\mathrm{par}} = \{A_c, L_c, F_x, F_y\}$ for LPV and $\overrightarrow{\mathrm{par}} = \{\widehat{A}_c^{lj}, \widehat{B}_c^{l}, \widehat{C}_c^{j}, \widehat{D}_c\}$ for qusi-LPV. The dynamic OFRMPC aims at solving, at each k,

$$\min_{\{\gamma, M, \overrightarrow{\mathrm{par}}\}(k)} \left\{ \max_{[A\mid B\mid C\mid D\mid E\mid \mathcal{C}\mid \mathcal{E}\mid \mathcal{F}\mid \mathcal{G}](k+i)\in\Omega, \|w(k+i)\|\leq 1} \gamma(k) \right\}, \quad \text{s.t.} \quad (22), (57), (48) \text{ and } (55). \tag{69}$$

Lemma 2 (see firstly [19] for LPV and [18, 20] for quasi-LPV) (recursive feasibility). *Assume that the state x is measurable. At each time $k \geq 0$, solve (69) and implement $u(k)$. Problem (69) is feasible for any $k>0$ if and only if it is feasible at $k=0$.*

Theorem 5 (see firstly [20] for quasi-LPV and [19] for LPV) (stability). *Assume that the state x is measurable. At each time $k \geq 0$, solve (69) and implement $u(k)$. If (69) is feasible at $k=0$, then with the evolution of k, $\{\gamma, z', x_c, u\}$ will converge to a neighborhood of the origin, and stay in this neighborhood thereafter, and the constraints in (22) are satisfied for all $k \geq 0$.*

According to the above section, (69) is transformed into (equivalently in the sense for any $\tilde{x}(k+i\mid k)$ and $w(k+i)$)

$$\min_{\{\gamma, \alpha_{lj}, M, \overrightarrow{\mathrm{par}}\}(k)} \left\{ \max_{[A\mid B\mid C\mid D\mid E\mid \mathcal{C}\mid \mathcal{E}\mid \mathcal{F}\mid \mathcal{G}](k+i)\in\Omega} \gamma(k) \right\}, \quad \text{s.t.} \quad (22), (57), (50) \text{ and } (61), \tag{70}$$

with recursive feasibility and stability properties retained.

5.1. Handling Physical Constraints. In [21, 22], the following lemma is utilized to handle the physical constraints (e.g., the magnitude constraints on x, y, and u).

Lemma 3. *Suppose a and b are vectors with appropriate dimensions. Then for any scalar $\eta \in (0,1)$, $\|a+b\|^2 \le (1-\eta)\|a\|^2 + (1/\eta)\|b\|^2$.*

In [5, 7, 23, 24], it is found that applying the above lemma, although simple, can greatly reduce the conservativeness for physical constraint handling. In essence, the physical constraints are handled based on the invariance of $\tilde{x}(k+i\,|\,k)$ within $\varepsilon_{M(k)}$.

Theorem 6 (see firstly [5, 23] for LPV and [9] for quasi-LPV). *Suppose at time k, there exist scalars $\alpha(i,k) \in (0,1)$ and η_{rs}, and matrix $M(k) > 0$, such that (57) and (50) hold, and*

$$\begin{bmatrix} M(k) & \star & \star \\ 0 & I & \star \\ \frac{1}{\sqrt{1-\eta_{1s}}}\xi_s\left[F_y(k)C(k+i) \;\; F_x(k)\right] & \frac{1}{\sqrt{\eta_{1s}}}\xi_s F_y(k)E(k+i) & \overline{u}_s^2 \end{bmatrix} \ge 0, \quad s = 1,\ldots,n_u,\, i \ge 0, \tag{71}$$

$$\begin{bmatrix} M(k) & \star & \star \\ 0 & I & \star \\ \frac{1}{\sqrt{(1-\eta_{2s})(1-\eta_{3s})}}\Psi_s\mathscr{C}(k+i+1)\Phi^1(i,k) & \frac{1}{\sqrt{(1-\eta_{2s})\eta_{3s}}}\Psi_s\mathscr{C}(k+i+1)\Gamma^1(i,k) & \overline{\psi}_s^2 - \frac{1}{\eta_{2s}}\Psi_s\mathscr{E}(k+i+1)\mathscr{E}(k+i+1)^{\mathrm{T}}\Psi_s^{\mathrm{T}} \end{bmatrix} \ge 0, \quad s = 1,\ldots,q,\, i \ge 0, \tag{72}$$

where $\Phi^1(i,k)$ ($\Gamma^1(i,k)$) is the first of the two rows of $\Phi(i,k)$ ($\Gamma(i,k)$). Take care of the special cases:

(a) If $\varepsilon(k+i+1) = 0$, then take $(1/\eta_{2s})\Psi_s\mathscr{E}(k+i+1)\mathscr{E}(k+i+1)^{\mathrm{T}}\Psi_s^{\mathrm{T}} = 0$ and $\eta_{2s} = 0$

(b) If $E(k+i) = 0$, then take $(1/\sqrt{\eta_{1s}})\xi_s F_y(k)E(k+i) = 0$ and $\eta_{1s} = 0$

(c) If $D(k+i) = 0$ and $E(k+i) = 0$, then take $(1/\sqrt{\eta_{3s}})\Psi_s\mathscr{C}(k+i+1)\Gamma^1(i,k) = 0$ and $\eta_{3s} = 0$

Then, (22) is satisfied.

In the above theorem, one may want to choose η_{rs} be time-varying. However, we have not found a good method to online optimize η_{rs}, so we take η_{rs} as time-invariant.

According to Theorem 6, the problem (70) is approximated as (by no means equivalent to)

$$\min_{\{\gamma,\alpha_{lj},M,\overrightarrow{\mathrm{par}}\}(k)} \left\{\max_{[A|B|C|D|E|\mathscr{C}|\mathscr{E}|\mathscr{F}|\mathscr{G}](k+i)\in\Omega} \gamma(k)\right\}, \quad \text{s.t.} \quad (57), (50), (61), (71) \text{ and } (72), \tag{73}$$

with recursive feasibility and stability properties retained. In (73), η_{rs} is prespecified (see firstly [5, 23] for LPV and [9] for quasi-LPV).

5.2. Current Augmented State. The condition (57) (i.e., $\|\tilde{x}(k)\|^2_{M(k)} \le 1$ or $\tilde{x}(k) \in \varepsilon_{M(k)}$) is the current condition on the augmented state. At time k, in $\tilde{x}(k) = [x(k)^{\mathrm{T}}, x_c(k)^{\mathrm{T}}]^{\mathrm{T}}$, $x(k)$ can be unmeasurable, while $x_c(k)$ is always known. When $x(k)$ is unmeasurable, we need to remove it from (57) for the sake of solving (73).

Let us define an error signal

$$e(k) = x(k) - x_0(k), \tag{74}$$

where

$$x_0(k) = U(k)x_c(k), \tag{75}$$

with $U(k)$ being a known transformation matrix. When $U(k) = I$, defining $e(k)$ is usual; when $U(k) = E_0^T$ is fixed, see firstly [7, 25]; when $U(k)$ is online refreshed, see firstly [5, 26] for LPV and [9] for quasi-LPV. When $x(k)$ is unmeasurable, $e(k)$ is unknown (nondeterministic). If we can obtain the outer bounding set of $e(k)$, say $\mathscr{D}_e(k)$, then we can utilize $x_0(k) \oplus \mathscr{D}_e(k)$ to replace $x(k)$. Since $\mathscr{D}_e(k)$ is known (deterministic), by replacing $x(k)$ by $x_0(k) \oplus \mathscr{D}_e(k)$, (57) becomes deterministic.

Define

$$M = \begin{bmatrix} M_1 & M_2^{\mathrm{T}} \\ M_2 & M_3 \end{bmatrix}. \tag{76}$$

Using $x = e + Ux_c$, we obtain

$$\begin{aligned} \tilde{x}^{\mathrm{T}} M \tilde{x} &= (e + Ux_c)^{\mathrm{T}} M_1 (e + Ux_c) + 2(e + Ux_c)^{\mathrm{T}} M_2^{\mathrm{T}} x_c + x_c^{\mathrm{T}} M_3 x_c \\ &= e^{\mathrm{T}} M_1 e + 2e^{\mathrm{T}}\left(M_1 U + M_2^{\mathrm{T}}\right)x_c + x_c^{\mathrm{T}}\left(U^{\mathrm{T}} M_1 U + 2U^{\mathrm{T}} M_2^{\mathrm{T}} + M_3\right)x_c. \end{aligned} \tag{77}$$

If we can remove the cross item $2e^{\mathrm{T}}(M_1U + M_2^{\mathrm{T}})x_c$, then the treatment of (57) will become easier, and the treatment of recursive feasibility of the resultant optimization will become simpler.

Lemma 4. *In order to remove the cross item $2e^T(M_1U + M_2^T)x_c$ in $\tilde{x}^T M\tilde{x}$, we need to take $U = -M_1^{-1}M_2^T$.*

For quasi-LPV, [12, 18] firstly impose $M_2 = -M_1$ and [9] firstly imposes $U = -M_1^{-1}M_2^{\mathrm{T}}$, both removing the cross item. For LPV, [19] firstly imposes $M_2 = -M_1$, [7, 25] firstly impose $M_2 = -E_0M_1$, and [5, 26] firstly impose $U = -M_1^{-1}M_2^{\mathrm{T}}$, all removing the cross item.

By substituting $U = -M_1^{-1}M_2^{\mathrm{T}}$ into (77), we obtain

$$\tilde{x}^{\mathrm{T}}M\tilde{x} = e^{\mathrm{T}}M_1e + x_c^{\mathrm{T}}\left(M_3 - U^{\mathrm{T}}M_1U\right)x_c. \tag{78}$$

By introducing a scalar $\varrho(k)$ and imposing

$$e(k)^{\mathrm{T}}M_1(k)e(k) \le \varrho(k), \tag{79}$$

$$x_c(k)^{\mathrm{T}}\left[M_3(k) - U(k)^{\mathrm{T}}M_1(k)U(k)\right]x_c(k) \le 1 - \varrho(k), \tag{80}$$

it is apparent that (57) is guaranteed. Condition (79) is guaranteed by (7) if we can firstly guarantee that

$$e(k) \in \varepsilon_{M_e(k)}. \tag{81}$$

The condition $e(k) \in \varepsilon_{M_e(k)}$ can be guaranteed by appropriately refreshing $M_e(k)$ at each $k > 0$. However, for the initial time $k = 0$, $e(k) \in \varepsilon_{M_e(k)}$ has to be assumed.

Assumption 3. $e(0) = x(0) - x_0(0) \in \varepsilon_{M_e(0)}$.

Based on Assumption 3 and the fact that $e(k) \in \varepsilon_{M_e(k)}$, problem (73) is approximated as (by no means equivalent to)

$$\min_{\{\gamma,\alpha_{1j},\varrho,M,\overrightarrow{\mathrm{par}}\}(k)} \left\{\max_{[A|B|C|D|E|\mathscr{C}|\mathscr{E}|\mathscr{F}|\mathscr{G}](k+i)\in\Omega} \gamma(k)\right\}, \quad \text{s.t.} \quad (80), (7), (50), (61), (71) \text{ and } (72), \tag{82}$$

with recursive feasibility and stability properties retained in case $M_e(k)$ is appropriately refreshed.

Lemma 5 (for quasi-LPV, see [20] firstly with $M_2 = -M_1$, and [9] firstly with $U = -M_1^{-1}M_2^{\mathrm{T}}$; for LPV, see [19] firstly with $M_2 = -M_1$, [7, 25] firstly with $M_2 = -E_0M_1$, and [5, 26] firstly with $U = -M_1^{-1}M_2^{\mathrm{T}}$). *At each $k > 0$, if we choose (12)–(14), then at time k, (80) and (7) can be satisfied with equalities, i.e.,*

$$x_c(k)^{\mathrm{T}}\left[M_3(k) - U(k)^{\mathrm{T}}M_1(k)U(k)\right]x_c(k) = 1 - \varrho(k), \quad M_1(k) = \varrho(k)M_e(k). \tag{83}$$

Remark 4. In the above, the ellipsoidal bound on e or x has been discussed. We have also utilized polyhedral outer bounding sets of x, e.g.,

(a) Polyhedron with plane representation (see firstly [27] for LPV), i.e.,

$$x(k) \in \mathscr{P}_x(k) := \{x \mid -G(k)\bar{e} \le Hx - \check{x}(k) \le G(k)\bar{e}\}, \tag{84}$$

where $\check{x}$ is a bias item, $H = \begin{bmatrix} H_a \\ H_b \end{bmatrix}$ a prespecified transformation matrix with H_a being nonsingular, $G(k)$ a diagonal matrix, and $\bar{e} = [\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_p]^{\mathrm{T}}$ with $p > n_x$ and $\bar{e}_j > 0$ (for all $j = 1, \ldots, p$) being prespecified

(b) Polyhedron with vertex representation, i.e.,

$$x(k) \in \overline{\mathscr{P}}_x(k) = \mathrm{Co}\left\{\vartheta_j(k) \mid j = 1, 2, \ldots, n_\vartheta(k)\right\}, \tag{85}$$

(see, e.g., [20] for quasi-LPV and [19] for LPV) which is a general formulation of convex polyhedron.

We will not discuss the details for utilizing polyhedral bounds, but the following points are promising:

(i) For the output feedback MPC, $\mathscr{P}_x(k)$ in (84) is a general formulation of convex polyhedron, which includes the other polyhedral sets (e.g., [11–13, 18, 22]) as special cases, and is equivalent to the expression $\mathscr{P}_x(k) = \left\{\xi \mid \mathscr{H}\xi \le G(k)\overrightarrow{1}\right\}$ (with $\mathscr{H} \in \mathfrak{R}^{p\times n_x}$ being prespecified, and $\overrightarrow{1} = [1, 1, \ldots, 1]^{\mathrm{T}}$) of [10].

(ii) Before [28], either ellipsoidal bound or polyhedral bound is solely applied in the optimization problem. The recursive feasibility is guaranteed by a simple refreshment of the ellipsoidal bound but might be lost by applying polyhedral bound. In [28], it utilizes either the ellipsoidal bound or the polyhedral bound, the latter being used if and only if it is contained in the former. Moreover, [28] shows the sufficient conditions under which some approaches based on polyhedral bound preserve the property of recursive feasibility. In [29], the potentiality of applying both ellipsoidal and polyhedral bounds is further explored.

5.3. Some Usual Transformations. In order to solve (82), we need to transform (50) and (61) into familiar forms (comparing with, e.g., [3]). Define $Q = M^{-1}$ and

$$Q = \begin{bmatrix} Q_1 & Q_2^{\mathrm{T}} \\ Q_2 & Q_3 \end{bmatrix}. \tag{86}$$

By applying the Schur complement, (50) is transformed into

$$\begin{bmatrix} (1-\alpha(i,k))M(k) & \star & \star \\ 0 & \alpha(i,k)I & \star \\ \Phi(i,k) & \Gamma(i,k) & Q(k) \end{bmatrix} \ge 0, \quad i \ge 0. \tag{87}$$

By applying the Schur complement, (61) is transformed into

$$\begin{bmatrix} M(k) & \star & \star & \star \\ \Phi(i,k) & Q(k) & \star & \star \\ \mathcal{Q}^{1/2}\operatorname{diag}\{\mathcal{F}(k+i), I\} & 0 & \gamma(k)I & \star \\ \mathcal{R}^{1/2}\left[F_y(k)C(k+i)\ \ F_x(k)\right] & 0 & 0 & \gamma(k)I \end{bmatrix} \geq 0, \quad i \geq 0. \tag{88}$$

Then, we need to remove or handle the convex combinations in {(87), (88), (71), (72)}. By invoking the double convex combinations (DbCCs), (87) and (88) are equivalent to, respectively,

$$\begin{gathered} \sum_{l=1}^{L}\sum_{j=1}^{L}\lambda_l(k+i)\lambda_j(k+i)\Upsilon_{lj}^{\mathrm{QB}}(k) \geq 0, \quad i \geq 0, \\ \sum_{l=1}^{L}\sum_{j=1}^{L}\lambda_l(k+i)\lambda_j(k+i)\Upsilon_{lj}^{\mathrm{opt}}(k) \geq 0, \quad i \geq 0, \end{gathered} \tag{89}$$

where

$$\Upsilon_{lj}^{\mathrm{QB}}(k) = \begin{bmatrix} \left(1-\alpha_{lj}\right)M(k) & \star & \star \\ 0 & \alpha_{lj}I & \star \\ \Phi_{lj}(k) & \Gamma_{lj}(k) & Q(k) \end{bmatrix},$$

$$\Upsilon_{lj}^{\mathrm{opt}}(k) = \begin{bmatrix} M(k) & \star & \star & \star \\ \Phi_{lj}(k) & Q(k) & \star & \star \\ \mathcal{Q}^{1/2}\operatorname{diag}\{\mathcal{F}_j, I\} & 0 & \gamma(k)I & \star \\ \mathcal{R}^{1/2}\left[F_y(k)C_j\ \ F_x(k)\right] & 0 & 0 & \gamma(k)I \end{bmatrix}. \tag{90}$$

By removing the single convex combination, (71) is guaranteed by

$$\Upsilon_j^u(k) \geq 0, \quad j = 1,\ldots,L,\ s = 1,\ldots,n_u, \tag{91}$$

where

$$\Upsilon_j^u(k) = \begin{bmatrix} M(k) & \star & \star \\ 0 & I & \star \\ \dfrac{1}{\sqrt{1-\eta_{1s}}}\xi_s\left[F_y(k)C_j\ \ F_x(k)\right] & \dfrac{1}{\sqrt{\eta_{1s}}}\xi_s F_y(k)E_j & \overline{u}_s^2 \end{bmatrix}. \tag{92}$$

By removing the single convex combination, and invoking DbCC, (72) is guaranteed by

$$\begin{gathered} \sum_{l=1}^{L}\sum_{j=1}^{L}\lambda_l(k+i)\lambda_j(k+i)\Upsilon_{hlj}^{z}(k) \geq 0, \\ h = 1,\ldots,L,\ s = 1,\ldots,q,\ i \geq 0, \end{gathered} \tag{93}$$

where

$$\Upsilon_{hlj}^{z}(k) = \begin{bmatrix} M(k) & \star & \star \\ 0 & I & \star \\ \dfrac{1}{\sqrt{\left(1-\eta_{2s}\right)\left(1-\eta_{3s}\right)}}\Psi_s\mathcal{C}_h\Phi_{lj}^1(k) & \dfrac{1}{\sqrt{\left(1-\eta_{2s}\right)\eta_{3s}}}\Psi_s\mathcal{C}_h\Gamma_{lj}^1(k) & \overline{\psi}_s^2 - \dfrac{1}{\eta_{2s}}\Psi_s\mathcal{C}_h\mathcal{C}_h^{\mathrm{T}}\Psi_s^{\mathrm{T}} \end{bmatrix}. \tag{94}$$

In summary, problem (85) is approximated as (not strictly equivalent to)

$$\begin{aligned} &\min_{\left\{\gamma,\alpha_{lj},\mathcal{Q},M,Q,\overrightarrow{\mathrm{par}}\right\}(k)} \left\{\max_{[A|B|C|D|E|\mathcal{C}|\mathcal{E}|\mathcal{F}|\mathcal{G}](k+i)\in\Omega}\gamma(k)\right\} \\ &\text{s.t.} \quad (80),\ (7),\ (89),\ (91),\ (93)\ \text{and}\ Q = M^{-1}, \end{aligned} \tag{95}$$

with recursive feasibility and stability properties retained in case $M_e(k)$ is appropriately refreshed.

5.4. Handling Double Convex Combinations. In the literature of fuzzy control based on Takagi–Sugeno model and robust feedback control, the double convex combinations as in {(89), (93)} have been extensively studied. Some well-known examples include [30] (being invoked by MPC in [11, 18]), [31, 32] (being invoked by MPC firstly in [10, 25]), and [1] (being invoked by MPC firstly in [12]).

By analogy to "Theorem 1" in [31], the following result can be obtained.

Lemma 6 (see firstly [10, 25]). *The conditions*

$$\sum_{l=1}^{L}\sum_{j=1}^{L}\lambda_l(k+i)\lambda_j(k+i)\Upsilon_{lj}(k) \geq 0, \quad i \geq 0, \tag{96}$$

hold if and only if there exists a sufficiently large $d \geq 0$ such that

$$\sum_{l=1}^{L}\mathscr{C}_l^{\ell}(d,2)\Upsilon_{ll}(k)+\sum_{l=1}^{L-1}\sum_{j=l+1}^{L}\mathscr{C}_{lj}^{\ell}(d,1,1)\left[\Upsilon_{lj}(k)+\Upsilon_{jl}(k)\right]\geq 0,$$
$$\ell\in\{1,\ldots,|\mathscr{K}(d+2)|\}. \quad (97)$$

Moreover, if (97) holds for $d=\widehat{d}$, then they hold for any $d>\widehat{d}$.

This lemma has been utilized in Theorem 1. In this lemma, $\Upsilon_{lj}(k)\in\left\{\Upsilon_{lj}^{QB}(k),\Upsilon_{lj}^{\text{opt}}(k),\Upsilon_{hlj}^{z}(k)\right\}$. Equivalently, the techniques for the positivity of DbCC, as in [1], can be exactly utilized to obtain finite dimensional sufficient conditions for the nonnegativity of DbCC in (96). For example, (96) is guaranteed by any one set of the following sets of conditions (see "Proposition 2" of [1]):

Set 1: ($n=2$) (i) $\Upsilon_{ll}(k)\geq 0$, $l\in\{1,\ldots,L\}$, (ii) $\Upsilon_{lj}(k)+\Upsilon_{jl}(k)\geq 0$, $j>l$, $l,j\in\{1,\ldots,L\}$

Set 2: ($n=3$) (i) $\Upsilon_{ll}(k)\geq 0$, $l\in\{1,\ldots,L\}$, (ii) $\Upsilon_{ll}(k)+\Upsilon_{lj}(k)+\Upsilon_{jl}(k)\geq 0$, $j\neq l$, $l,j\in\{1,\ldots,L\}$, (iii) $\Upsilon_{lj}(k)+\Upsilon_{jl}(k)+\Upsilon_{jt}(k)+\Upsilon_{tj}(k)+\Upsilon_{tl}(k)+\Upsilon_{lt}(k)\geq 0$, $t>j>l$, $l,j,t\in\{1,\ldots,L\}$

In Sets 1 and 2, n is the complexity parameter of [1]. With a larger n, the conditions are less conservative but the computational burden is heavier. There exists a finite n such that necessary and sufficient conditions for satisfaction of (96) can be obtained for a concrete model.

6. Solutions to Output Feedback MPC

For solving (95), LPV is much more difficult than quasi-LPV. For quasi-LPV, by setting

$$Q=\begin{bmatrix} Q_1 & -(Q_1-M_1^{-1})M_1M_2^{-1} \\ -M_2^{-\mathrm{T}}M_1(Q_1-M_1^{-1}) & M_2^{-\mathrm{T}}M_1(Q_1-M_1^{-1})M_1M_2^{-1}\end{bmatrix}, \quad (98)$$

$$M=\begin{bmatrix} M_1 & M_2^{\mathrm{T}} \\ M_2 & M_2(M_1-Q_1^{-1})^{-1}M_2^{\mathrm{T}}\end{bmatrix}, \quad (99)$$

which naturally satisfies $M=Q^{-1}$, and using the transformation (equivalent to (5), with "(k)" being omitted for brevity)

$$\begin{cases} \widehat{D}_c=\overline{D}_c \\ \widehat{C}_c^{j}=\overline{D}_cC_jQ_1+\overline{C}_c^{j}Q_2 \\ \widehat{B}_c^{l}=M_1B_l\overline{D}_c+M_2^{\mathrm{T}}\overline{B}_c^{l} \\ \widehat{A}_c^{lj}=M_1A_lQ_1+M_1B_l\overline{D}_cC_jQ_1+M_2^{\mathrm{T}}\overline{B}_c^{l}C_jQ_1 \\ \qquad +M_1B_l\overline{C}_c^{j}Q_2+M_2^{\mathrm{T}}\overline{A}_c^{lj}Q_2 \end{cases}, \quad (100)$$

a solution to (95) can be obtained through a single optimization problem (6)–(11). For the prespecified $\{\alpha,\eta_{1s},\eta_{2s},\eta_{3s}\}$, (6)–(11) is an LMI optimization problem. Before [9], for the quasi-LPV, some special solutions to (95) can be found in [20, 28]. For LPV, even with prespecified $\{\alpha_{lj},\eta_{1s},\eta_{2s},\eta_{3s}\}$, one cannot find all the parameters $\{A_c,L_c,F_x,F_y\}(k)$ in a single LMI optimization problem. In the following, we give two solutions to (95) for LPV.

6.1. Full Online Method for LPV. By applying the block-matrix inversion on $Q=\begin{bmatrix} Q_1 & Q_2^{\mathrm{T}} \\ Q_2 & Q_3\end{bmatrix}$, it is easy to show that

$$M=\begin{bmatrix} M_1 & -M_1Q_2^{\mathrm{T}}Q_3^{-1} \\ -Q_3^{-1}Q_2M_1 & Q_3^{-1}+Q_3^{-1}Q_2M_1Q_2^{\mathrm{T}}Q_3^{-1}\end{bmatrix}. \quad (101)$$

Take $U=-M_1^{-1}M_2^{\mathrm{T}}$. Then, it is easy to show that $U=-Q_2^{\mathrm{T}}Q_3^{-1}$ and

$$\tilde{x}(k)^{\mathrm{T}}M(k)\tilde{x}(k)=\left[x(k)-x^0(k)\right]^{\mathrm{T}}M_1(k)\left[x(k)-x^0(k)\right]$$
$$+x_c(k)^{\mathrm{T}}Q_3(k)^{-1}x_c(k). \quad (102)$$

Lemma 7. *Let Assumption 3 hold and at each $k>0$, find $\{x^0,M_e\}(k)$ such that $x(k)-x^0(k)\in\varepsilon_{M_e(k)}$. Choose $\{U,x_c\}(0)$ such that $U(0)x_c(0)=x^0(0)$ and at each $k>0$, $U(k)$ such that $U(k)x_c(k)=x^0(k)$. Then, condition (80) holds if*

$$\begin{bmatrix} 1-\varrho(k) & \star \\ x_c(k) & Q_3(k)\end{bmatrix}\geq 0. \quad (103)$$

Further define $N_1=M_1^{-1}$ and $P_3=Q_3^{-1}$. Then,

$$Q=\begin{bmatrix} N_1+UQ_3U^{\mathrm{T}} & UQ_3 \\ Q_3U^{\mathrm{T}} & Q_3\end{bmatrix},$$
$$M=\begin{bmatrix} M_1 & -M_1U \\ -U^{\mathrm{T}}M_1 & P_3+U^{\mathrm{T}}M_1U\end{bmatrix}, \quad (104)$$

which naturally satisfies $M=Q^{-1}$. By applying (104), problem (95) becomes (equivalently)

$$\begin{aligned} &\min_{\{\gamma,\alpha_{lj},\varrho,N_1,M_1,P_3,Q_3,A_c,L_c,F_x,F_y\}(k)} && \left\{\max_{[A|B|C|D|E|\mathscr{C}|\mathscr{E}|\mathscr{F}|\mathscr{G}](k+i)\in\Omega} \gamma(k)\right\}, \\ &\text{s.t.} && (103), (7), (89), (91), (93), (104), N_1(k) = M_1(k)^{-1}, P_3(k) = Q_3(k)^{-1}. \end{aligned} \tag{105}$$

This approach is proposed in [6, 7] where $U(k) = E_0^{\mathrm{T}}$, and hence,

$$\begin{aligned} Q &= \begin{bmatrix} Q_1 & E_0^{\mathrm{T}}Q_3 \\ Q_3E_0 & Q_3 \end{bmatrix}, \\ M &= \begin{bmatrix} M_1 & -M_1E_0^{\mathrm{T}} \\ -E_0M_1 & M_3 \end{bmatrix}. \end{aligned} \tag{106}$$

In solving (105), usually $\alpha_{lj}(k) = \alpha(k)$ can be prespecified. One can line-search $\alpha(k)$ over the interval $(0, 1)$. Indeed, we found that the improvement on control performance is negligible by online optimizing $\alpha(k)$. The problem (105) has been solved by the iterative cone-complementary approach (ICCA) (see firstly in [10, 25]). ICCA has two major loops. The inner loop is the cone-complementary approach (CCA) which minimizes $\mathrm{Trace}\{M_1(k) N_1(k) + N_1(k)M_1(k) + Q_3(k)P_3(k) + P_3(k)Q_3(k)\}$ in order to achieve $N_1(k) = M_1(k)^{-1}$ and $P_3(k) = Q_3(k)^{-1}$. The outer loop gradually reduces $\gamma(k)$. Note that, even with $\alpha(k)$ being prespecified, (105) cannot be transformed into LMI optimization problem.

In Algorithm 1, while first and second equations in step (c) are natural for refreshing the bound of $x(k)$, third equation in step (c) is imposed for the recursive feasibility of (105). Finding $M_e'(k)$ satisfying equations in step (c) in Algorithm 1 can be achieved via LMI techniques.

Theorem 7 (see [5, 26]). *Adopt Algorithm 1. Suppose that Assumption 3 holds, and (105) is feasible at time $k = 0$. Then,*

(i) (105) will be feasible at each $k > 0$

(ii) $\{\gamma, z', x_c, u\}$ will converge to a neighborhood of 0, and the constraints in (2) are satisfied for all $k \geq 0$

6.2. Partial Online Method for LPV. In order to alleviate the computational burden, we can prespecify $\{L_c, F_y\}$ in (105). In this way, $\{M_1, P_3\}$ are no longer the decision variables. Therefore, {(7), (89), (91), (93)} will be modified accordingly.

By applying the Schur complement, (7) is equivalent to

$$\begin{bmatrix} \varrho(k)M_e(k) & I \\ I & N_1(k) \end{bmatrix} \geq 0. \tag{107}$$

Taking congruence transformations via $\mathrm{diag}\{Q(k), I\}$ on (89) yields

$$\begin{aligned} &\sum_{l=1}^{L}\lambda_l(k+i)\sum_{j=1}^{L}\lambda_j(k+i)\begin{bmatrix} (1-\alpha_{lj}(k))Q(k) & \star & \star \\ 0 & \alpha_{lj}(k)I & \star \\ \breve{\Phi}_{lj}(k) & \Gamma_{lj}(k) & Q(k) \end{bmatrix} \geq 0, \quad i \geq 0, \\ &\sum_{l=1}^{L}\lambda_l(k+i)\sum_{j=1}^{L}\lambda_j(k+i)\begin{bmatrix} Q(k) & \star & \star \\ \breve{\Phi}_{lj}(k) & Q(k) & \star \\ \spadesuit & 0 & \gamma(k)I \end{bmatrix} \geq 0, \quad i \geq 0, \end{aligned} \tag{108}$$

$$\begin{aligned} \spadesuit &= \begin{bmatrix} \mathscr{Q}_1^{1/2}\mathscr{F}_j\heartsuit & \mathscr{Q}_1^{1/2}\mathscr{F}_jU(k)Q_3(k) \\ \mathscr{Q}_2^{1/2}Q_3(k)U(k)^{\mathrm{T}} & \mathscr{Q}_2^{1/2}Q_3(k) \\ \mathscr{R}^{1/2}\left[F_yC_j\heartsuit + \breve{F}_x(k)U(k)^{\mathrm{T}}\right] & \mathscr{R}^{1/2}\left[F_yC_jU(k)Q_3(k) + \breve{F}_x(k)\right] \end{bmatrix}, \\ \heartsuit &= \left[N_1(k) + U(k)Q_3(k)U(k)^{\mathrm{T}}\right], \end{aligned} \tag{109}$$

where

$$\begin{aligned} \breve{\Phi}_{lj}(k) &= \begin{bmatrix} \diamond Q_1(k) + B_l\breve{F}_x(k)U(k)^{\mathrm{T}} & \diamond Q_2(k)^{\mathrm{T}} + B_l\breve{F}_x(k) \\ L_cC_jQ_1(k) + \breve{A}_c(k)U(k)^{\mathrm{T}} & L_cC_jQ_2(k)^{\mathrm{T}} + \breve{A}_c(k) \end{bmatrix}, \\ \breve{\Phi}_{lj}^1(k) &= \begin{bmatrix} \diamond Q_1(k) + B_l\breve{F}_x(k)U(k)^{\mathrm{T}} & \diamond Q_2(k)^{\mathrm{T}} + B_l\breve{F}_x(k) \end{bmatrix}, \\ \diamond &= \left(A_l + B_lF_yC_j\right), \breve{A}_c(k) = A_c(k)Q_3(k), \breve{F}_x(k) = F_x(k)Q_3(k). \end{aligned} \tag{110}$$

At each $k \geq 0$,
(a) for $k = 0$, take $U(0) = I$;
(b) for $k > 0$, calculate $x_c(k) = A_c(k-1)x_c(k-1) + L_c(k-1)y(k-1)$, and refresh $\{M_e, U, x^0\}(k)$ as in (15)–(17);
(c) for $k > 0$, find $M_e'(k)$ satisfying (the same as (19)–(20))
$\{x(k-1) - x^0(k-1) \in \varepsilon_{M_e(k-1)}, \|w(k-1)\| \leq 1\} \Longrightarrow x(k) - x'^0(k) \in \varepsilon_{M_e'(k)}$,
$x'^0(k) \geq U'(k)x_c(k)$,
$M_e'(k) \geq M_e(k)$,
and, if equations in step (c) in Algorithm 1 are feasible, then change $M_e(k) = M_e'(k)$, $U(k) = U'(k)$ and $x^0(k) = U'(k)x_c(k)$;
(d) solve (105) to find $\{A_c, L_c, F_x, F_y, M_1, N_1, Q_3, P_3\}^*(k)$;
(e) implement $u(k) = F_x(k)x_c(k) + F_y(k)y(k)$.

Algorithm 1: Full dynamic OFRMPC.

Taking congruence transformations on (91) and (93) via $\mathrm{diag}\{Q(k), I\}$, and applying the Schur complement, yields

$$\begin{bmatrix} Q(k) & \star \\ \frac{1}{\sqrt{1-\eta_{1s}}}\xi_s\left[\spadesuit \quad F_yC_jU(k)Q_3(k) + \breve{F}_x(k)\right] & \overline{u}_s^2 - \frac{1}{\eta_{1s}}\xi_sF_yE_jE_j^{\mathrm{T}}F_y^{\mathrm{T}}\xi_s^{\mathrm{T}} \end{bmatrix} \geq 0,$$
$$\spadesuit = F_yC_j\left[N_1(k) + U(k)Q_3(k)U(k)^T\right] + \breve{F}_x(k)U(k)^{\mathrm{T}},$$
$$j = 1, \ldots, L, \quad s = 1, \ldots, n_u, \tag{111}$$

$$\sum_{l=1}^{L}\sum_{j=1}^{L}\lambda_l(k+i)\lambda_j(k+i)\begin{bmatrix} Q(k) & \star & \star \\ 0 & I & \star \\ \spadesuit_1 & \spadesuit_2 & \spadesuit_3 \end{bmatrix} \geq 0,$$
$$\spadesuit_1 = \frac{1}{\sqrt{(1-\eta_{2s})(1-\eta_{3s})}}\Psi_s\mathscr{C}_h\breve{\Phi}_{lj}^1(k),$$
$$\spadesuit_2 = \frac{1}{\sqrt{(1-\eta_{2s})\eta_{3s}}}\Psi_s\mathscr{C}_h\Gamma_{lj}^1(k),$$
$$\spadesuit_3 = \overline{\psi}_s^2 - \frac{1}{\eta_{2s}}\Psi_s\mathscr{C}_h\mathscr{C}_h^{\mathrm{T}}\Psi_s^{\mathrm{T}}, \qquad h = 1, \ldots, L, s = 1, \ldots, q, i \geq 0. \tag{112}$$

In summary, problem (105) is simplified as

$$\min_{\{\gamma, \alpha_{lj}, \mathcal{Q}, N_1, Q_3, \breve{A}_c, \breve{F}_x\}(k)} \left\{\max_{[A|B|C|D|E|\mathscr{C}|\mathscr{E}|\mathscr{F}|\mathscr{G}](k+i)\in\Omega} \gamma(k)\right\},$$
$$\text{s.t.} \quad (103), (107) \text{ and } (108), (111), (112), \tag{113}$$

with $\{A_c(k), F_x(k)\}$ calculated by

$$A_c(k) = \breve{A}_c(k)Q_3(k)^{-1},$$
$$F_x(k) = \breve{F}_x(k)Q_3(k)^{-1}. \tag{114}$$

The solution to (113) can be obtained by LMI toolbox. Since CCA is not involved, it is computationally less expensive than (105).

Theorem 8 (see [5, 26]). *Adopt Algorithm 2. Suppose that Assumption 3 holds, and (113) is feasible at time $k = 0$. Then*

(i) (113) will be feasible at each $k > 0$;

(ii) $\{\gamma, z', x_c, u\}$ will converge to a neighborhood of 0, and the constraints in (2) are satisfied for all $k \geq 0$.

6.3. Prespecifying Relaxation Scalars. The scalars η_{rs} appear nonaffine and nonlinear in (105) and (113). Although it is suggested that η_{rs} can be line-searched over the interval $(0, 1)$ for online optimizations, in this way, the computational burden will be considerably increased. An alternative is to offline optimize η_{rs}. In [5, 26], we offline calculated η_{rs} by applying the norm-bounding technique.

At each $k \geq 0$,
(a) see steps (a)–(c) in Algorithm 1;
(b) solve (113) to find $\{\breve{A}_c, \breve{F}_x, N_1, Q_3\}^*(k)$;
(c) calculate $\{A_c, F_x\}^*(k)$ via (114), and implement $u(k) = F_x(k)x_c(k) + F_y y(k)$.

ALGORITHM 2: Partial dynamic OFRMPC.

The condition (111) is satisfied if

$$\begin{bmatrix} Q(k) & \star \\ \xi_s\left[\spadesuit \;\; F_yC_jU(k)Q_3(k) + \breve{F}_x(k)\right] & \tilde{u}_s^2 \end{bmatrix} \geq 0,$$

$$\spadesuit = F_yC_j\left[N_1(k) + U(k)Q_3(k)U(k)^{\mathrm{T}}\right] + \breve{F}_x(k)U(k)^{\mathrm{T}},$$

$$j = 1,\ldots,L,\; s = 1,\ldots,n_u, \tag{115}$$

$$\frac{1}{1-\eta_{1s}}\tilde{u}_s^2 + \frac{1}{\eta_{1s}}(\zeta_s^u)^2 \leq \bar{u}_s^2, \tag{116}$$

where $\zeta_s^u = \max\left\{(\xi_sF_yE_jE_j^{\mathrm{T}}F_y^{\mathrm{T}}\xi_s^{\mathrm{T}})^{1/2} \mid j = 1,\ldots,L\right\}$. The maximum $\tilde{u}_s$ satisfying (116) is calculated by

$$\tilde{u}_s = \bar{u}_s - \zeta_s^u, \tag{117}$$

by taking $\eta_{1s} = \zeta_s^u/\bar{u}_s$.

The condition (112) is satisfied if

$$\sum_{l=1}^{L}\lambda_l(k+i)\sum_{j=1}^{L}\lambda_j(k+i)\begin{bmatrix} Q(k) & \star \\ \Psi_s\mathscr{C}_h\breve{\Phi}_{lj}^1(k) & \tilde{\psi}_s^2 \end{bmatrix} \geq 0,$$

$$h = 1,\ldots,L,\; s = 1,\ldots,q,\; i \geq 0, \tag{118}$$

$$\frac{1}{(1-\eta_{2s})(1-\eta_{3s})}\tilde{\psi}_s^2 + \frac{1}{(1-\eta_{2s})\eta_{3s}}\left(\bar{\zeta}_s^z\right)^2 + \frac{1}{\eta_{2s}}(\zeta_s^z)^2 \leq \bar{\psi}_s^2, \tag{119}$$

where $\zeta_s^z = \max\left\{(\Psi_s\mathscr{E}_h\mathscr{E}_h^{\mathrm{T}}\Psi_s^{\mathrm{T}})^{1/2} \mid h = 1,\ldots,L\right\}$ and

$$\bar{\zeta}_s^z = \min_{\bar{\zeta}_s^z} \bar{\zeta}_s^z$$

$$\text{s.t.} \quad \sum_{l=1}^{L}\lambda_l(k+i)\sum_{j=1}^{L}\lambda_j(k+i) \cdot \begin{bmatrix} \bar{\zeta}_s^z I & \star \\ \Psi_s\mathscr{C}_h\Gamma_{lj}^1(k) & \bar{\zeta}_s^z \end{bmatrix} \geq 0, h = 1,\ldots,L, \quad i \geq 0. \tag{120}$$

The maximum $\tilde{\psi}_s$ satisfying (119) is calculated by

$$\tilde{\psi}_s = \bar{\psi}_s - \zeta_s^z - \bar{\zeta}_s^z, \tag{121}$$

by taking $\eta_{2s} = \zeta_s^z/\bar{\psi}_s$ and $\eta_{3s} = \bar{\zeta}_s^z/(\bar{\psi}_s - \zeta_s^z)$.

In the above, since ζ_s^u and $\left\{\zeta_s^z, \bar{\zeta}_s^z\right\}$ are the norms of the disturbance-related items, the method for optimizing $\{\eta_{1s}, \eta_{2s}, \eta_{3s}\}$ has been called the norm-bounding technique. In this way, we obtain "the second best" values of η_{rs} (though may not be the best).

The constraints (115) and (118) will not be utilized in the optimization problems (though they could be utilized), since they are more conservative than (111) and (112). In (111) the item $\bar{u}_s^2 - (1/\eta_{1s})\xi_sF_yE_jE_j^{\mathrm{T}}F_y^{\mathrm{T}}\xi_s^{\mathrm{T}}$ applies for each j, while in (115), the item $\tilde{u}_s^2$ imposes for all $j = 1,\ldots,L$. Similarly, in (112), the item $[(1/(\sqrt{(1-\eta_{2s})\eta_{3s}}))\Psi_s\mathscr{C}_h\Gamma_{lj}^1(k), \bar{\psi}_s^2 - (1/\eta_{2s})\Psi_s\varepsilon_h\varepsilon_h^{\mathrm{T}}\Psi_s^{\mathrm{T}}]$ applies for each pair of $\{l, j\}$, while in (118), the item $\tilde{\psi}_s^2$ imposes for all $l, j = 1,\ldots,L$.

6.4. Alternative Transformations for LPV. Take $n_x = n_{x_c}$ and $\mathscr{Q}_2 = 0$ in this subsection. Based on (98) and (99), let us define

$$\begin{aligned}
N_1 &:= M_1^{-1}, \\
P_1 &:= Q_1^{-1}, \\
U &:= -M_1^{-1}M_2^{\mathrm{T}}, \\
e(k) &:= x(k) - Ux_c(k), \\
\hat{A}_c &:= -UA_cQ_2, \\
\hat{L}_c &:= -UL_c, \\
\hat{F}_x &= F_xQ_2, \\
\bar{A}_c &:= -UA_cM_2^{-\mathrm{T}}(M_1 - P_1), \\
\bar{F}_x &:= F_xM_2^{-\mathrm{T}}(M_1 - P_1), \\
T_0 &:= \begin{bmatrix} I & 0 \\ 0 & M_2^{-\mathrm{T}}(M_1 - P_1) \end{bmatrix}, \\
T_1 &:= \begin{bmatrix} Q_1 & N_1 \\ Q_2 & 0 \end{bmatrix}, \\
T_2 &:= \begin{bmatrix} I & 0 \\ 0 & -U^{\mathrm{T}} \end{bmatrix}, \\
\mathscr{M}_P &:= \begin{bmatrix} M_1 & \star \\ M_1 - P_1 & M_1 - P_1 \end{bmatrix}, \\
\mathscr{Q}_N &:= \begin{bmatrix} Q_1 & \star \\ N_1 - Q_1 & Q_1 - N_1 \end{bmatrix}, \\
\mathscr{N}_Q &:= \begin{bmatrix} Q_1 & \star \\ N_1 & N_1 \end{bmatrix}, \\
\bar{\Phi}_{lj} &:= \begin{bmatrix} A_l + B_lF_yC_j & B_l\bar{F}_x \\ \hat{L}_cC_j & \bar{A}_c \end{bmatrix}, \\
\hat{\Phi}_{lj} &:= \begin{bmatrix} (A_l + B_lF_yC_j)Q_1 + B_l\hat{F}_x & (A_l + B_lF_yC_j)N_1 \\ \hat{L}_cC_jQ_1 + \hat{A}_c & \hat{L}_cC_jN_1 \end{bmatrix}, \\
\hat{\Gamma}_{lj} &:= \begin{bmatrix} D_l + B_lF_yE_j \\ \hat{L}_cE_j \end{bmatrix}, \\
\bar{\Phi}_{lj}^1 &:= \begin{bmatrix} A_l + B_lF_yC_j & B_l\bar{F}_x \end{bmatrix}, \\
\hat{\Gamma}_{lj}^1 &:= D_l + B_lF_yE_j, \\
\hat{\Phi}_{lj}^1 &= \left[(A_l + B_lF_yC_j)Q_1 + B_l\hat{F}_x \;\; (A_l + B_lF_yC_j)N_1\right].
\end{aligned} \tag{122}$$

Based on these notations, we have

$$\begin{aligned} A_c &= -U^{-1}\overline{A}_c(M_1-P_1)^{-1}M_2^{\mathrm{T}},\\ L_c &= -U^{-1}\widehat{L}_c,\\ F_x &= \overline{F}_x(M_1-P_1)^{-1}M_2^{\mathrm{T}},\\ M_2 &= -U^{\mathrm{T}}M_1, \end{aligned} \tag{123}$$

$$\begin{aligned} A_c &= -U^{-1}\widehat{A}_cQ_2^{-1},\\ L_c &= -U^{-1}\widehat{L}_c,\\ F_x &= \widehat{F}_xQ_2^{-1},\\ Q_2 &= U^{-1}(Q_1-N_1). \end{aligned} \tag{124}$$

According to (98), we have $Q_3 = U^{-1}(Q_1-N_1)U^{-\mathrm{T}}$. Applying a congruence transformation on (103), via diag $\{I, U(k)^{\mathrm{T}}\}$, yields

$$\begin{bmatrix} 1-\varrho(k) & \star \\ U(k)x_c(k) & Q_1(k)-N_1(k) \end{bmatrix} \geq 0. \tag{125}$$

Based on (98) and (99), applying congruence transformations on (89), via diag$\{T_0, I, T_2\}$ and diag$\{T_0, T_2, I, I\}$, respectively, yields

$$\sum_{l=1}^{L}\lambda_l(k+i)\sum_{j=1}^{L}\lambda_j(k+i)\Upsilon_{lj}^{\mathrm{QB}} \geq 0, \quad i\geq 0, \tag{126}$$

$$\sum_{l=1}^{L}\lambda_l(k+i)\sum_{j=1}^{L}\lambda_j(k+i)\Upsilon_{lj}^{\mathrm{opt}} \geq 0, \quad i\geq 0, \tag{127}$$

where $\Upsilon_{lj}^{\mathrm{QB}} := \begin{bmatrix} (1-\alpha_{lj})\mathscr{M}_P & \star & \star \\ 0 & \alpha_{lj}I & \star \\ \overline{\Phi}_{lj} & \widehat{\Gamma}_{lj} & \mathscr{Q}_N \end{bmatrix}$, $\Upsilon_{lj}^{\mathrm{opt}} := \begin{bmatrix} \mathscr{M}_P & \star & \star & \star \\ \overline{\Phi}_{lj} & \mathscr{Q}_N & \star & \star \\ \left[\mathscr{Q}_1^{1/2}\mathscr{F}_j \;\; 0\right] & 0 & \gamma I & \star \\ \mathscr{R}^{1/2}\left[F_yC_j \;\; \overline{F}_x\right] & 0 & 0 & \gamma I \end{bmatrix}$. Applying congruence transformations on (91) and (93), via diag$\{T_0, I\}$, yields

$$\begin{bmatrix} \mathscr{M}_P & \star & \star \\ 0 & I & \star \\ \dfrac{1}{\sqrt{1-\eta_{1s}}}\xi_s\left[F_yC_j \;\; \overline{F}_x\right] & \dfrac{1}{\sqrt{\eta_{1s}}}\xi_sF_yE_j & \overline{u}_s^2 \end{bmatrix} \geq 0, \quad j=1,\ldots,L,\; s=1,\ldots,n_u, \tag{128}$$

$$\sum_{l=1}^{L}\sum_{j=1}^{L}\lambda_l(k+i)\lambda_j(k+i)\times\begin{bmatrix} \mathscr{M}_P & \star & \star \\ 0 & I & \star \\ \dfrac{1}{\sqrt{(1-\eta_{2s})(1-\eta_{3s})}}\Psi_s\mathscr{C}_h\overline{\Phi}_{lj}^1 & \dfrac{1}{\sqrt{(1-\eta_{2s})\eta_{3s}}}\Psi_s\mathscr{C}_h\widehat{\Gamma}_{lj}^1 & \overline{\psi}_s^2-\dfrac{1}{\eta_{2s}}\Psi_s\varepsilon_h\varepsilon_h^{\mathrm{T}}\Psi_s^{\mathrm{T}} \end{bmatrix} \geq 0, \quad h=1,\ldots,L,\; s=1,\ldots,q,\; i\geq 0. \tag{129}$$

Summarizing the above, an equivalent transformation of (105) is (see [5])

$$\begin{aligned} &\min_{\left\{\gamma,\alpha_{lj},\varrho,M_1,N_1,Q_1,P_1,\overline{A}_c,\widehat{L}_c,\overline{F}_x,F_y\right\}(k)} \left\{\max_{[A|B|C|D|E|\mathscr{C}|\mathscr{E}|\mathscr{F}|\mathscr{G}](k+i)\in\Omega}\gamma(k)\right\},\\ &\text{s.t.} \quad (125),\,(7),\,(126)-(129)\text{ and } M_1(k)=N_1(k)^{-1}, Q_1(k)=P_1(k)^{-1}, \end{aligned} \tag{130}$$

with $\{A_c, L_c, F_x\}(k)$ calculated by (123). The optimization problem (130) is nonconvex, but its near-optimal solution arbitrarily close to the theoretically optimal one can be found by applying ICCA.

Based-on (98) and (99), applying congruence transformations on (89) and (90), via diag$\{T_1, I, T_2\}$ and diag$\{T_1, T_2, I, I\}$ respectively, yields

$$\sum_{l=1}^{L}\lambda_l(k+i)\sum_{j=1}^{L}\lambda_j(k+i)\begin{bmatrix}(1-\alpha_{lj})\mathcal{N}_Q & \star & \star \\ 0 & \alpha_{lj}I & \star \\ \widehat{\Phi}_{lj} & \widehat{\Gamma}_{lj} & \mathcal{Q}_N\end{bmatrix}\geq 0,\quad i\geq 0, \tag{131}$$

$$\sum_{l=1}^{L}\lambda_l(k+i)\sum_{j=1}^{L}\lambda_j(k+i)\begin{bmatrix}\mathcal{N}_Q & \star & \star \\ \widehat{\Phi}_{lj} & \mathcal{Q}_N & \star \\ \begin{bmatrix}\mathcal{Q}_1^{1/2}\mathcal{F}_jQ_1 & \mathcal{Q}_1^{1/2}\mathcal{F}_jN_1 \\ \mathcal{R}^{1/2}(F_yC_jQ_1+\widehat{F}_x) & \mathcal{R}^{1/2}F_yC_jN_1\end{bmatrix} & 0 & \gamma I\end{bmatrix}\geq 0,\quad i\geq 0. \tag{132}$$

Applying congruence transformations on (91) and (93), via $\operatorname{diag}\{T_1, I\}$, yields

$$\begin{bmatrix}\mathcal{N}_Q & \star \\ \frac{1}{\sqrt{1-\eta_{1s}}}\xi_s\begin{bmatrix}F_yC_jQ_1+\widehat{F}_x & F_yC_jN_1\end{bmatrix} & \overline{u}_s^2-\frac{1}{\eta_{1s}}\xi_sF_yE_jE_j^{\mathrm{T}}F_y^{\mathrm{T}}\xi_s^{\mathrm{T}}\end{bmatrix}\geq 0,\quad j=1,\ldots,L,\quad s=1,\ldots,n_u, \tag{133}$$

$$\sum_{l=1}^{L}\sum_{j=1}^{L}\lambda_l(k+i)\lambda_j(k+i)\begin{bmatrix}\mathcal{N}_Q & \star & \star \\ 0 & I & \star \\ \frac{1}{\sqrt{(1-\eta_{2s})(1-\eta_{3s})}}\Psi_s\mathscr{C}_h\widehat{\Phi}_{lj}^1 & \frac{1}{\sqrt{(1-\eta_{2s})\eta_{3s}}}\Psi_s\mathscr{C}_h\widehat{\Gamma}_{lj}^1 & \overline{\psi}_s^2-\frac{1}{\eta_{2s}}\Psi_s\mathscr{E}_h\varepsilon_h^{\mathrm{T}}\Psi_s^{\mathrm{T}}\end{bmatrix}\geq 0, \tag{134}$$
$$h=1,\ldots,L, s=1,\ldots,q, i\geq 0.$$

In summary, an equivalent transformation of (113) is (see [5])

$$\min_{\{\gamma,\alpha_{lj},\mathcal{Q},N_1,Q_1,\widehat{A}_c,\widehat{F}_x\}(k)}\left\{\max_{[A|B|C|D|E|\mathscr{C}|\mathscr{E}|\mathscr{F}|\mathscr{G}](k+i)\in\Omega}\gamma(k)\right\},$$
$$\text{s.t.}\quad (125),\ (107)\ \text{and}\ (131)-(134), \tag{135}$$

with $\{A_c, L_c, F_x\}(k)$ calculated by (124) and $\{\widehat{L}_c, F_y\}$ prespecified. The solution to (135) can be obtained by LMI toolbox.

7. Conclusion

We have summarized the existing results for dynamic output feedback robust MPC for the polytopic LPV model with additive bounded disturbance. This kind of research is still undergoing. For example, the free control moves are not included satisfactorily as in the disturbance-free case when x is measurable (e.g., the partial feedback MPC, feedback MPC, open-loop MPC, and parameter-dependent open-loop MPC). The summary in this paper may pave the way for future research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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