Research Article

An Uncertain Alternating Renewal Insurance Risk Model

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The claim process in an insurance risk model with uncertainty is traditionally described by an uncertain renewal reward process. However, the claim process actually includes two processes, which are called the report process and the payment process, respectively. An alternative way is to describe the claim process by an uncertain alternating renewal reward process. Therefore, this paper proposes an insurance risk model under uncertain measure in which the claim process is supposed to be an alternating renewal reward process and the premium process is regarded as a renewal reward process. Then, the paper also gives the inverse uncertainty distribution of the insurance risk process. The expression of ruin index and the uncertainty distribution of the ruin time are derived which both have explicit expressions based on given uncertainty distributions. Finally, several examples are provided to illustrate the modeling ideas.

1. Introduction

The classical insurance risk models and their extended models generally assume that claim numbers and claim amounts are random variables. They also suppose that the time of the accident and the time of payment are consistent; that is, the insurance company immediately pays compensation to the insured when the accident occurs. Then, many types of insurance risk models are presented by means of stochastic process based on the probability theory; several scholars, for example, Dickson and Hipp [1], Li and Garrido [2], Dickson and Hipp [3], Chun [4], Gerber and Shiu [5], Sundt and Teugels [6], Paulsen and Gjessing [7], Albrecher and Hipp [8], and Yu et al. [9], extend the insurance risk model by considering inflation, dividend, and tax. Using Lévy process to model insurance risk processes and other insurance product has become popular, see, for example, Griffin [10], Biffis and Kyprianou [11], Zhang et al. [12], and Yu et al. [13].

However, for a new insurance product, it usually lacks historical data to estimate probability distributions. In this situation, we often use the belief degrees of the claim numbers and claim amounts estimated by some experienced

domain experts to describe the indeterminacy. Kahneman and Tversky [14] find that humans tend to place too much emphasis on unlikely events. Thus, probability theory is difficult to model the belief degree unless we get enough historical data, and it is unreasonable to describe an insurance risk process with stochastic process. The research about the insurance risk process that is described by a stochastic process has been gradually challenged by many scholars. De Wit [15] first developed an insurance risk process under fuzzy theory. Then, the insurance risk processes with fuzziness are studied by Lemaire [16], Cummins and Derrig [17], Derrig and Ostaszewski [18], Yu [19], Shapiro [20], and Li et al. [21]. Furthermore, Huang et al. [22] and Shapiro [23] regard the claim amounts as fuzzy random variable and propose the fuzzy random risk model.

When the estimated distributions are not close enough to the real frequencies, Liu [24] invented the uncertainty theory, which is used to model human indeterminacy due to belief degrees. Now, the theory has been applied to construct insurance risk models. Considering the human uncertainty in running an insurance company, Li et al. [25] propose a premium principle under uncertain measure via the distortion function. To study the evolution of uncertain

phenomenon over time, Liu [26] develops the concept of uncertain process. Meanwhile, Liu also presents the uncertain renewal process as a special and important case. After that, Liu [27] researches the uncertain renewal reward process. Yao and Relescu [28] apply the uncertain renewal process to analyze an age replacement policy. In addition, Yao [29] studies the uncertain calculus of the uncertain renewal process by proposing the integration and differentiation about the renewal process. Yao and Li [30] regard off-times and on-times as uncertain variables and propose an uncertain alternating renewal process. Zhang et al. [31] also show a delayed renewal process for uncertain interarrival times, where the first interarrival time is completely different from other times. Recently, Liu [32] provides an uncertain insurance risk model by applying the renewal reward process and derives the ruin index. In the uncertain insurance risk model, Liu assumes that the premium is a real function proportional to time and the claim amount obeys an uncertain renewal reward process. Based on Liu's insurance risk process, Yao and Zhou [33] further investigate the uncertainty distribution of the ruin time. Yao and Qin [34] point out that the premiums follow the renewal reward process in actual applications rather than a real function, so they propose an uncertain insurance risk process in which both the premiums and claims follow the uncertain renewal reward process. Liu et al. [35] extend Liu's model and discuss an uncertain insurance risk model with a variational lower limit. Liu and Yang [36] establish an uncertain insurance risk process considering an insurance company having multiple claims with uncertainty theory.

In the above studies, the claim process is regarded as a renewal process or a renewal reward process. In fact, the moment of the accident and the moment of the payment are not simultaneous, and in many types of insurance, the time interval between a claim event and the determination of the payment for the claim can be very long [37]. So, the claim process should include two processes: the report process and the payment process. The report process refers to the insured formally notifying the insurance company about an event. The payment process refers to the process whereby the insurer reviews the claim and sees whether the event or situation falls within the risks covered by the policy. That is to say, the insurer will need to determine that the claim meets the terms and conditions of the insurance policy. Obviously, the two processes should follow different uncertainty distributions. Therefore, it is more reasonable to view the claim process as an uncertain alternating renewal reward process. At present, few studies have considered the claim process in an insurance risk model as an uncertain alternating renewal process.

Inspired by the ideas we have reviewed, this paper proposes an insurance risk model in which the claims follow an uncertain alternating renewal reward process, while the premiums follow an uncertain renewal reward process. The inverse uncertainty distribution of insurance risk process, ruin index, and ruin time are derived. We also compare our model with Yao and Qin's model through numerical

examples and explain the significance of describing the claim process by uncertain alternating renewal reward process.

The rest of the paper is organized as follows. Section 2 presents an uncertain alternating renewal insurance risk model and gives the expressions of ruin index and ruin time. In Section 3, several examples are provided to clarify the modeling idea of the insurance risk model. Finally, conclusions will be listed at the end.

2. An Uncertain Alternating Renewal Insurance Risk Model

Next, we study an insurance risk process in an uncertain environment. The premium process and the claim process are regarded as an uncertain renewal reward process and an uncertain alternating renewal reward process, respectively. In the following discussions, we give an assumption that the new claim event will not occur during the period of reviewing the current claim. Let the premium process be an uncertain renewal reward process:

$$R_{1t} = \sum_{i=1}^{N_{1t}} P_i, \quad (1)$$

where $P_1, P_2, \dots$ are independent uncertain premium amounts, and

$$N_{1t} = \max_{n \geq 0} \{n \mid \xi_{11} + \xi_{12} + \dots + \xi_{1n} \leq t\}, \quad (2)$$

is an uncertain renewal process with independent uncertain interarrival times $\xi_{11}, \xi_{12}, \dots$

The claim process is an uncertain alternating renewal reward process:

$$R_{2t} = \sum_{i=1}^{N_{2t}} C_i, \quad (3)$$

where $C_1, C_2, \dots$ are independent uncertain claim amounts. Consider the claim process as an uncertain alternating renewal reward process, and the claim process can be described as the following process. The first event happens as well as the insured reports claim for an uncertain time ξ_{21} . After an uncertain time ξ_{21} , the insurer reviews your claim and provides a payout for an uncertain time η_{21} and at an uncertain claim amount C_1 . Next, the second event happens and the insured reports claim for an uncertain time ξ_{22} . After an uncertain time ξ_{22} , the insurer reviews your claim and provides a payout for an uncertain time η_{22} and at an uncertain claim amount C_2 . The process continues infinitely (see Figure 1). Then, let

$$\begin{aligned} N_{2t} &= \max_{n \geq 0} \{n \mid (\xi_{21} + \eta_{21}) + (\xi_{22} + \eta_{22}) + \dots + (\xi_{2n} + \eta_{2n}) \leq t\} \\ &= \sup_{n \geq 0} \{n \mid 0 \leq (\xi_{21} + \eta_{21}) + (\xi_{22} + \eta_{22}) + \dots + (\xi_{2n} + \eta_{2n}) \\ &\leq t < (\xi_{21} + \eta_{21}) + (\xi_{22} + \eta_{22}) + \dots + (\xi_{2n} + \eta_{2n}) + \xi_{2n+1}\}, \end{aligned} \quad (4)$$

be an uncertain alternating renewal process, where $\xi_{21}, \xi_{22}, \dots$ and $\eta_{21}, \eta_{22}, \dots$ are independent uncertain

FIGURE 1: Alternating renewal process.

interarrival times. $S_n = (\xi_{21} + \eta_{21}) + (\xi_{22} + \eta_{22}) + \dots + (\xi_{2n} + \eta_{2n})$ denotes the moment of the payment of the n th claim.

Let a be the initial capital, then the capital of an insurance company at time t is

$$Z_t = a + R_{1t} - R_{2t}. \quad (5)$$

The Z_t is an insurance risk model with an uncertain alternating renewal reward process. Apparently, once $Z_t < 0$, the insurance company faces the risk of ruin.

For an uncertain insurance risk model Z_t , in order to obtain some important theorems, the following notations will be used:

Φ : the uncertainty distribution of the premium amount P_1

Ψ : the uncertainty distribution of the claim amount C_1

μ_1 : the uncertainty distribution of the interarrival time ξ_{11}

μ_2 : the uncertainty distribution of the interarrival time ξ_{21}

λ : the uncertainty distribution of the interarrival time η_{21}

2.1. Ruin Index. The ruin index can be defined as the uncertain measure that the capital $Z_t < 0$ at time t . This section derives the explicit forms of the ruin index. Firstly, we get the inverse uncertainty distribution of an uncertain insurance risk process.

Theorem 1. For an uncertain insurance risk process $Z_t = a + R_{1t} - R_{2t}$, the inverse uncertainty distribution of Z_t can be derived as

$$Y_t^{-1}(\alpha) = a + \frac{\Phi^{-1}(\alpha)}{\mu_1^{-1}(1-\alpha)}t - \frac{\Psi^{-1}(\alpha)}{\mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)}t. \quad (6)$$

Proof. Note that, for any

$$\gamma \in \bigcup_{i=1}^{\infty} \left\{ (\xi_{1i} \geq \mu_1^{-1}(1-\alpha)) \cap (P_i \leq \Phi^{-1}(\alpha)) \cap (\xi_{2i} + \eta_{2i} \leq \mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)) \cap (C_i \geq \Psi^{-1}(1-\alpha)) \right\}, \quad (7)$$

we have

$$\begin{aligned} Z_t(\gamma) &= a + \sum_{i=1}^{N_{1t}(\gamma)} P_i - \sum_{i=1}^{N_{2t}(\gamma)} C_i \\ &\leq a + \frac{\Phi^{-1}(\alpha)}{\mu_1^{-1}(1-\alpha)}t - \frac{\Psi^{-1}(\alpha)}{\mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)}t, \\ &\left\{ Z_t(\gamma) \leq a + \frac{\Phi^{-1}(\alpha)}{\mu_1^{-1}(1-\alpha)}t - \frac{\Psi^{-1}(\alpha)}{\mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)}t \right\} \\ &\supset \bigcup_{i=1}^{\infty} \left\{ (\xi_{1i} \geq \mu_1^{-1}(1-\alpha)) \cap (P_i \leq \Phi^{-1}(\alpha)) \cap (\xi_{2i} + \eta_{2i} \leq \mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)) \cap (C_i \geq \Psi^{-1}(1-\alpha)) \right\}. \end{aligned} \quad (8)$$

Since the uncertain variables are independent and according to the property of uncertain measure, we can obtain

$$\begin{aligned} &M \left\{ \bigcap_{i=1}^{\infty} \left(\xi_{1i} \geq \mu_1^{-1}(1-\alpha) \right) \cap \left(P_i \leq \Phi^{-1}(\alpha) \right) \cap \left(\xi_{2i} + \eta_{2i} \leq \mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha) \right) \cap \left(C_i \geq \Psi^{-1}(1-\alpha) \right) \right\} \\ &= \bigwedge_{i=1}^{\infty} M \left\{ \xi_{1i} \geq \mu_1^{-1}(1-\alpha) \right\} \wedge M \left\{ P_i \leq \Phi^{-1}(\alpha) \right\} \wedge M \left\{ \xi_{2i} + \eta_{2i} \leq \mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha) \right\} \wedge M \left\{ C_i \geq \Psi^{-1}(1-\alpha) \right\} \\ &= \bigwedge_{i=1}^{\infty} \alpha \wedge \alpha \wedge \alpha \wedge \alpha \\ &= \alpha. \end{aligned} \quad (9)$$

Furthermore, it follows from the monotonicity of uncertain measure that

$$M\left\{Z_t(\gamma) \leq a + \frac{\Phi^{-1}(\alpha)}{\mu_1^{-1}(1-\alpha)}t - \frac{\Psi^{-1}(\alpha)}{\mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)}t\right\} \geq \alpha. \quad (10)$$

In addition, since for any

$$\gamma \in \bigcap_{i=1}^{\infty} \left\{ (\xi_{1i} < \mu_1^{-1}(1-\alpha)) \cap (P_i > \Phi^{-1}(\alpha)) \cap (\xi_{2i} + \eta_{2i} > \mu_2^{-1}(\alpha) + \lambda_2^{-1}(\alpha)) \cap (C_i < \Psi^{-1}(1-\alpha)) \right\}, \quad (11)$$

we have

$$\begin{aligned} Z_t(\gamma) &= a + \sum_{i=1}^{N_{1t}(\gamma)} P_i - \sum_{i=1}^{N_{2t}(\gamma)} C_i \\ &> a + \frac{\Phi^{-1}(\alpha)}{\mu_1^{-1}(1-\alpha)}t - \frac{\Psi^{-1}(\alpha)}{\mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)}t, \\ &\left\{ Z_t(\gamma) > a + \frac{\Phi^{-1}(\alpha)}{\mu_1^{-1}(1-\alpha)}t - \frac{\Psi^{-1}(\alpha)}{\mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)}t \right\} \\ &\supset \bigcap_{i=1}^{\infty} \left\{ (\xi_{1i} < \mu_1^{-1}(1-\alpha)) \cap (P_i > \Phi^{-1}(\alpha)) \cap (\xi_{2i} + \eta_{2i} > \mu_2^{-1}(\alpha) + \lambda_2^{-1}(\alpha)) \cap (C_i < \Psi^{-1}(1-\alpha)) \right\}. \end{aligned} \quad (12)$$

According to the independence of these uncertain variables again that

$$\begin{aligned} &M\left\{ \bigcup_{i=1}^{\infty} (\xi_{1i} < \mu_1^{-1}(1-\alpha)) \cap (P_i > \Phi^{-1}(\alpha)) \cap (\xi_{2i} + \eta_{2i} > \mu_2^{-1}(\alpha) + \lambda_2^{-1}(\alpha)) \cap (C_i < \Psi^{-1}(1-\alpha)) \right\} \\ &= \bigwedge_{i=1}^{\infty} M\{\xi_{1i} < \mu_1^{-1}(1-\alpha)\} \wedge M\{P_i > \Phi^{-1}(\alpha)\} \wedge M\{\xi_{2i} + \eta_{2i} > \mu_2^{-1}(\alpha) + \lambda_2^{-1}(\alpha)\} \wedge M\{C_i < \Psi^{-1}(1-\alpha)\} \\ &= \bigwedge_{i=1}^{\infty} (1-\alpha) \wedge (1-\alpha) \wedge (1-\alpha) \wedge (1-\alpha) \\ &= 1-\alpha, \end{aligned} \quad (13)$$

we get

$$M\left\{Z_t(\gamma) > a + \frac{\Phi^{-1}(\alpha)}{\mu_1^{-1}(1-\alpha)}t - \frac{\Psi^{-1}(\alpha)}{\mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)}t\right\} \geq 1-\alpha, \quad (14)$$

which is equivalent to the following form (see duality of uncertain measure):

$$M\left\{Z_t(\gamma) \leq a + \frac{\Phi^{-1}(\alpha)}{\mu_1^{-1}(1-\alpha)}t - \frac{\Psi^{-1}(\alpha)}{\mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)}t\right\} \leq \alpha. \quad (15)$$

Above all, $M\{Z_t(\gamma) \leq a + ((\Phi^{-1}(\alpha))/(\mu_1^{-1}(1-\alpha)))t - ((\Psi^{-1}(\alpha))/(\mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)))t\} = \alpha$.

Theorem 1 is proved. $\square$

Theorem 2. Let $Z_t = a + R_{1t} - R_{2t}$, the ruin index can be expressed as

$$\text{ruin} = \max_{m \geq 0, n \geq 1} \alpha_{m,n}, \quad (16)$$

where m and n are nonnegative integers and

$$\alpha_{m,n} = \sup\{\alpha \in [0, 1] \mid n\mu_2^{-1}(\alpha) + n\lambda^{-1}(\alpha) - (m+1)\mu_1^{-1}(1-\alpha) < 0, a + m\Phi^{-1}(\alpha) - n\Psi^{-1}(1-\alpha) < 0\}. \quad (17)$$

Proof. Obviously, the ruin risk can be calculated by

$$\left\{ \inf_{t \geq 0} Z_t < 0 \right\} = \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \left\{ \left(\sum_{j=1}^n (\xi_{2j} + \eta_{2j}) < \sum_{i=1}^{m+1} \xi_{1i} \right) \cap \left(a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right) \right\}. \quad (18)$$

(1) For given nonnegative m and n ,

$$\begin{aligned} & \left\{ \left(\sum_{j=1}^n (\xi_{2j} + \eta_{2j}) < \sum_{i=1}^{m+1} \xi_{1i} \right) \cap \left(a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right) \right\} \\ & \supset \bigcap_{i=1}^{m+1} \left\{ \xi_{1i} > \mu_1^{-1}(1 - \alpha_{m,n}) \right\} \cap \bigcap_{i=1}^m \left\{ P_i \leq \Phi^{-1}(\alpha_{m,n}) \right\} \cap \bigcap_{j=1}^n \left\{ (\xi_{2j} + \eta_{2j} \leq \mu_2^{-1}(\alpha_{m,n}) + \lambda^{-1}(\alpha_{m,n})) \cap (C_j > \Psi^{-1}(\alpha_{m,n})) \right\}. \end{aligned} \quad (19)$$

Then, we have

$$\begin{aligned} & M \left\{ \left(\sum_{j=1}^n (\xi_{2j} + \eta_{2j}) < \sum_{i=1}^{m+1} \xi_{1i} \right) \cap \left(a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right) \right\} \\ & \geq \bigcap_{i=1}^{m+1} \left\{ \xi_{1i} > \mu_1^{-1}(1 - \alpha_{m,n}) \right\} \cap \bigcap_{i=1}^m \left\{ P_i \leq \Phi^{-1}(\alpha_{m,n}) \right\} \cap \bigcap_{j=1}^n \left\{ (\xi_{2j} + \eta_{2j} \leq \mu_2^{-1}(\alpha_{m,n}) + \lambda^{-1}(\alpha_{m,n})) \cap (C_j > \Psi^{-1}(\alpha_{m,n})) \right\} \\ & \geq \bigwedge_{i=1}^{m+1} M \left\{ \xi_{1i} > \mu_1^{-1}(1 - \alpha_{m,n}) \right\} \wedge \bigwedge_{i=1}^m M \left\{ P_i \leq \Phi^{-1}(\alpha_{m,n}) \right\} \wedge \bigwedge_{j=1}^n M \left\{ (\xi_{2j} + \eta_{2j} \leq \mu_2^{-1}(\alpha_{m,n}) + \lambda^{-1}(\alpha_{m,n})) \cap (C_j > \Psi^{-1}(\alpha_{m,n})) \right\} \\ & = \alpha_{m,n}, \\ & M \left\{ \inf_{t \geq 0} Z_t < 0 \right\} = M \left\{ \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \left(\sum_{j=1}^n (\xi_{2j} + \eta_{2j}) < \sum_{i=1}^{m+1} \xi_{1i} \right) \cap \left(a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right) \right\} \\ & \geq \max_{m \geq 0, n \geq 1} M \left\{ \left(\sum_{j=1}^n (\xi_{2j} + \eta_{2j}) < \sum_{i=1}^{m+1} \xi_{1i} \right) \cap \left(a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right) \right\} \\ & = \max_{m \geq 0, n \geq 1} \alpha_{m,n}. \end{aligned} \quad (20)$$

(2) For given nonnegative m and n ,

$$\begin{aligned} & \left\{ \left(\sum_{j=1}^n (\xi_{2j} + \eta_{2j}) < \sum_{i=1}^{m+1} \xi_{1i} \right) \cap \left(a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right) \right\} \\ & \subset \bigcup_{i=1}^{m+1} \left\{ \xi_{1i} > \mu_1^{-1}(1 - \alpha_{m,n}) \right\} \cup \bigcup_{i=1}^m \left\{ P_i \leq \Phi^{-1}(\alpha_{m,n}) \right\} \cup \bigcup_{j=1}^n \left\{ (\xi_{2j} + \eta_{2j} \leq \mu_2^{-1}(\alpha_{m,n}) + \lambda^{-1}(\alpha_{m,n})) \cup (C_j > \Psi^{-1}(\alpha_{m,n})) \right\}. \end{aligned} \quad (21)$$

Then, we also have

$$\begin{aligned}
& \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \left\{ \left(\sum_{j=1}^n (\xi_{2j} + \eta_{2j}) < \sum_{i=1}^{m+1} \xi_{1i} \right) \cap \left(a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right) \right\} \\
& \subset \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=i-1}^{\infty} \{ \xi_{1i} > \mu_1^{-1}(1 - \alpha_{m,n}) \} \cup \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=i-1}^{\infty} \{ P_i \leq \Phi^{-1}(\alpha_{m,n}) \} \\
& \cup \bigcup_{j=1}^{\infty} \bigcup_{m=0}^{\infty} \bigcup_{n=j}^{\infty} \{ (\xi_{2j} + \eta_{2j} \leq \mu_2^{-1}(\alpha_{m,n}) + \lambda^{-1}(\alpha_{m,n})) \cup (C_j > \Psi^{-1}(\alpha_{m,n})) \} \\
& = \bigcup_{i=1}^{\infty} \{ \xi_{1i} > \bigwedge_{n=1}^{\infty} \bigwedge_{m=i-1}^{\infty} \mu_1^{-1}(1 - \alpha_{m,n}) \} \cup \bigcup_{i=1}^{\infty} \{ P_i \leq \bigvee_{n=1}^{\infty} \bigvee_{m=i-1}^{\infty} \Phi^{-1}(\alpha_{m,n}) \} \\
& \cup \bigcup_{j=1}^{\infty} \left\{ \left(\xi_{2j} + \eta_{2j} \leq \bigvee_{m=0}^{\infty} \bigvee_{n=j}^{\infty} (\mu_2^{-1}(\alpha_{m,n}) + \lambda^{-1}(\alpha_{m,n})) \right) \cup \left(C_j > \bigwedge_{m=0}^{\infty} \bigwedge_{n=j}^{\infty} \Psi^{-1}(\alpha_{m,n}) \right) \right\}.
\end{aligned} \tag{22}$$

Similarly, it follows from definition of uncertain measure that

$$\begin{aligned}
M \left\{ \inf_{t \geq 0} Z_t < 0 \right\} & \leq \bigvee_{i=1}^{\infty} M \left\{ \xi_{1i} > \bigwedge_{n=1}^{\infty} \bigwedge_{m=i-1}^{\infty} \mu_1^{-1}(1 - \alpha_{m,n}) \right\} \vee \bigvee_{i=1}^{\infty} M \left\{ P_i \leq \bigvee_{n=1}^{\infty} \bigvee_{m=i-1}^{\infty} \Phi^{-1}(\alpha_{m,n}) \right\} \\
& \vee \bigvee_{j=1}^{\infty} M \left\{ \xi_{2j} + \eta_{2j} \leq \bigvee_{m=0}^{\infty} \bigvee_{n=j}^{\infty} (\mu_2^{-1}(\alpha_{m,n}) + \lambda^{-1}(\alpha_{m,n})) \right\} \vee \bigvee_{j=1}^{\infty} M \left\{ C_j > \bigwedge_{m=0}^{\infty} \bigwedge_{n=j}^{\infty} \Psi^{-1}(\alpha_{m,n}) \right\} \\
& = \max_{m,n \geq 0} \alpha_{m,n}.
\end{aligned} \tag{23}$$

From (1) and (2), we can draw

$$M \left\{ \inf_{t \geq 0} Z_t < 0 \right\} = \max_{m,n \geq 0} \alpha_{m,n}. \tag{24}$$

Theorem 2 is proved. $\square$

Theorem 3. Let $Z_t = a + R_{1t} - R_{2t}$ be an uncertain alternating renewal insurance risk process. Then, the ruin index can be calculated by the following form:

$$\begin{aligned}
\text{ruin} & = \max_{m \geq 0, n \geq 1} \sup_{x, y \geq 0} \left(1 - \mu_1 \left(\frac{x}{m+1} \right) \right) \wedge \mu_2 \left(\frac{x}{2n} \right) \wedge \lambda \left(\frac{x}{2n} \right) \\
& \quad \wedge \Phi \left(\frac{y}{m} \right) \wedge \left(1 - \Psi \left(\frac{a+y}{n} \right) \right).
\end{aligned} \tag{25}$$

Proof. It is obvious that

$$\begin{aligned}
& M \left\{ \left(\sum_{j=1}^n (\xi_{2j} + \eta_{2j}) < \sum_{i=1}^{m+1} \xi_{1i} \right) \cap \left(a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right) \right\} \\
& M \left\{ \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) < \sum_{i=1}^{m+1} \xi_{1i} \right\} \wedge M \left\{ a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right\}.
\end{aligned} \tag{26}$$

Because we assume the uncertain interarrival times are independent, we have

$$\begin{aligned}
& M \left\{ \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) < \sum_{i=1}^{m+1} \xi_{1i} \right\} \\
& = M \left\{ \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) - \sum_{i=1}^{m+1} \xi_{1i} < 0 \right\} \\
& = \sup_{x \geq 0} \left(1 - \mu_1 \left(\frac{x}{m+1} \right) \right) \wedge \mu_2 \left(\frac{x}{2n} \right) \wedge \lambda \left(\frac{x}{2n} \right), \\
& M \left\{ a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right\} \\
& = \sup_{y \geq 0} \Phi \left(\frac{y}{m} \right) \wedge \left(1 - \Psi \left(\frac{a+y}{n} \right) \right).
\end{aligned} \tag{27}$$

Hence,

$$\begin{aligned}
\text{ruin} & = M \left\{ \inf_{t \geq 0} Z_t < 0 \right\} = \max_{m \geq 0, n \geq 1} \sup_{x, y \geq 0} \left(1 - \mu_1 \left(\frac{x}{m+1} \right) \right) \\
& \quad \wedge \mu_2 \left(\frac{x}{2n} \right) \wedge \lambda \left(\frac{x}{2n} \right) \wedge \Phi \left(\frac{y}{m} \right) \wedge \left(1 - \Psi \left(\frac{a+y}{n} \right) \right).
\end{aligned} \tag{28}$$

Theorem 3 is proved. $\square$

When uncertain variables P_1 , ξ_{11} , C_1 , ξ_{21} , and η_{22} have determinate uncertainty distributions and corresponding inverse uncertainty distributions exist, we can calculate the

crisp expressions of ruin index through Theorem 3 and Theorem 2, respectively.

2.2. Ruin Time. In addition to the ruin index, ruin time can also be used to measure the risk of an insurance company. Next, the definition of ruin time will be given, and the uncertainty distribution of ruin time can be derived.

Let $Z_t = a + \sum_{i=1}^{N_{1t}} P_i - \sum_{j=1}^{N_{2t}} C_j$ be the uncertain alternating renewal insurance risk process of an insurance company. Then, the ruin time of the insurance company can be defined as

$$\tau = \inf\{t \geq 0 \mid Z_t < 0\}. \quad (29)$$

It is easy to know that $\tau = +\infty$ means that the insurance company will not ruin. Therefore, for any $t \geq 0$, the ruin index also can be expressed in the following form:

$$M\{\tau < +\infty\} = \lim_{t \rightarrow \infty} M\{\tau \leq t\}, \quad (30)$$

where $M\{\tau \leq t\}$ denotes the uncertainty distribution of the ruin time.

We assume that the n th claim occurs at the instant $S_n = \sum_{i=1}^n (\xi_{2i} + \eta_{2i})$ and $N_{1t} = m$ at this time. Thus, the capital of the insurance company at the n th claim is

$$Y_{m,n} = a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j. \quad (31)$$

Then, we have

$$\{\tau \leq t\} = \left\{ \inf_{0 \leq s \leq t} Z_s < 0 \right\} = \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, Y_{m,n} < 0 \right\}. \quad (32)$$

The uncertain event $\left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, Y_{m,n} < 0 \right\}$ means that the n th claim occurs before the instant t , and the capital of the insurance company is less than 0 at this time.

Theorem 4. Suppose that the inverse uncertainty distributions of all the uncertain variables exist in the insurance risk process. Then,

where

$$\begin{aligned} \alpha_{m,n}(t) = & \sup\{\alpha \in [0, 1] \mid m\mu_1^{-1}(\alpha) \leq t\} \wedge \sup\{\alpha \in [0, 1] \mid n\mu_2^{-1}(\alpha) + n\lambda^{-1}(\alpha) \leq t\} \\ & \wedge \sup\{\alpha \in [0, 1] \mid a + m\Phi^{-1}(\alpha) - n\Psi^{-1}(1 - \alpha) \leq 0\}. \end{aligned} \quad (34)$$

Proof. Since

$$\begin{aligned} & M\left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, Y_{m,n} < 0 \right\}, \\ & M\left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right\}, \\ \alpha_{m,n}(t) = & \sup\{\alpha \in [0, 1] \mid m\mu_1^{-1}(\alpha) \leq t\} \wedge \sup\{\alpha \in [0, 1] \mid n\mu_2^{-1}(\alpha) + n\lambda^{-1}(\alpha) \leq t\} \\ & \wedge \sup\{\alpha \in [0, 1] \mid a + m\Phi^{-1}(\alpha) - n\Psi^{-1}(1 - \alpha) \leq 0\}, \end{aligned} \quad (35)$$

then, we have

$$\begin{aligned} & \left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, Y_{m,n} < 0 \right\}, \\ & \supset \bigcap_{i=1}^m \{\xi_{1i} \leq \mu_1^{-1}(\alpha_{m,n}(t))\} \cap \bigcap_{j=1}^n \{(\xi_{2j} + \eta_{2j}) \leq \mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t))\} \\ & \cap \bigcap_{i=1}^m \{P_i \leq \Phi^{-1}(\alpha_{m,n}(t))\} \cap \bigcap_{j=1}^n \{C_j \geq \Psi^{-1}(1 - \alpha_{m,n}(t))\}. \end{aligned} \quad (36)$$

According to the monotonicity of uncertain measure, it is obtained that

$$\begin{aligned}
 & M \left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right\} \\
 & \geq \bigwedge_{i=1}^m M \{ \xi_{1i} \leq \mu_1^{-1}(\alpha_{m,n}(t)) \} \wedge \bigwedge_{j=1}^n M \{ (\xi_{2j} + \eta_{2j}) \leq \mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t)) \} \\
 & \wedge \bigwedge_{i=1}^m M \{ P_i \leq \Phi^{-1}(\alpha_{m,n}(t)) \} \wedge \bigwedge_{j=1}^n \{ C_j \geq \Psi^{-1}(1 - \alpha_{m,n}(t)) \} \\
 & = \alpha_{m,n}(t).
 \end{aligned} \tag{37}$$

Additionally, we have

$$\begin{aligned}
 & \left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, Y_{m,n} < 0 \right\} \\
 & \subset \bigcup_{i=1}^m \{ \xi_{1i} \leq \mu_1^{-1}(\alpha_{m,n}(t)) \} \cup \bigcup_{j=1}^n \{ (\xi_{2j} + \eta_{2j}) \leq \mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t)) \} \\
 & \cup \bigcup_{i=1}^m \{ P_i \leq \Phi^{-1}(\alpha_{m,n}(t)) \} \cup \bigcup_{j=1}^n \{ C_j \geq \Psi^{-1}(1 - \alpha_{m,n}(t)) \}.
 \end{aligned} \tag{38}$$

Similarly, we can obtain

$$\begin{aligned}
 & M \left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right\} \\
 & \leq \bigvee_{i=1}^m M \{ \xi_{1i} \leq \mu_1^{-1}(\alpha_{m,n}(t)) \} \vee \bigvee_{j=1}^n M \{ (\xi_{2j} + \eta_{2j}) \leq \mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t)) \} \\
 & \vee \bigvee_{i=1}^m M \{ P_i \leq \Phi^{-1}(\alpha_{m,n}(t)) \} \vee \bigvee_{j=1}^n \{ C_j \geq \Psi^{-1}(1 - \alpha_{m,n}(t)) \} \\
 & = \alpha_{m,n}(t).
 \end{aligned} \tag{39}$$

Above all, we get

$$M \left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, Y_{m,n} < 0 \right\} = \alpha_{m,n}(t). \tag{40}$$

Theorem 4 is proved. $\square$

Theorem 5. Suppose that the inverse uncertainty distributions of all the uncertain variables exist in the insurance risk process. Then, the ruin time τ has an uncertainty distribution:

$$M\{\tau \leq t\} = \max_{m \geq 0, n \geq 1} \alpha_{m,n}(t), \tag{41}$$

where

$$\begin{aligned}
 \alpha_{m,n}(t) = & \sup \{ \alpha \in [0, 1] \mid m\mu_1^{-1}(\alpha) \leq t \} \wedge \sup \{ \alpha \in [0, 1] \mid n\mu_2^{-1}(\alpha) + n\lambda^{-1}(\alpha) \leq t \} \\
 & \wedge \sup \{ \alpha \in [0, 1] \mid a + m\Phi^{-1}(\alpha) - n\Psi^{-1}(1 - \alpha) \leq 0 \}.
 \end{aligned} \tag{42}$$

Proof. Since

$$\{\tau \leq t\} = \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right\}, \quad (43)$$

then, we have

$$\begin{aligned} & \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right\} \\ & \supset \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{i=1}^m \{ \xi_{1i} \leq \mu_1^{-1}(\alpha_{m,n}(t)) \} \cap \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{j=1}^n \{ (\xi_{2j} + \eta_{2j}) \leq \mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t)) \} \\ & \cap \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{i=1}^m \{ P_i \leq \Phi^{-1}(\alpha_{m,n}(t)) \} \cap \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{j=1}^n \{ C_j \geq \Psi^{-1}(1 - \alpha_{m,n}(t)) \}. \end{aligned} \quad (44)$$

According to the monotonicity of uncertain measure, it is obtained

$$\begin{aligned} M\{\tau \leq t\} &= M \left\{ \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \left(\sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right) \right\} \\ &\geq M \left\{ \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{i=1}^m (\xi_{1i} \leq \mu_1^{-1}(\alpha_{m,n}(t))) \right\} \wedge M \left\{ \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{j=1}^n ((\xi_{2j} + \eta_{2j}) \leq \mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t))) \right\} \\ &\wedge M \left\{ \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{i=1}^m (P_i \leq \Phi^{-1}(\alpha_{m,n}(t))) \right\} \wedge M \left\{ \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{j=1}^n (C_j \geq \Psi^{-1}(1 - \alpha_{m,n}(t))) \right\} \\ &= \bigvee_{m=0}^{\infty} \bigvee_{n=1}^{\infty} \bigwedge_{i=1}^m M \{ \xi_{1i} \leq \mu_1^{-1}(\alpha_{m,n}(t)) \} \wedge \bigvee_{m=0}^{\infty} \bigvee_{n=1}^{\infty} \bigwedge_{j=1}^n M \{ (\xi_{2j} + \eta_{2j}) \leq \mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t)) \} \\ &\wedge \bigvee_{m=0}^{\infty} \bigvee_{n=1}^{\infty} \bigwedge_{i=1}^m M \{ P_i \leq \Phi^{-1}(\alpha_{m,n}(t)) \} \wedge \bigvee_{m=0}^{\infty} \bigvee_{n=1}^{\infty} \bigwedge_{j=1}^n M \{ C_j \geq \Psi^{-1}(1 - \alpha_{m,n}(t)) \} \\ &= \bigvee_{m=0}^{\infty} \bigvee_{n=1}^{\infty} \alpha_{m,n}(t) \\ &= \max_{m \geq 0, n \geq 1} \alpha_{m,n}(t). \end{aligned} \quad (45)$$

Also, since

$$\begin{aligned} & \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \left\{ \sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0 \right\} \\ & \subset \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{i=1}^m \{ \xi_{1i} \leq \mu_1^{-1}(\alpha_{m,n}(t)) \} \cup \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{j=1}^n \{ (\xi_{2j} + \eta_{2j}) \leq \mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t)) \} \\ & \cup \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{i=1}^m \{ P_i \leq \Phi^{-1}(\alpha_{m,n}(t)) \} \cup \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{j=1}^n \{ C_j \geq \Psi^{-1}(1 - \alpha_{m,n}(t)) \} \\ & = \bigcup_{i=1}^{\infty} \bigcup_{m=i}^{\infty} \bigcup_{n=1}^{\infty} \{ \xi_{1i} \leq \mu_1^{-1}(\alpha_{m,n}(t)) \} \cup \bigcup_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \bigcup_{m=0}^{\infty} \{ (\xi_{2j} + \eta_{2j}) \leq \mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t)) \} \\ & \cup \bigcup_{i=1}^{\infty} \bigcup_{m=i}^{\infty} \bigcup_{n=1}^{\infty} \{ P_i \leq \Phi^{-1}(\alpha_{m,n}(t)) \} \cup \bigcup_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \bigcup_{m=0}^{\infty} \{ C_j \geq \Psi^{-1}(1 - \alpha_{m,n}(t)) \} \\ & = \bigcup_{i=1}^{\infty} \left\{ \xi_{1i} \leq \bigvee_{m=i}^{\infty} \bigvee_{n=1}^{\infty} \mu_1^{-1}(\alpha_{m,n}(t)) \right\} \cup \bigcup_{j=1}^{\infty} \left\{ (\xi_{2j} + \eta_{2j}) \leq \bigvee_{n=j}^{\infty} \bigvee_{m=0}^{\infty} (\mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t))) \right\} \\ & \cup \bigcup_{i=1}^{\infty} \left\{ P_i \leq \bigvee_{m=i}^{\infty} \bigvee_{n=1}^{\infty} \Phi^{-1}(\alpha_{m,n}(t)) \right\} \cup \bigcup_{j=1}^{\infty} \left\{ C_j \geq \bigwedge_{n=j}^{\infty} \bigwedge_{m=0}^{\infty} \Psi^{-1}(1 - \alpha_{m,n}(t)) \right\}, \end{aligned} \quad (46)$$

then, we have

$$\begin{aligned}
M\{\tau \leq t\} &= M\left\{\bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \left(\sum_{i=1}^m \xi_{1i} \leq t, \sum_{j=1}^n (\xi_{2j} + \eta_{2j}) \leq t, a + \sum_{i=1}^m P_i - \sum_{j=1}^n C_j < 0\right)\right\} \\
&\leq M\left\{\bigcup_{i=1}^{\infty} \left(\xi_{1i} \leq \bigvee_{m=i}^{\infty} \bigvee_{n=1}^{\infty} \mu_1^{-1}(\alpha_{m,n}(t))\right)\right\} \cup M\left\{\bigcup_{j=1}^{\infty} \left((\xi_{2j} + \eta_{2j}) \leq \bigvee_{n=j}^{\infty} \bigvee_{m=0}^{\infty} (\mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t)))\right)\right\} \\
&\cup M\left\{\bigcup_{i=1}^{\infty} \left(P_i \leq \bigvee_{m=i}^{\infty} \bigvee_{n=1}^{\infty} \Phi^{-1}(\alpha_{m,n}(t))\right)\right\} \cup M\left\{\bigcup_{j=1}^{\infty} \left(C_j \geq \bigwedge_{n=j}^{\infty} \bigwedge_{m=0}^{\infty} \Psi^{-1}(1 - \alpha_{m,n}(t))\right)\right\} \\
&= \bigvee_{i=1}^{\infty} M\left\{\xi_{1i} \leq \bigvee_{m=i}^{\infty} \bigvee_{n=1}^{\infty} \mu_1^{-1}(\alpha_{m,n}(t))\right\} \vee \bigvee_{j=1}^{\infty} M\left\{(\xi_{2j} + \eta_{2j}) \leq \bigvee_{n=j}^{\infty} \bigvee_{m=0}^{\infty} (\mu_2^{-1}(\alpha_{m,n}(t)) + \lambda^{-1}(\alpha_{m,n}(t)))\right\} \\
&\vee \bigvee_{i=1}^{\infty} M\left\{P_i \leq \bigvee_{m=i}^{\infty} \bigvee_{n=1}^{\infty} \Phi^{-1}(\alpha_{m,n}(t))\right\} \vee \bigvee_{j=1}^{\infty} M\left\{C_j \geq \bigwedge_{n=j}^{\infty} \bigwedge_{m=0}^{\infty} \Psi^{-1}(1 - \alpha_{m,n}(t))\right\} \\
&= \bigvee_{i=1}^{\infty} \bigvee_{m=i}^{\infty} \bigvee_{n=1}^{\infty} \alpha_{m,n}(t) \vee \bigvee_{j=1}^{\infty} \bigvee_{n=j}^{\infty} \bigvee_{m=0}^{\infty} \alpha_{m,n}(t) \\
&= \max_{m \geq 0, n \geq 1} \alpha_{m,n}(t).
\end{aligned} \tag{47}$$

Thus, we get $M\{\tau \leq t\} = \max_{m \geq 0, n \geq 1} \alpha_{m,n}(t)$.

Theorem 5 is proved. $\square$

3. Numerical Examples

We will illustrate the above conclusions through the following five numerical examples and present how to calculate the ruin index and ruin time. Firstly, we compare our model with Yao and Qin's model [34] in Example 1 and compute the ruin index by inverse uncertainty distributions. Example 2 presents how to calculate the expected claim rate and the

report rate according to the properties of uncertain renewal reward process [27] and uncertain alternating renewal process [30]. Example 3 and Example 4 give the uncertain measure that an insurance company ruins before time t . We compute the uncertainty distribution of ruin time in Example 5.

Example 1. Assume that $\mu_1 = N(e_1, \sigma_1)$, $\mu_2 = N(e_2, \sigma_2)$, $\lambda = N(e_3, \sigma_3)$, $\Phi = N(e'_1, \sigma'_1)$, and $\Psi = N(e'_2, \sigma'_2)$ follow normal uncertainty distributions. Then, we have

$$\begin{aligned}
&n\mu_2^{-1}(\alpha) + n\lambda^{-1}(\alpha) - (m+1)\mu_1^{-1}(1-\alpha) \\
&= n\left(e_2 + \frac{\sigma_2\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right) + n\left(e_3 + \frac{\sigma_3\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right) - (m+1)\left(e_1 + \frac{\sigma_1\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right), \\
&a + m\Phi^{-1}(\alpha) - n\Psi^{-1}(1-\alpha) \\
&= a + m\left(e'_1 + \frac{\sigma'_1\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right) - n\left(e'_2 + \frac{\sigma'_2\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha}\right).
\end{aligned} \tag{48}$$

According to Theorem 2, we can obtain that

$$\begin{aligned}
\alpha_{m,n} &= \left(1 + \exp\left(\frac{\pi}{\sqrt{3}} \cdot \frac{n(e_2 + e_3) - (m+1)e_1}{n(\sigma_2 + \sigma_3) + (m+1)\sigma_1}\right)\right)^{-1} \\
&\wedge \left(1 + \exp\left(\frac{\pi}{\sqrt{3}} \cdot \frac{a + me'_1 - ne'_2}{m\sigma'_1 + n\sigma'_2}\right)\right)^{-1}.
\end{aligned} \tag{49}$$

Furthermore, the ruin index can be calculated as

$$\begin{aligned}
\text{ruin} &= \max_{m \geq 0, n \geq 1} \left(1 + \exp\left(\frac{\pi}{\sqrt{3}} \cdot \frac{n(e_2 + e_3) - (m+1)e_1}{n(\sigma_2 + \sigma_3) + (m+1)\sigma_1}\right)\right)^{-1} \\
&\wedge \left(1 + \exp\left(\frac{\pi}{\sqrt{3}} \cdot \frac{a + me'_1 - ne'_2}{m\sigma'_1 + n\sigma'_2}\right)\right)^{-1}.
\end{aligned} \tag{50}$$

We set $e_1 = e_2 = e_3 = 1$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $e'_1 = e'_2 = 3$, $\sigma'_1 = \sigma'_2 = 3$, $a = 1000$, and $m = n = 80$ and calculate the risk

index of the insurance company to be ruin = 0.0223. In addition, when the initial capital a varies from 0 to 1200, we calculate the ruin index and present the results in Figure 2. The results show that the ruin index falls from 0.3556 to 0.0106 when the initial capital a increases from 0 to 1200.

Note that if $e_3 = 0$ and $\sigma_3 = 0$ in Example 1, the ruin index degenerates into the ruin index of the modified insurance risk model proposed by Yao and Qin [34], that is, $\text{ruin} = \max_{m \geq 0, n \geq 1} (1 + \exp((\pi/\sqrt{3}) \cdot ((ne_2 - (m+1)e_1)/(n\sigma_2 + (m+1)\sigma_1))))^{-1} \wedge (1 + \exp((\pi/\sqrt{3}) \cdot ((a + me'_1 - ne'_2)/(m\sigma'_1 + n\sigma'_2))))^{-1}$. When other parameters are fixed, $(1 + \exp((\pi/\sqrt{3}) \cdot ((a + me'_1 - ne'_2)/(m\sigma'_1 + n\sigma'_2))))^{-1}$ decreases as the initial capital a increases. The ruin index is equal to $\max_{m \geq 0, n \geq 1} (1 + \exp((\pi/\sqrt{3}) \cdot ((a + me'_1 - ne'_2)/(m\sigma'_1 + n\sigma'_2))))^{-1}$ when the initial capital a is large enough. We can analyze that, with the increase in initial capital a , the ruin index calculated by the uncertain alternating renewal insurance risk model will tend to be the same as that computed by the model proposed by Yao and Qin.

Example 2. Assume that $\mu_1 = L(0, 1)$, $\mu_2 = L(1, 4)$, $\lambda = L(2, 3)$, $P_1 = L(0, 10)$, and $C_1 = L(0, 15)$ are linear uncertain variables. Thus, the expected claim rate is

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{E[N_{2t}]}{t} &= E\left[\frac{1}{\xi_{21} + \eta_{21}}\right] \\ &= \int_0^1 \frac{1}{\mu_2^{-1}(\alpha) + \lambda^{-1}(\alpha)} d\alpha \\ &= 0.212, \end{aligned} \quad (51)$$

and the report rate is

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{E[A_t]}{t} &= E\left[\frac{\xi_{21}}{\xi_{21} + \eta_{21}}\right] \\ &= \int_0^1 \frac{\mu_2^{-1}(\alpha)}{\mu_2^{-1}(\alpha) + \lambda^{-1}(1 - \alpha)} d\alpha \\ &= 0.486. \end{aligned} \quad (52)$$

Example 3. Assume that $\mu_1 = L(0, 1)$, $\mu_2 = L(1, 4)$, $\lambda = L(2, 3)$, $P_1 = L(0, 10)$, $C_1 = L(0, 15)$, the initial capital $a = 1000$, and premium amount $m = 80$. We calculate the company ruins with an uncertain measure 0.1 before the instant $t = 300$ at the $n = 80$ th claim. Figure 3 shows the uncertain measure that the insurance company ruins before time t .

Example 4. Assume that $\mu_1 = L(0, 1)$, $\mu_2 = L(1, 4)$, $\lambda = L(2, 3)$, $P_1 = L(0, 10)$, and $C_1 = L(0, 15)$ are linear uncertain variables. According to Theorem 5, if the initial capital a is 1000, then we can calculate that $M\{\tau \leq 300\} = 0.1524$; i.e., the insurance company ruins with an uncertain measure 0.1524 before the time $t = 300$.

FIGURE 2: The ruin index with different initial capital a .

FIGURE 3: Risk measure of insurance company before the time t .

FIGURE 4: Uncertainty distribution of ruin time in Example 5.

Example 5. Similar to Example 4, we assume that $\mu_1 = L(0, 1)$, $\mu_2 = L(1, 4)$, $\lambda = L(2, 3)$, $P_1 = L(0, 10)$, and $C_1 = L(0, 15)$ are linear uncertain variables. According to Theorem 5, if the initial capital a is 5000, the uncertainty distribution of the ruin time τ is shown in Figure 4.

4. Conclusions

This paper extends an insurance risk model with an alternating renewal process in uncertain environment. We propose an uncertain insurance risk process in which the claim process is regarded as an uncertain alternating renewal reward process, and this process is more in line with the claim process in real life. Moreover, we provide the inverse uncertainty distribution of the uncertain insurance risk process and the ruin index. The explicit form of uncertainty distribution of ruin time is also derived. At last, several examples are provided to illustrate the proven results. In future research, some criteria such as inflation, dividend, and tax can be considered to further extend the uncertain insurance risk process.

Data Availability

The data presented in Examples 1–5 and Figures 2–4 in this paper, which are used to support the findings of this study, are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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