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Research Article

An Optimal Control Problem Governed by Nonlinear First Order Dynamic Equation on Time Scales

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In this paper, we are concerned with a class of optimal control problem governed by nonlinear first order dynamic equation on time scales. By imposing some suitable conditions on the related functions, for any given control policy, we first obtain the existence of a unique solution for the nonlinear controlled system. Then, we study the existence of an optimal solution for the optimal control problem.

1. Introduction

The theory of time scales was introduced by Hilger in [1] in order to unify discrete and continuous analysis. Some foundational definitions and results from the calculus on time scales will be defined in Section 2. For more details, one can see [2–4].

In recent years, the calculus of variations and optimal control problems on time scales have attracted the attention of some researchers. For example, [5–8] discussed the calculus of variations on time scales and [9–12] studied some maximum principles on time scales, while [13–16] investigated the existence of optimal solutions or the necessary conditions of optimality for some optimal control problems on time scales.

In 2017, Guo [17] studied the projective synchronization problem of a class of chaotic systems in arbitrary dimensions. Firstly, a necessary and sufficient condition for the existence of the projective synchronization problem was presented. Secondly, an algorithm was proposed to obtain all the solutions of the projective synchronization problem. Thirdly, a simple and physically implementable controller was designed to ensure the realization of the projective synchronization. Finally, some numerical examples were provided to verify the effectiveness and the validity of the proposed results.

In 2020, Xu and Zhang [18] investigated general mean-field linear-quadratic (LQ) games of stochastic large-population system, where the individual diffusion coefficient could depend on both the state and the control of the agent, and the control weight in the cost functional could be indefinite. The asymptotic suboptimality property of the decentralized strategies for the LQ games was derived through the consistency condition. A pricing problem was also studied, for which the decentralized suboptimal price was obtained.

Throughout this paper, we always assume that $\mathbb{T}$ is a time scale, $T > 0$ is fixed, $0, T \in \mathbb{T}$ and $\sigma^2(T) = \sigma(T)$. For each interval $\mathbf{I}$ of $\mathbb{R}$, we denote by $\mathbf{I}_{\mathbb{T}} = \mathbf{I} \cap \mathbb{T}$.

Suppose that there is a flock of sheep in a pasture. We consider the changes in the number of sheep during a time interval $[0, \sigma(T)]_{\mathbb{T}}$. It is well known that the supply of herbage, which influences growth rate and reproductive ability of sheep, is one of the main ways to control the number of sheep. Now, we define some related functions as follows:

$x(t)$ is the number of sheep at time t

$r(t)$ is the number of births per unit of time at time t

$p(t)$ is the number of sales per sheep per unit of time at time t

$u(t)$ is the amount of herbage supplied at time t

$q(t)$ is the number of sheep converted by per unit of herbage supplied per unit of time at time t

Let U_{ad} be the admissible control set. Then, for any given control policy $u \in U_{\text{ad}}$, it is easy to know that the changes in the number of sheep can be described by the following linear dynamic equation:

$$x^{\Delta}(t) + p(t)x(\sigma(t)) = r(t) + q(t)u(t), \quad t \in [0,T]_{\mathbb{T}}. \tag{1}$$

At the same time, in order to keep steady development, we may assume that the number of sheep at the beginning is equal to that at the end, that is,

$$x(0) = x(\sigma(T)). \tag{2}$$

Suppose that x_u is the solution of the controlled systems (1) and (2) corresponding to the control policy u and x_d is the desired value. Recently, the authors [19] considered the optimal control problem (P_0). Find a $u_0 \in U_{\text{ad}}$ such that

$$J(u_0) \le J(u), \quad \text{for all } u \in U_{\text{ad}}, \tag{3}$$

where

$$J(u) = \int_0^T [x_u(\sigma(t)) - x_d(t)]^2 \Delta t + \int_0^T u^2(t)\Delta t, \quad u \in U_{\text{ad}}, \tag{4}$$

is the quadratic cost functional.

Motivated greatly by the abovementioned works, in this paper, we suppose that the controlled system is governed by the following more general nonlinear periodic boundary value problem:

$$\begin{cases} x^{\Delta}(t) + p(t)x(\sigma(t)) = f(t, x(t), x(\sigma(t))) + g(u(t)), & t \in [0,T]_{\mathbb{T}}, \\ x(0) = x(\sigma(T)). \end{cases} \tag{5}$$

First, by imposing some suitable conditions on p, f, and g, for any given control policy $u \in U_{\text{ad}}$, we obtain the existence of a unique solution x_u for the nonlinear controlled system (5). Then, we study the optimal control problem (P). Find a $u_0 \in U_{\text{ad}}$ such that

$$J(u_0) \le J(u), \quad \text{for all } u \in U_{\text{ad}}, \tag{6}$$

where

$$J(u) = \int_0^T [x_u(\sigma(t)) - x_d(t)]^2 \Delta t + \int_0^T h(u(t))\Delta t, \quad u \in U_{\text{ad}}, \tag{7}$$

where x_d is the desired value and $h: \mathbb{R} \longrightarrow [0,\infty)$ is continuous.

2. Preliminaries

In this section, we will provide some foundational definitions and results from the calculus on time scales.

Definition 1. We define the forward jump operator $\sigma: \mathbb{T} \longrightarrow \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T}: s > t\}, \quad \text{for all } t \in \mathbb{T}, \tag{8}$$

while the backward jump operator $\rho: \mathbb{T} \longrightarrow \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T}: s < t\}, \quad \text{for all } t \in \mathbb{T}. \tag{9}$$

In this definition, we put $\inf\varnothing = \sup\mathbb{T}$ and $\sup\varnothing = \inf\mathbb{T}$, where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf\mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum m, then we define $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. Finally, the graininess function $\mu: \mathbb{T} \longrightarrow [0,\infty)$ is defined by

$$\mu(t) := \sigma(t) - t, \quad \text{for all } t \in \mathbb{T}. \tag{10}$$

Definition 2. Assume $f: \mathbb{T} \longrightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then, we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta)_{\mathbb{T}}$ for some $\delta > 0$) such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \le \varepsilon |\sigma(t) - s|, \quad \text{for all } s \in U. \tag{11}$$

We call $f^{\Delta}(t)$ the delta derivative of f at t.

Moreover, we say that f is delta differentiable (or in short, differentiable) on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\Delta}: \mathbb{T}^k \longrightarrow \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^k$. A function $F: \mathbb{T} \longrightarrow \mathbb{R}$ is called an antiderivative of $f: \mathbb{T} \longrightarrow \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in \mathbb{T}^k. \tag{12}$$

If $F: \mathbb{T} \longrightarrow \mathbb{R}$ is an antiderivative of $f: \mathbb{T} \longrightarrow \mathbb{R}$, then we define the Cauchy integral by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \text{for all } a, b \in \mathbb{T}. \tag{13}$$

Definition 3. A function $f: \mathbb{T} \longrightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 4. We say that a function $p: \mathbb{T} \longrightarrow \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \ne 0, \quad \text{for all } t \in \mathbb{T}^k, \tag{14}$$

holds. The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R}$. We define the set of positively regressive functions $\mathscr{R}^+$ as the set consisting of those $p \in \mathscr{R}$ satisfying

$$1 + \mu(t)p(t) > 0, \quad \text{for all } t \in \mathbb{T}. \tag{15}$$

Lemma 1. *Let $p \in \mathscr{R}$, $t_0 \in \mathbb{T}$, and $e_p(\cdot, t_0)$ be the exponential function on $\mathbb{T}$. Then,*

(i) $e_p(t,t) \equiv 1$ for all $t \in \mathbb{T}$

(ii) $e_p^{\Delta}(t,t_0) = p(t)e_p(t,t_0)$ for all $t \in \mathbb{T}^k$

Moreover, if $p \in \mathscr{R}^+$, then

$$e_p(t,t_0) > 0, \quad \text{for all } t \in \mathbb{T}. \tag{16}$$

Lemma 2. *Let f be a continuous function on $[a,b]_{\mathbb{T}}$ that is differentiable on $[a,b)_{\mathbb{T}}$. Then, f is increasing, decreasing, nondecreasing, and nonincreasing on $[a,b]_{\mathbb{T}}$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a,b)_{\mathbb{T}}$, respectively.*

Lemma 3. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Δ integrable functions on $[a,b]_{\mathbb{T}}$ and suppose that $f_n \longrightarrow f$ uniformly on $[a,b)_{\mathbb{T}}$ for a function f defined on $[a,b]_{\mathbb{T}}$. Then, f is Δ integrable from a to b and*

$$\int_a^b f(t)\Delta t = \lim_{n\longrightarrow\infty} \int_a^b f_n(t)\Delta t. \tag{17}$$

In the remainder of this paper, we always assume that Banach space

$$C([a,b]_{\mathbb{T}}, \mathbb{R}) = \{y \mid y\colon [a,b]_{\mathbb{T}} \longrightarrow \mathbb{R} \text{ is continuous}\}, \tag{18}$$

is equipped with the norm $\|y\| = \max_{t\in[a,b]_{\mathbb{T}}} |y(t)|$, $p\colon [0,T]_{\mathbb{T}} \longrightarrow (0,\infty)$ is rd-continuous, and denote

$$M = \frac{1}{e_p(\sigma(T),0) - 1}. \tag{19}$$

Then, it is easy to see that $M > 0$.

Lemma 4 (see [20]). *For any $y \in C([0,T]_{\mathbb{T}}, \mathbb{R})$, the following first order linear periodic boundary value problem*

$$\begin{cases} x^{\Delta}(t) + p(t)x(\sigma(t)) = y(t), & t \in [0,T]_{\mathbb{T}}, \\ x(0) = x(\sigma(T)), \end{cases} \tag{20}$$

has a unique solution

$$x(t) = \frac{1}{e_p(t,0)} \left[\int_0^t e_p(s,0)y(s)\Delta s + M \int_0^{\sigma(T)} e_p(s,0)y(s)\Delta s \right], \quad t \in [0,\sigma(T)]_{\mathbb{T}}. \tag{21}$$

3. Main Results

First, we list the following two conditions which we shall use in the sequel.

(A_1) $f\colon [0,T]_{\mathbb{T}} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and there exists $0 < L < (M/2(1+M)^2\sigma(T))$ such that

$$\left|f(t,\omega_1,\nu_1) - f(t,\omega_2,\nu_2)\right| \leq L\left(\left|\omega_1 - \omega_2\right| + \left|\nu_1 - \nu_2\right|\right), \quad t \in [0,T]_{\mathbb{T}}, \omega_1, \omega_2, \nu_1, \nu_2 \in \mathbb{R}. \tag{22}$$

(A_2) $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ and there exists $K > 0$ such that

$$|g(\omega) - g(\nu)| \leq K|\omega - \nu|, \quad \omega, \nu \in \mathbb{R}. \tag{23}$$

From now on, we always suppose that the control space is $C([0,T]_{\mathbb{T}}, \mathbb{R})$ and the admissible control set U_{ad} is a compact subset of $C([0,T]_{\mathbb{T}}, \mathbb{R})$.

Lemma 5. *Assume that conditions (A_1) and (A_2) are satisfied. Then, for any given control policy $u \in U_{ad}$, the nonlinear controlled system (5) has a unique solution x_u and*

$$\begin{aligned} x_u(t) = \frac{1}{e_p(t,0)} \Bigg\{ &\int_0^t e_p(s,0)[f(s,x_u(s),x_u(\sigma(s))) + g(u(s))]\Delta s \\ &+ M \int_0^{\sigma(T)} e_p(s,0)[f(s,x_u(s),x_u(\sigma(s))) + g(u(s))]\Delta s \Bigg\}, \quad t \in [0,\sigma(T)]_{\mathbb{T}}. \end{aligned} \tag{24}$$

Proof. For any fixed $u \in U_{\mathrm{ad}}$, we define an operator $\Phi_u\colon C([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R}) \longrightarrow C([0,\sigma(T)]_{\mathbb{T}}, \mathbb{R})$ as follows:

$$\begin{aligned} (\Phi_u x)(t) = \frac{1}{e_p(t,0)} \Bigg\{ &\int_0^t e_p(s,0)[f(s,x(s),x(\sigma(s))) + g(u(s))]\Delta s \\ &+ M \int_0^{\sigma(T)} e_p(s,0)[f(s,x(s),x(\sigma(s))) + g(u(s))]\Delta s \Bigg\}, \quad t \in [0,\sigma(T)]_{\mathbb{T}}. \end{aligned} \tag{25}$$

Obviously, x is a solution of the nonlinear controlled system (5) if and only if x is a fixed point of Φ_u in $C([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$.

Let $x, y \in C([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. Then, in view of Lemmas 1 and 2 and (A_1), we have

$$\begin{aligned}
\left|(\Phi_u x)(t) - (\Phi_u y)(t)\right| &= \frac{1}{e_p(t,0)}\bigg|\int_0^t e_p(s,0)[f(s,x(s),x(\sigma(s))) - f(s,y(s),y(\sigma(s)))]\Delta s \\
&\quad + M\int_0^{\sigma(T)} e_p(s,0)[f(s,x(s),x(\sigma(s))) - f(s,y(s),y(\sigma(s)))]\Delta s\bigg| \\
&\le \frac{1}{e_p(t,0)}\bigg\{\int_0^t e_p(s,0)|f(s,x(s),x(\sigma(s))) - f(s,y(s),y(\sigma(s)))|\Delta s \\
&\quad + M\int_0^{\sigma(T)} e_p(s,0)|f(s,x(s),x(\sigma(s))) - f(s,y(s),y(\sigma(s)))|\Delta s\bigg\} \\
&\le (1+M)\int_0^{\sigma(T)} e_p(s,0)|f(s,x(s),x(\sigma(s))) - f(s,y(s),y(\sigma(s)))|\Delta s \\
&\le L(1+M)\int_0^{\sigma(T)} e_p(s,0)[|x(s)-y(s)| + |x(\sigma(s)) - y(\sigma(s))|]\Delta s \\
&\le 2L(1+M)\|x-y\|\int_0^{\sigma(T)} e_p(s,0)\Delta s \\
&\le \frac{2L(1+M)^2\sigma(T)}{M}\|x-y\|, \quad t \in [0,\sigma(T)]_{\mathbb{T}},
\end{aligned} \tag{26}$$

so

$$\|\Phi_u x - \Phi_u y\| \le \frac{2L(1+M)^2\sigma(T)}{M}\|x-y\|, \tag{27}$$

which together with $0 < L < (M/2(1+M)^2\sigma(T))$ implies that Φ_u: $C([0,\sigma(T)]_{\mathbb{T}},\mathbb{R}) \longrightarrow C([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$ is a contraction mapping.

Therefore, it follows from Banach contraction principle that Φ_u has a unique fixed point $x_u \in C([0,\sigma(T)]_{\mathbb{T}},\mathbb{R})$. This indicates that the nonlinear controlled system (5) has a unique solution x_u and

$$\begin{aligned}
x_u(t) &= \frac{1}{e_p(t,0)}\bigg\{\int_0^t e_p(s,0)[f(s,x_u(s),x_u(\sigma(s))) + g(u(s))]\Delta s \\
&\quad + M\int_0^{\sigma(T)} e_p(s,0)[f(s,x_u(s),x_u(\sigma(s))) + g(u(s))]\Delta s\bigg\}, \quad t \in [0,\sigma(T)]_{\mathbb{T}}.
\end{aligned} \tag{28}$$

□

Theorem 1. *Assume that conditions (A_1) and (A_2) are satisfied and h: $\mathbb{R} \longrightarrow [0,\infty)$ is continuous. Then, the optimal control problem (P) has an optimal solution $u_0 \in U_{ad}$.*

Proof. First, it follows from Lemma 5 that, for any given control policy $u \in U_{\mathrm{ad}}$, the nonlinear controlled system (5) has a unique solution x_u and

$$\begin{aligned}
x_u(t) &= \frac{1}{e_p(t,0)}\bigg\{\int_0^t e_p(s,0)[f(s,x_u(s),x_u(\sigma(s))) + g(u(s))]\Delta s \\
&\quad + M\int_0^{\sigma(T)} e_p(s,0)[f(s,x_u(s),x_u(\sigma(s))) + g(u(s))]\Delta s\bigg\}, \quad t \in [0,\sigma(T)]_{\mathbb{T}}.
\end{aligned} \tag{29}$$

Next, in view of $J(u)=\int_0^T[x_u(\sigma(t))-x_d(t)]^2\Delta t+\int_0^T h(u(t))\Delta t$, $u\in U_{\text{ad}}$, it is obvious that $\inf_{u\in U_{\text{ad}}}J(u)$ exists. Thus, by the definition of infimum, we know that there exists a sequence $\{u_n\}_{n=1}^{\infty}\subset U_{\text{ad}}$ such that

$$\lim_{n\longrightarrow\infty}J(u_n)=\inf_{u\in U_{\text{ad}}}J(u). \tag{30}$$

On the one hand, since U_{ad} is a compact subset of $C([0,T]_{\mathbb{T}},\mathbb{R})$ and $\{u_n\}_{n=1}^{\infty}\subset U_{\text{ad}}$, $\{u_n\}_{n=1}^{\infty}$ has a convergent subsequence in U_{ad}. Without loss of generality, we may assume that $\{u_n\}_{n=1}^{\infty}$ converges in U_{ad}, that is, there exists $u_0\in U_{\text{ad}}$ such that

$$\lim_{n\longrightarrow\infty}u_n=u_0. \tag{31}$$

On the other hand, in view of Lemmas 1 and 2, (A_1) and (A_2), for any $n=1,2,\ldots$, we have

$$\begin{aligned}
\left|x_{u_n}(t)-x_{u_0}(t)\right| &= \frac{1}{e_p(t,0)}\left|\int_0^t e_p(s,0)\left[f\left(s,x_{u_n}(s),x_{u_n}(\sigma(s))\right)-f\left(s,x_{u_0}(s),x_{u_0}(\sigma(s))\right)+g(u_n(s))-g(u_0(s))\right]\Delta s\right.\\
&\quad\left.+M\int_0^{\sigma(T)} e_p(s,0)\left[f\left(s,x_{u_n}(s),x_{u_n}(\sigma(s))\right)-f\left(s,x_{u_0}(s),x_{u_0}(\sigma(s))\right)+g(u_n(s))-g(u_0(s))\right]\Delta s\right|\\
&\le \int_0^t e_p(s,0)\left|f\left(s,x_{u_n}(s),x_{u_n}(\sigma(s))\right)-f\left(s,x_{u_0}(s),x_{u_0}(\sigma(s))\right)+g(u_n(s))-g(u_0(s))\right|\Delta s\\
&\quad+M\int_0^{\sigma(T)} e_p(s,0)\left|f\left(s,x_{u_n}(s),x_{u_n}(\sigma(s))\right)-f\left(s,x_{u_0}(s),x_{u_0}(\sigma(s))\right)+g(u_n(s))-g(u_0(s))\right|\Delta s\\
&\le (1+M)\int_0^{\sigma(T)} e_p(s,0)\left|f\left(s,x_{u_n}(s),x_{u_n}(\sigma(s))\right)-f\left(s,x_{u_0}(s),x_{u_0}(\sigma(s))\right)+g(u_n(s))-g(u_0(s))\right|\Delta s\\
&\le (1+M)\int_0^{\sigma(T)} e_p(s,0)\left[\left|f\left(s,x_{u_n}(s),x_{u_n}(\sigma(s))\right)-f\left(s,x_{u_0}(s),x_{u_0}(\sigma(s))\right)\right|+\left|g(u_n(s))-g(u_0(s))\right|\right]\Delta s\\
&\le (1+M)\int_0^{\sigma(T)} e_p(s,0)\left\{L\left[\left|x_{u_n}(s)-x_{u_0}(s)\right|+\left|x_{u_n}(\sigma(s))-x_{u_0}(\sigma(s))\right|\right]+K\left|u_n(s)-u_0(s)\right|\right\}\Delta s\\
&\le (1+M)\left(2L\left\|x_{u_n}-x_{u_0}\right\|+K\left\|u_n-u_0\right\|\right)\int_0^{\sigma(T)} e_p(s,0)\Delta s\\
&\le \frac{2L(1+M)^2\sigma(T)}{M}\left\|x_{u_n}-x_{u_0}\right\|+\frac{K(1+M)^2\sigma(T)}{M}\left\|u_n-u_0\right\|,\quad t\in[0,\sigma(T)]_{\mathbb{T}},
\end{aligned} \tag{32}$$

so, for any $n=1,2,\ldots$, we obtain

$$\left\|x_{u_n}-x_{u_0}\right\|\le\frac{K(1+M)^2\sigma(T)}{M-2L(1+M)^2\sigma(T)}\left\|u_n-u_0\right\|, \tag{33}$$

which together with (31) implies that

$$\lim_{n\longrightarrow\infty}x_{u_n}=x_{u_0}. \tag{34}$$

Thus, in view of Lemma 3, (31), and (34), we obtain

$$\begin{aligned}
\lim_{n\longrightarrow\infty}J(u_n) &= \lim_{n\longrightarrow\infty}\left(\int_0^T\left[x_{u_n}(\sigma(t))-x_d(t)\right]^2\Delta t+\int_0^T h(u_n(t))\Delta t\right)\\
&= \int_0^T\lim_{n\longrightarrow\infty}\left[x_{u_n}(\sigma(t))-x_d(t)\right]^2\Delta t+\int_0^T\lim_{n\longrightarrow\infty}h(u_n(t))\Delta t\\
&= \int_0^T\left[x_{u_0}(\sigma(t))-x_d(t)\right]^2\Delta t+\int_0^T h(u_0(t))\Delta t\\
&= J(u_0),
\end{aligned} \tag{35}$$

which together with (30) indicates that

$$J(u_0) = \inf_{u\in U_{ad}} J(u). \tag{36}$$

Therefore, $J(u_0) \le J(u)$ for all $u \in U_{ad}$. This shows that u_0 is an optimal solution of the optimal control problem (P). □

Example 1. Let $\mathbb{T} = [0,1] \cup [2,3]$. We suppose that the controlled system is governed by the following nonlinear periodic boundary value problem

$$\begin{cases} x^{\Delta}(t) + x(\sigma(t)) = Dt^2\left[x(t)\arctan x(t) - \frac{1}{2}\ln\left(1 + x^2(t)\right) + \frac{\pi}{4}\sin^2 x(\sigma(t))\right] + u(t), & t \in [0,3]_{\mathbb{T}}, \\ x(0) = x(3), \end{cases} \tag{37}$$

where

$$D = \frac{2(2e^2 - 1)}{9\pi[28e^4 - 4e^2 + 1]}. \tag{38}$$

In view of $\mathbb{T} = [0,1] \cup [2,3]$, $T = 3$ and $p(t) \equiv 1$ for $t \in [0,3]_{\mathbb{T}}$, it is not difficult to obtain that

$$M = \frac{1}{2e^2 - 1}. \tag{39}$$

Since $f(t,\omega,v) = Dt^2[\omega\arctan\omega - (1/2)\ln(1+\omega^2) + (\pi/4)\sin^2 v]$ for $(t,\omega,v) \in [0,3]_{\mathbb{T}} \times \mathbb{R}^2$ and $g(\omega) = \omega$ for $\omega \in \mathbb{R}$, it is obvious that $f\colon [0,3]_{\mathbb{T}} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and (A_2) is satisfied. Moreover, if we choose $L = (9\pi D/2)$, then $0 < L < (M/2(1+M)^2\sigma(T))$ and it follows from Lagrange mean value theorem that

$$\begin{aligned} &\left|f(t,\omega_1,v_1) - f(t,\omega_2,v_2)\right| \\ &= Dt^2\left|\omega_1\arctan\omega_1 - \frac{1}{2}\ln\left(1+\omega_1^2\right) + \frac{\pi}{4}\sin^2 v_1 - \omega_2\arctan\omega_2 + \frac{1}{2}\ln\left(1+\omega_2^2\right) - \frac{\pi}{4}\sin^2 v_2\right| \\ &\le 9D\left[\left|\omega_1\arctan\omega_1 - \frac{1}{2}\ln\left(1+\omega_1^2\right) - \omega_2\arctan\omega_2 + \frac{1}{2}\ln\left(1+\omega_2^2\right)\right| + \frac{\pi}{4}\left|\sin^2 v_1 - \sin^2 v_2\right|\right] \\ &\le \frac{9\pi D}{2}\left(\left|\omega_1 - \omega_2\right| + \left|v_1 - v_2\right|\right) \\ &= L\left(\left|\omega_1 - \omega_2\right| + \left|v_1 - v_2\right|\right), \quad t \in [0,3]_{\mathbb{T}}, \omega_1, \omega_2, v_1, v_2 \in \mathbb{R}. \end{aligned} \tag{40}$$

This shows that (A_1) is fulfilled.

For any given constant $N > 0$, let $U_{ad} = \{u \in C([0,3]_{\mathbb{T}}, \mathbb{R}) \mid u(0) = 0 \text{ and } |u(t_1) - u(t_2)| \le N|t_1 - t_2| \text{ for all } t_1, t_2 \in [0,3]_{\mathbb{T}}\}$. Then, it is easy to verify that U_{ad} is a compact subset of $C([0,3]_{\mathbb{T}}, \mathbb{R})$.

By Lemma 5, we know that, for any given control policy $u \in U_{ad}$, the nonlinear controlled system (37) has a unique solution x_u.

Now, we consider the optimal control problem (P^*). Find a $u_0 \in U_{ad}$ such that

$$J(u_0) \le J(u), \quad \text{for all } u \in U_{ad}, \tag{41}$$

where

$$J(u) = \int_0^3 [x_u(\sigma(t)) - x_d(t)]^2\Delta t + \int_0^3 u^2(t)\Delta t, \quad u \in U_{ad}, \tag{42}$$

where x_d is the desired value.

Since $h(\omega) = \omega^2$ for $\omega \in \mathbb{R}$, $h\colon \mathbb{R} \longrightarrow [0,\infty)$ is continuous, thus, all the conditions of Theorem 1 are satisfied. Therefore, it follows from Theorem 1 that the optimal control problem (P^*) has an optimal solution $u_0 \in U_{ad}$.

4. Conclusions

In this paper, we consider a class of optimal control problem governed by nonlinear first order dynamic equation on time scales. First, by imposing some suitable conditions on the related functions and applying Banach contraction principle, for any given control policy, we obtain the existence of a unique solution for the nonlinear controlled system. Next, we prove that the optimal control problem has an optimal solution in the admissible control set. Finally, an example is also given to illustrate the main result of this paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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