Research Article

Structured Rectangular Tensors and Rectangular Tensor Complementarity Problems

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In this paper, some properties of structured rectangular tensors are presented, and the relationship among these structured rectangular tensors is also given. It is shown that all the V-singular values of rectangular P-tensors are positive. Some necessary and/or sufficient conditions for a rectangular tensor to be a rectangular P-tensor are also obtained. A new subclass of rectangular tensors, which is called rectangular S-tensors, is introduced and it is proved that rectangular S-tensors can be defined by the feasible vectors of the corresponding rectangular tensor complementarity problem.

1. Introduction

Consider the following m degree homogeneous polynomial:

$$f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m, \quad (1)$$

where $\mathcal{A}\mathbf{x}^m = \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_1} \dots x_{i_m}$ and $\mathcal{A} \in \mathbb{R}^{[m, n]}$ is an m th order n -dimensional real square tensor. When m is even, the positive definiteness of $f(\mathbf{x})$ in (1) plays an important role in automatic control [1]. In order to verify the positive definiteness of $f(\mathbf{x})$ in (1), Qi introduced the definitions of H-eigenvalue and Z-eigenvalue of $\mathcal{A}$ and showed that when m is even, $\mathcal{A}$ is positive definite (i.e., $f(\mathbf{x})$ in (1) is positive definite) if and only if all H-eigenvalues or Z-eigenvalues of $\mathcal{A}$ are positive [2–4].

One important structured tensor is called copositive tensor, which can be viewed as a generalization of copositive matrices and plays an important role in tensor complementarity problem [5] and polynomial optimization problems [6]. In [7], Qi introduced the definition of copositive tensors and obtained some necessary and sufficient conditions for a real symmetric tensor to be a copositive tensor. In [6], a general characterization of the class of polynomial

optimization problems that can be formulated as a conic program over the cone of completely positive tensors is presented. Che et al. [5] showed that the tensor complementarity problem with a strictly copositive tensor has a nonempty and compact solution set. Song and Qi [8] proved that a real symmetric tensor is semipositive if and only if it is copositive. A numerical algorithm for copositivity of square tensors is proposed in [9].

Another important structured tensor is called P-tensor. The P-tensors and P_0 -tensors are first introduced by Song and Qi [10], which can be viewed as generalizations of the P-matrices and P_0 -matrices [11]. The authors in [10] also showed that a symmetric tensor with even order is positive definite if and only if it is a P-tensor and a symmetric tensor with even order is positive semidefinite if and only if it is a P_0 -tensor. Another definition of P-tensors (P_0 -tensors) is presented, which includes many important structured tensors with odd order [12], and the authors also showed that the complementarity problem with a P-tensor has a nonempty compact solution set.

Consider the following $p+q$ degree homogeneous polynomial:

$$f(\mathbf{x}, \mathbf{y}) = \mathcal{A} \mathbf{x}^p \mathbf{y}^q, \quad (2)$$

where

$$\mathcal{A} \mathbf{x}^p \mathbf{y}^q = \sum_{i_2, \dots, i_p=1}^m \sum_{j_1, \dots, j_q=1}^n a_{i_1 \dots i_p j_1 \dots j_q} x_{i_1} \dots x_{i_p} y_{j_1} \dots y_{j_q}. \quad (3)$$

$\mathcal{A} = (a_{i_1 \dots i_p j_1 \dots j_q}) \in \mathbb{R}^{[p; q; m; n]}$ is a (p, q) th order $(m \times n)$ -dimensional real rectangular tensor. $\mathcal{A}$ is called a real partially symmetric rectangular tensor, if $a_{i_1 \dots i_p j_1 \dots j_q}$ is invariant under any permutation of indices among $i_1 \dots i_p$, and any permutation of indices among $j_1 \dots j_q$, i.e.,

$$a_{\pi(i_1 \dots i_p) \sigma(j_1 \dots j_q)} = a_{i_1 \dots i_p j_1 \dots j_q}, \quad \pi \in S_p, \sigma \in S_q, \quad (4)$$

where S_r is the permutation group of r indices. Let $\mathcal{A} \in \mathbb{R}^{[p; q; m; n]}$ be a partially symmetric rectangular tensor, and p and q are even. Then, $\mathcal{A}$ is positive definite if and only if all of its H-singular values (or V-singular values) are positive [13–19].

The definition of copositive rectangular tensors is introduced in [20], which can be viewed as a generalization of copositive square tensors, and some necessary and sufficient conditions for a real partially symmetric rectangular tensor to be a copositive rectangular tensor are also given in [20]. Based on the criteria for identifying copositive rectangular tensors, a numerical method for identifying the copositeness of a partially symmetric rectangular tensor is obtained [21].

The rest of this first part is organized as follows. In Section 2, some preliminaries are given. In Section 3, we intend to introduce two new classes of rectangular tensors which are called rectangular P-tensors and rectangular P_0 -tensors. Moreover, we prove that all the V-singular values of rectangular P-tensors (rectangular P_0 -tensors) are positive (nonnegative). We also discuss some properties of quantities for rectangular P-tensors, and a necessary and sufficient condition for a rectangular tensor to be a rectangular P-tensor is also obtained. In Section 4, we extend the S-tensors to rectangular S-tensors, and some properties of rectangular S-tensors are also given. In Section 5, we introduce the rectangular tensor complementarity problem (RTCP), which can be used to define the rectangular S-tensors, and the relationship among positive definite rectangular tensors, strictly copositive rectangular tensors, rectangular P-tensors, and rectangular S-tensors is also presented.

2. Notation and Preliminaries

In this section, we list some definitions related to rectangular tensors, which are needed in the subsequent analysis.

Let $\mathbb{R}$ and $\mathbb{C}$ be the real field and complex field, $[m] = \{1, 2, \dots, m\}$. We use small letters $a, b, \dots$ for scalars, small bold letters $\mathbf{x}, \mathbf{y}, \dots$ for vectors, capital letters $A, B, \dots$ for matrices, and calligraphic letters $\mathcal{A}, \mathcal{B}, \dots$ for tensors. The i th entry of a vector $\mathbf{x}$ is denoted by x_i , the (i, j) th entry of a matrix A is denoted by a_{ij} , and the

$(i_1, \dots, i_p, j_1, \dots, j_q)$ th entry of a rectangular tensor $\mathcal{A}$ is denoted by $a_{i_1 \dots i_p j_1 \dots j_q}$. Let $\mathbb{R}^n$ be the n -dimensional real Euclidean space and the set of all nonnegative vectors in $\mathbb{R}^n$ be denoted by $\mathbb{R}_+^n$.

Definition 1. A rectangular $\mathcal{A} \in \mathbb{R}^{[p; q; m; n]}$ is said to be

- (a) A positive definite rectangular tensor [13, 14], iff $\mathcal{A} \mathbf{x}^p \mathbf{y}^q > 0$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$
- (b) A copositive rectangular tensor [21], iff $\mathcal{A} \mathbf{x}^p \mathbf{y}^q \geq 0$ for all $\mathbf{x} \in \mathbb{R}_+^m$ and $\mathbf{y} \in \mathbb{R}_+^n$
- (c) A strictly copositive rectangular tensor [21], iff $\mathcal{A} \mathbf{x}^p \mathbf{y}^q > 0$ for all $\mathbf{x} \in \mathbb{R}_+^m \setminus \{\mathbf{0}\}$ and $\mathbf{y} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$

In order to verify the positive definiteness of a (p, q) th order $(m \times n)$ -dimensional partially symmetric rectangular tensor, the definition of a singular value of rectangular tensors is introduced by Chang et al. [13].

Definition 2 (see [13]). Let $\mathcal{A} \in \mathbb{R}^{[p; q; m; n]}$, if there exist a number $\lambda \in \mathbb{R}$, vectors $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that

$$\begin{aligned} \mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q &= \lambda \mathbf{x}^{[M-1]}, \\ \mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1} &= \lambda \mathbf{y}^{[M-1]}, \end{aligned} \quad (5)$$

where $\mathbf{x}^{[M-1]} = [x_1^{[M-1]}, \dots, x_m^{[M-1]}]^T$, $M = p + q$, and

$$\begin{aligned} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_i &= \sum_{i_2, \dots, i_p=1}^m \sum_{j_1, \dots, j_q=1}^n a_{i i_2 \dots i_p j_1 \dots j_q} x_{i_2} \dots x_{i_p} y_{j_1} \dots y_{j_q}, \\ (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_j &= \sum_{i_1, \dots, i_p=1}^m \sum_{j_2, \dots, j_q=1}^n a_{i_1 \dots i_p j j_2 \dots j_q} x_{i_1} \dots x_{i_p} y_{j_2} \dots y_{j_q}, \end{aligned} \quad (6)$$

then λ is called the H-singular value of $\mathcal{A}$ and $(\mathbf{x}, \mathbf{y})$ is the left and right H-eigenvectors pair of $\mathcal{A}$, associated with λ .

Some sufficient conditions for the positive definiteness of a (p, q) th order $(m \times n)$ -dimensional partially symmetric rectangular tensor are given in [14], based on the following definition of the V-singular value.

Definition 3 (see [14]). Let $\mathcal{A} \in \mathbb{R}^{[p; q; m; n]}$; if there exist a number $\lambda \in \mathbb{R}$, vectors $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$, $p, q \geq 2$, and $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that

$$\begin{aligned} \mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q &= \lambda \mathbf{x}^{[p-1]}, \\ \mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1} &= \lambda \mathbf{y}^{[q-1]}, \end{aligned} \quad (7)$$

then λ is called the V-singular value of $\mathcal{A}$ and $(\mathbf{x}, \mathbf{y})$ is the left and right V-eigenvectors pair of $\mathcal{A}$, associated with λ .

The definitions of P-tensors and P_0 -tensors are listed as follows.

Definition 4 (see [10]). A tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m; n]}$ is called a P-tensor if for each nonzero $\mathbf{x} \in \mathbb{R}^n$, there exists some index i such that

$$x_i^{m-1} (\mathcal{A} \mathbf{x}^{m-1})_i > 0, \quad (8)$$

where $\mathcal{A}\mathbf{x}^{m-1} = \sum_{i_2, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_2} \dots x_{i_m}$. A tensor $\mathcal{A} = (a_{i_1, \dots, i_m}) \in \mathbb{R}^{[m, n]}$ is called a P_0 -tensor if for each nonzero $\mathbf{x} \in \mathbb{R}^n$, there exists some index i such that

$$x_i^{m-1} (\mathcal{A}\mathbf{x}^{m-1})_i \geq 0. \quad (9)$$

3. Rectangular P-Tensors and Rectangular P_0 -Tensors

We now introduce the definitions of rectangular P-tensors and rectangular P_0 -tensors.

Definition 5. A rectangular tensor $\mathcal{A} = (a_{i_1 i_2, \dots, i_p j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$ is called a rectangular P-tensor if for each $\mathbf{x} \in \mathbb{R}^m / \{\mathbf{0}\}$ and $\mathbf{y} \in \mathbb{R}^n / \{\mathbf{0}\}$, there exists some indices $i \in [m]$, $j \in [n]$ such that

$$\begin{aligned} x_i^{p-1} (\mathcal{A}\mathbf{x}^{p-1} \mathbf{y}^q)_i &> 0, \\ y_j^{q-1} (\mathcal{A}\mathbf{x}^p \mathbf{y}^{q-1})_j &> 0. \end{aligned} \quad (10)$$

A rectangular tensor $\mathcal{A} = (a_{i_1 i_2, \dots, i_p j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$ is called a rectangular P_0 -tensor if for each $\mathbf{x} \in \mathbb{R}^m / \{\mathbf{0}\}$ and $\mathbf{y} \in \mathbb{R}^n / \{\mathbf{0}\}$, there exists some indices $i \in [m]$, $j \in [n]$ such that

$$\begin{aligned} x_i^{p-1} (\mathcal{A}\mathbf{x}^{p-1} \mathbf{y}^q)_i &\geq 0, \\ y_j^{q-1} (\mathcal{A}\mathbf{x}^p \mathbf{y}^{q-1})_j &\geq 0. \end{aligned} \quad (11)$$

The following result is given to show the positivity (nonnegativity) of the V-singular values for a rectangular P-tensor (P_0 -tensor).

Theorem 1. Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_p j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$ be a rectangular P-tensor (P_0 -tensor); then, all the V-singular values of $\mathcal{A}$ are positive (nonnegative).

Proof. If $\mathcal{A}$ is a rectangular P-tensor, λ is a V-singular value of $\mathcal{A}$ with eigenvectors pair $(\mathbf{x}, \mathbf{y})$; then, we have

$$\begin{aligned} \mathcal{A}\mathbf{x}^{p-1} \mathbf{y}^q &= \lambda \mathbf{x}^{[p-1]}, \\ \mathcal{A}\mathbf{x}^p \mathbf{y}^{q-1} &= \lambda \mathbf{y}^{[q-1]}, \end{aligned} \quad (12)$$

and then, there exists some indices $i \in [m]$, $j \in [n]$ such that

$$\begin{aligned} x_i^{p-1} (\mathcal{A}\mathbf{x}^{p-1} \mathbf{y}^q)_i &= \lambda x_i^{2(p-1)} > 0, \\ y_j^{q-1} (\mathcal{A}\mathbf{x}^p \mathbf{y}^{q-1})_j &= \lambda y_j^{2(q-1)} > 0. \end{aligned} \quad (13)$$

By the definition of rectangular P-tensors, we have $\lambda > 0$. The case for rectangular P_0 -tensors can be obtained similarly.

A rectangular tensor $\mathcal{C} \in \mathbb{R}^{[p; q; r_m; r_n]}$ is called a principal rectangular subtensor of a rectangular tensor $\mathcal{A} = (a_{i_1 i_2, \dots, i_p j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$ iff the sets $I \subseteq [m]$, $J \subseteq [n]$ contain r_m and r_n elements such that

$$\begin{aligned} \mathcal{C} &= \mathcal{A}_{r_m, r_n}^{I, J} = \left(c_{i_1 i_2, \dots, i_p j_1 j_2, \dots, j_q} \right), \quad \text{for all } i_1, i_2, \dots, i_p \in I, j_1, \\ &\quad j_2, \dots, j_q \in J. \end{aligned} \quad (14)$$

Let $\mathbf{x}_I$ be a r_m -dimensional subvector of a vector $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y}_J$ be a r_n -dimensional subvector of a vector $\mathbf{y} \in \mathbb{R}^n$. Note that, for $r_m = r_n = 1$, the principal rectangular subtensors are just the diagonal entries. $\square$

Lemma 1. Let $\mathcal{A} = (a_{i_0, \dots, i_0 j_0, \dots, j_0}) \in \mathbb{R}^{[p; q; 1; 1]}$ with V-singular value λ . Then,

$$\lambda = a_{i_0, \dots, i_0 j_0, \dots, j_0}. \quad (15)$$

Proof. If λ is a V-singular value of $\mathcal{A}$ with eigenvectors pair $(\mathbf{x}, \mathbf{y})$, then we have

$$\begin{aligned} \mathcal{A}\mathbf{x}^{p-1} \mathbf{y}^q &= \lambda \mathbf{x}^{[p-1]}, \\ \mathcal{A}\mathbf{x}^p \mathbf{y}^{q-1} &= \lambda \mathbf{y}^{[q-1]}, \end{aligned} \quad (16)$$

which implies

$$\begin{aligned} a_{i_0, \dots, i_0 j_0, \dots, j_0} x_{i_0}^{p-1} y_{j_0}^q &= \lambda x_{i_0}^{p-1}, \\ a_{i_0, \dots, i_0 j_0, \dots, j_0} x_{i_0}^p y_{j_0}^{q-1} &= \lambda y_{j_0}^{q-1}, \end{aligned} \quad (17)$$

where $\mathbf{x} = (x_{i_0})$, $\mathbf{y} = (y_{j_0})$, and $x_{i_0} \neq 0$, $y_{j_0} \neq 0$. Then, by $x_{i_0}^p = 1$ and $y_{j_0}^q = 1$, we have

$$\lambda = a_{i_0, \dots, i_0 j_0, \dots, j_0}. \quad (18) \quad \square$$

Theorem 2. Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_p j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$ be a rectangular P-tensor (P_0 -tensor). Then, every principal rectangular subtensor of $\mathcal{A}$ is a rectangular P-tensor (P_0 -tensor). In particular, all the diagonal entries of a rectangular P-tensor (P_0 -tensor) tensor are positive (nonnegative).

Proof. Let $\mathcal{A}_{r_m, r_n}^{I, J}$ be a principal rectangular subtensor of $\mathcal{A}$:

$$\begin{aligned} \mathbf{x}_I &= \left(x_{i_1}, \dots, x_{i_{r_m}} \right)^T \in \mathbb{R}^{r_m}, \\ \mathbf{y}_J &= \left(y_{j_1}, \dots, y_{j_{r_n}} \right)^T \in \mathbb{R}^{r_n}. \end{aligned} \quad (19)$$

$\mathbf{x}^* = (x_1^*, \dots, x_m^*)^T \in \mathbb{R}^m$ with $x_i^* = x_i$ if $i \in I$ and $x_i^* = 0$ if $i \notin I$, and $\mathbf{y}^* = (y_1^*, \dots, y_n^*)^T \in \mathbb{R}^n$ with $y_j^* = y_j$ if $j \in J$ and $y_j^* = 0$ if $j \notin J$. If $\mathcal{A} \in \mathbb{R}^{[p; q; m; n]}$ is a rectangular P-tensor, there exists some index $i \in [m]$, $j \in [n]$ such that

$$\begin{aligned} (x_i^*)^{p-1} (\mathcal{A}(\mathbf{x}^*)^{p-1} (\mathbf{y}^*)^q)_i &= x_i^{p-1} (\mathcal{A}_{r_m, r_n}^{I, J}(\mathbf{x}_I)^{p-1} (\mathbf{y}_J)^q)_i > 0, \\ (y_j^*)^{q-1} (\mathcal{A}(\mathbf{x}^*)^p (\mathbf{y}^*)^{q-1})_j &= y_j^{q-1} (\mathcal{A}_{r_m, r_n}^{I, J}(\mathbf{x}_I)^p (\mathbf{y}_J)^{q-1})_j > 0, \end{aligned} \quad (20)$$

which implies that $\mathcal{A}_{r_m, r_n}^{I, J}$ is a rectangular P-tensor. The case for rectangular P_0 -tensors can be obtained similarly.

A sufficient and necessary condition for a rectangular tensor to be a rectangular P-tensor is given as follows. $\square$

Theorem 3. Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_{p-1} j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$. Then, $\mathcal{A}$ is a rectangular P-tensor if and only if for each nonzero $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$, there exists positive diagonal matrices $D_{\mathbf{x}}, D_{\mathbf{y}}$ such that

$$\begin{aligned} (\mathbf{x}^{[p-1]})^T D_{\mathbf{x}} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q) &> 0, \\ (\mathbf{y}^{[q-1]})^T D_{\mathbf{y}} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1}) &> 0. \end{aligned} \quad (21)$$

Proof. If $\mathcal{A}$ is a rectangular P-tensor, then there exists some index $i \in [m]$, $j \in [n]$ such that

$$\begin{aligned} x_k^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_k &> 0, \\ y_l^{q-1} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_l &> 0. \end{aligned} \quad (22)$$

Then, for enough small $\mu, \nu > 0$, we have

$$\begin{aligned} x_k^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_k + \mu \left(\sum_{i=1, i \neq k}^m x_i^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_i \right) &> 0, \\ y_l^{q-1} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_l + \nu \left(\sum_{j=1, j \neq l}^n y_j^{q-1} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_j \right) &> 0. \end{aligned} \quad (23)$$

Therefore, we have

$$\begin{aligned} (\mathbf{x}^{[p-1]})^T D_{\mathbf{x}} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q) &> 0, \\ (\mathbf{y}^{[q-1]})^T D_{\mathbf{y}} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1}) &> 0, \end{aligned} \quad (24)$$

where $D_{\mathbf{x}} = \text{diag}(d_1, d_2, \dots, d_m)$ with $d_k = 1$ and $d_i = \mu$ for $i \neq k$ and $D_{\mathbf{y}} = \text{diag}(e_1, e_2, \dots, e_n)$ with $e_l = 1$ and $e_j = \nu$ for $j \neq l$.

On the contrary, if there exists positive diagonal matrices

$$\begin{aligned} D_{\mathbf{x}} &= \text{diag}(d_1, d_2, \dots, d_m), \\ D_{\mathbf{y}} &= \text{diag}(e_1, e_2, \dots, e_n), \end{aligned} \quad (25)$$

such that

$$\begin{aligned} (\mathbf{x}^{[p-1]})^T D_{\mathbf{x}} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q) &> 0, \\ (\mathbf{y}^{[q-1]})^T D_{\mathbf{y}} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1}) &> 0. \end{aligned} \quad (26)$$

Since $d_i > 0$ for all $i \in [m]$, $e_j > 0$ for all $j \in [n]$, then there exists $k \in [m]$ and $l \in [n]$ such that

$$\begin{aligned} x_k^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_k &> 0, \\ y_l^{q-1} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_l &> 0. \end{aligned} \quad (27)$$

Let $\|\mathbf{x}\|_{\infty} = \max\{|x_i|, i \in [n]\}$, and a quantity $\alpha(A)$ of a P-matrix A is introduced in [11]. In 2015, let

$$\begin{aligned} \alpha(T_{\mathcal{A}}) &= \min_{\|\mathbf{x}\|_{\infty}=1} \left\{ \max_{i \in [n]} x_i (T_{\mathcal{A}}(\mathbf{x}))_i \right\}, \\ \alpha(F_{\mathcal{A}}) &= \min_{\|\mathbf{x}\|_{\infty}=1} \left\{ \max_{i \in [n]} x_i (F_{\mathcal{A}}(\mathbf{x}))_i \right\}, \end{aligned} \quad (28)$$

where

$$\begin{aligned} T_{\mathcal{A}}(\mathbf{x}) &= \|\mathbf{x}\|_2^{2-m} \mathcal{A} \mathbf{x}^{m-1}, \\ F_{\mathcal{A}}(\mathbf{x}) &= (\mathcal{A} \mathbf{x}^{m-1})^{[1/m-1]}. \end{aligned} \quad (29)$$

Song and Qi introduced the definitions of quantities $\alpha(T_{\mathcal{A}})$ and $\alpha(F_{\mathcal{A}})$ for a P-tensor $\mathcal{A}$ and obtained monotonicity and boundedness of such two quantities, and they also showed that a tensor $\mathcal{A}$ is a P-tensor if and only if $\alpha(T_{\mathcal{A}})$ is positive, and a tensor $\mathcal{A}$ with even order is a P-tensor if and only if $\alpha(F_{\mathcal{A}})$ is positive [8]. We define the following two quantities for rectangular P-tensors:

$$\begin{aligned} \alpha_{\mathbf{x}}(\mathcal{A}) &= \min_{\|\mathbf{x}\|_{\infty}=1} \left\{ \max_{i \in [m]} x_i^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_i \right\}, \\ \alpha_{\mathbf{y}}(\mathcal{A}) &= \min_{\|\mathbf{y}\|_{\infty}=1} \left\{ \max_{j \in [n]} y_j^{q-1} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_j \right\}. \end{aligned} \quad (30)$$

We present some properties of quantities for rectangular P-tensors. $\square$

Theorem 4. Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_{p-1} j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$ be a rectangular P₀-tensor and $\mathcal{A}_{r_m, r_n}^{I, J}$ be a principal rectangular subtensor of $\mathcal{A}$. Then,

- (i) $\alpha_{\mathbf{x}}(\mathcal{A}) \leq \alpha_{\mathbf{x}}(\mathcal{A}_{r_m, r_n}^{I, J})$
- (ii) $\alpha_{\mathbf{y}}(\mathcal{A}) \leq \alpha_{\mathbf{y}}(\mathcal{A}_{r_m, r_n}^{I, J})$

Proof. Let $\mathcal{A}_{r_m, r_n}^{I, J}$ be a principal rectangular subtensor of $\mathcal{A}$:

$$\begin{aligned} \mathbf{x}_I &= (x_{i_1}, \dots, x_{i_m})^T \in \mathbb{R}^{r_m} / \{\mathbf{0}\}, \\ \mathbf{y}_J &= (y_{j_1}, \dots, y_{j_n})^T \in \mathbb{R}^{r_n} / \{\mathbf{0}\}. \end{aligned} \quad (31)$$

$\mathbf{x}^* = (x_1^*, \dots, x_m^*)^T \in \mathbb{R}^m$ with $x_i^* = x_i$ if $i \in I$ and $x_i^* = 0$ if $i \notin I$, and $\mathbf{y}^* = (y_1^*, \dots, y_n^*)^T \in \mathbb{R}^n$ with $y_j^* = y_j$ if $j \in J$ and $y_j^* = 0$ if $j \notin J$. Then, $\|\mathbf{x}^*\|_{\infty} = \|\mathbf{x}_I\|_{\infty}$ and $\|\mathbf{y}^*\|_{\infty} = \|\mathbf{y}_J\|_{\infty}$. Hence,

$$\begin{aligned}
\alpha_{\mathbf{x}}(\mathcal{A}) &= \min_{\|\mathbf{x}\|_{\infty}=1} \left\{ \max_{i \in [m]} x_i^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_i \right\} \\
&\leq \min_{\|\mathbf{x}^*\|_{\infty}=1} \left\{ \max_{i \in [m]} (x_i^*)^{p-1} (\mathcal{A} (\mathbf{x}^*)^{p-1} (\mathbf{y}^*)^q)_i \right\} \\
&= \min_{\|\mathbf{x}_I\|_{\infty}=1} \left\{ \max_{i \in I} (x_i)^{p-1} (\mathcal{A}_{r_m, r_n}^{I, J} (\mathbf{x}_I)^{p-1} (\mathbf{y}_J)^q)_i \right\} \\
&= \alpha_{\mathbf{x}}(\mathcal{A}_{r_m, r_n}^{I, J}), \\
\alpha_{\mathbf{y}}(\mathcal{A}) &= \min_{\|\mathbf{y}\|_{\infty}=1} \left\{ \max_{j \in [n]} y_j^{q-1} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_j \right\} \\
&\leq \min_{\|\mathbf{y}^*\|_{\infty}=1} \left\{ \max_{j \in [n]} (y_j^*)^{q-1} (\mathcal{A} (\mathbf{x}^*)^{p-1} (\mathbf{y}^*)^{q-1})_j \right\} \\
&= \min_{\|\mathbf{y}_J\|_{\infty}=1} \left\{ \max_{j \in J} (y_j)^{q-1} (\mathcal{A}_{r_m, r_n}^{I, J} (\mathbf{x}_I)^p (\mathbf{y}_J)^{q-1})_j \right\} \\
&= \alpha_{\mathbf{y}}(\mathcal{A}_{r_m, r_n}^{I, J}).
\end{aligned} \tag{32}$$

Let $\delta(\mathcal{A}) := \min\{\lambda(\mathcal{A}_{r_m, r_n}^{I, J}); I \subseteq [m], J \subseteq [n]\}$, where $\lambda(\mathcal{A})$ denotes the smallest of V-singular value (if any exists) of a rectangular P-tensor $\mathcal{A}$. Then, we have the following upper bounds for $\alpha_{\mathbf{x}}(\mathcal{A})$ and $\alpha_{\mathbf{y}}(\mathcal{A})$. Let $\mathcal{J}_R = (\epsilon_{i_1, \dots, i_p, j_1, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$ be the rectangular identity tensor with

$$\epsilon_{i_1, \dots, i_p, j_1, \dots, j_q} = \begin{cases} 1, & \text{if } i = \dots = i_p, j_1 = \dots = j_q, \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

□

Theorem 5. Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_p, j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$ be a rectangular P-tensor. Then,

$$\begin{aligned}
(i) \quad \alpha_{\mathbf{x}}(\mathcal{A}) &\leq \delta(\mathcal{A}) \leq \min_{i \in [m], j \in [n]} a_{i, \dots, i, j, \dots, j} \\
(ii) \quad \alpha_{\mathbf{y}}(\mathcal{A}) &\leq \delta(\mathcal{A}) \leq \min_{i \in [m], j \in [n]} a_{i, \dots, i, j, \dots, j}
\end{aligned}$$

Proof. Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_p, j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$ be a rectangular P-tensor, by Theorem 2, and we have

$$a_{i, \dots, i, j, \dots, j} > 0, \quad \text{for all } i \in [m], j \in [n]. \tag{34}$$

Since $\mathcal{A}_{1,1}^{I, J}$ is a principal rectangular subtensor of $\mathcal{A}$, by Lemma 1, $a_{i, \dots, i, j, \dots, j}$ is a V-singular value of $\mathcal{A}_{1,1}^{I, J}$; therefore,

$$\delta(\mathcal{A}) \leq a_{i, \dots, i, j, \dots, j}. \tag{35}$$

Furthermore, $\delta(\mathcal{A})$ is a V-singular value of $\mathcal{A}_{1,1}^{I, J}$; then, $\mathcal{A}_{1,1}^{I, J} - \delta(\mathcal{A})(\mathcal{J}_R)_{r_m, r_n}^{I, J}$ is not a rectangular P-tensor. By Theorem 2, $\mathcal{A} - \delta(\mathcal{A})\mathcal{J}_R$ is not a rectangular P-tensor. Then, there exists vectors $\mathbf{x}$ and $\mathbf{y}$ with $\|\mathbf{x}\|_{\infty} = 1$ and $\|\mathbf{y}\|_{\infty} = 1$ such that

$$\begin{aligned}
\max_{i \in [m]} x_i^{p-1} ((\mathcal{A} - \delta(\mathcal{A})\mathcal{J}_R) \mathbf{x}^{p-1} \mathbf{y}^q)_i &\leq 0, \\
\max_{j \in [n]} y_j^{q-1} ((\mathcal{A} - \delta(\mathcal{A})\mathcal{J}_R) \mathbf{x}^p \mathbf{y}^{q-1})_j &\leq 0.
\end{aligned} \tag{36}$$

Then,

$$\begin{aligned}
x_k^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_k - x_k^{p-1} (\delta(\mathcal{A}) \mathcal{J}_R \mathbf{x}^{p-1} \mathbf{y}^q)_k &\leq 0, \\
y_l^{q-1} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_l - y_l^{q-1} (\delta(\mathcal{A}) \mathcal{J}_R \mathbf{x}^p \mathbf{y}^{q-1})_l &\leq 0,
\end{aligned} \tag{37}$$

for all $k \in [m], l \in [n]$. Then,

$$\begin{aligned}
\max_{i \in [m]} x_i^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_i &= (x_{i_0})^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_{i_0} \\
&\leq (x_{i_0})^{p-1} (\delta(\mathcal{A}) \mathcal{J}_R \mathbf{x}^{p-1} \mathbf{y}^q)_{i_0} \\
&\leq \delta(\mathcal{A}) \|\mathbf{x}\|_{\infty} \|\mathbf{y}\|_{\infty} = \delta(\mathcal{A}).
\end{aligned} \tag{38}$$

Similarly, we have

$$\max_{j \in [n]} y_j^{q-1} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_j \leq \delta(\mathcal{A}). \tag{39}$$

Therefore,

$$\begin{aligned}
\alpha_{\mathbf{x}}(\mathcal{A}) &\leq \delta(\mathcal{A}) \leq \min_{i \in [m], j \in [n]} a_{i, \dots, i, j, \dots, j}, \\
\alpha_{\mathbf{y}}(\mathcal{A}) &\leq \delta(\mathcal{A}) \leq \min_{i \in [m], j \in [n]} a_{i, \dots, i, j, \dots, j}.
\end{aligned} \tag{40}$$

Based on the quantities $\alpha_{\mathbf{x}}(\mathcal{A})$ and $\alpha_{\mathbf{y}}(\mathcal{A})$, a necessary and sufficient conditions for a rectangular tensor to be a rectangular P-tensor is given as follows. □

Theorem 6. Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_p, j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$. Then, $\mathcal{A}$ is a rectangular P-tensor (P_0 -tensor) if and only if $\alpha_{\mathbf{x}}(\mathcal{A})$ and $\alpha_{\mathbf{y}}(\mathcal{A})$ are positive (nonnegative).

Proof. Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_p, j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$ be a rectangular P-tensor. Then, for each $\mathbf{x} \in \mathbb{R}^m \setminus \{0\}$ and $\mathbf{y} \in \mathbb{R}^n \setminus \{0\}$, there exists some index $i \in [m], j \in [n]$ such that

$$\begin{aligned}
x_i^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_i &> 0, \\
y_j^{q-1} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_j &> 0.
\end{aligned} \tag{41}$$

Then,

$$\begin{aligned}
\alpha_{\mathbf{x}}(\mathcal{A}) &= \min_{\|\mathbf{x}\|_{\infty}=1} \left\{ \max_{i \in [m]} x_i^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_i \right\} > 0, \\
\alpha_{\mathbf{y}}(\mathcal{A}) &= \min_{\|\mathbf{y}\|_{\infty}=1} \left\{ \max_{j \in [n]} y_j^{q-1} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_j \right\} > 0.
\end{aligned} \tag{42}$$

Conversely, if $\alpha_{\mathbf{x}}(\mathcal{A}) > 0$ and $\alpha_{\mathbf{y}}(\mathcal{A}) > 0$, we have

$$\begin{aligned}
x_i^{p-1} (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_i &> 0, \\
y_j^{q-1} (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_j &> 0,
\end{aligned} \tag{43}$$

which implies $\mathcal{A}$ is a rectangular P-tensor. The case for rectangular P_0 -tensors can be obtained similarly. □

4. Rectangular S-Tensor

Definition 6. A rectangular tensor $\mathcal{A} = (a_{i_1 i_2, \dots, i_p, j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$ is called a rectangular S-tensor if and only if there exists $\mathbf{0} < \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{0} < \mathbf{y} \in \mathbb{R}^n$ such that

$$\begin{aligned}\mathcal{A}\mathbf{x}^{p-1}\mathbf{y}^q &> \mathbf{0}, \\ \mathcal{A}\mathbf{x}^p\mathbf{y}^{q-1} &> \mathbf{0}.\end{aligned}\quad (44)$$

A rectangular tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_p j_1 j_2 \dots j_q}) \in \mathbb{R}^{[p;q;m;n]}$ is called a rectangular S_0 -tensor if and only if there exists $\mathbf{0} \leq \mathbf{x} \in \mathbb{R}^m / \{\mathbf{0}\}$ and $\mathbf{0} \leq \mathbf{y} \in \mathbb{R}^n / \{\mathbf{0}\}$ such that

$$\begin{aligned}\mathcal{A}\mathbf{x}^{p-1}\mathbf{y}^q &\geq \mathbf{0}, \\ \mathcal{A}\mathbf{x}^p\mathbf{y}^{q-1} &\geq \mathbf{0}.\end{aligned}\quad (45)$$

The conditions $\mathbf{0} < \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{0} < \mathbf{y} \in \mathbb{R}^n$ in the definition of rectangular S-tensors can be relaxed to $\mathbf{0} \leq \mathbf{x} \in \mathbb{R}^m / \{\mathbf{0}\}$ and $\mathbf{0} \leq \mathbf{y} \in \mathbb{R}^n / \{\mathbf{0}\}$.

Theorem 7. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_p j_1 j_2 \dots j_q}) \in \mathbb{R}^{[p;q;m;n]}$. Then, $\mathcal{A}$ is a rectangular S-tensor if and only if there exists $\mathbf{0} \leq \mathbf{x} \in \mathbb{R}^m / \{\mathbf{0}\}$ and $\mathbf{0} \leq \mathbf{y} \in \mathbb{R}^n / \{\mathbf{0}\}$ such that

$$\begin{aligned}\mathcal{A}\mathbf{x}^{p-1}\mathbf{y}^q &> \mathbf{0}, \\ \mathcal{A}\mathbf{x}^p\mathbf{y}^{q-1} &> \mathbf{0}.\end{aligned}\quad (46)$$

Proof. The necessity is obvious by the definition of rectangular S-tensors. We prove the sufficiency as follows.

If there exists $\mathbf{0} \leq \mathbf{x} \in \mathbb{R}^m / \{\mathbf{0}\}$ and $\mathbf{0} \leq \mathbf{y} \in \mathbb{R}^n / \{\mathbf{0}\}$ such that

$$\begin{aligned}\mathcal{A}\mathbf{x}^{p-1}\mathbf{y}^q &> \mathbf{0}, \\ \mathcal{A}\mathbf{x}^p\mathbf{y}^{q-1} &> \mathbf{0}.\end{aligned}\quad (47)$$

Let $\mathbf{e}_m = (1, \dots, 1)^T$ and $\mathbf{e}_n = (1, \dots, 1)^T$; for some small enough $t > 0$, we have

$$\begin{aligned}\mathcal{A}(\mathbf{x} + t\mathbf{e}_m)^{p-1}(\mathbf{y} + t\mathbf{e}_n)^q &> \mathbf{0}, \\ \mathcal{A}(\mathbf{x} + t\mathbf{e}_m)^p(\mathbf{y} + t\mathbf{e}_n)^{q-1} &> \mathbf{0},\end{aligned}\quad (48)$$

which means that $\mathcal{A}$ is a rectangular S-tensor.

From Theorem 2, we know that every principal rectangular subtensor of a rectangular P-tensor is a rectangular P-tensor. However, such a property does not always hold for rectangular S-tensor by the following example, i.e., the principal rectangular subtensor of a rectangular S-tensor is not always a rectangular S-tensor. $\square$

Example 1. Let $\mathcal{A} = (a_{i_1 i_2 j_1 j_2}) \in \mathbb{R}^{[2;2;2;2]}$, where

$$\begin{aligned}a_{1111} &= 1, \\ a_{1112} &= -1, \\ a_{2221} &= 1, \\ a_{2222} &= -1,\end{aligned}\quad (49)$$

and all other $a_{i_1 i_2 j_1 j_2} = 0$. Then, for any $\mathbf{0} \leq \mathbf{x} \in \mathbb{R}^m / \{\mathbf{0}\}$ and $\mathbf{0} \leq \mathbf{y} \in \mathbb{R}^n / \{\mathbf{0}\}$, we obtain

$$\begin{aligned}\mathcal{A}\mathbf{x}\mathbf{y}^2 &= \begin{pmatrix} x_1 y_1^2 - x_1 y_1 y_2 \\ x_2 y_2 y_1 - x_2 y_2^2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 (y_1 - y_2) \\ x_2 y_2 (y_1 - y_2) \end{pmatrix}, \\ \mathcal{A}\mathbf{x}^2 \mathbf{y} &= \begin{pmatrix} x_1^2 y_1 - x_1^2 y_2 \\ x_2^2 y_1 - x_2^2 y_2 \end{pmatrix} = \begin{pmatrix} x_1^2 (y_1 - y_2) \\ x_2^2 (y_1 - y_2) \end{pmatrix}.\end{aligned}\quad (50)$$

Then, for $\mathbf{y} = (2, 1)^T > \mathbf{0}$ and $\mathbf{x} = (1, 1)^T > \mathbf{0}$, $\mathcal{A}\mathbf{x}\mathbf{y}^2 > \mathbf{0}$ and $\mathcal{A}\mathbf{x}^2 \mathbf{y} > \mathbf{0}$, which implies, $\mathcal{A}$ is a rectangular S-tensor.

Let $\mathcal{A}_{r_m, r_n}^{I, J}$ be a principal rectangular subtensor of $\mathcal{A}$ with $I = J = \{2\}$; then, for any $x > 0$ and $y > 0$, we have

$$\begin{aligned}\mathcal{A}_{r_m, r_n}^{I, J} \mathbf{x} \mathbf{y}^2 &= (-1)x y^2 < 0, \\ \mathcal{A}_{r_m, r_n}^{I, J} \mathbf{x}^2 \mathbf{y} &= (-1)x^2 y < 0,\end{aligned}\quad (51)$$

which means that $\mathcal{A}_{r_m, r_n}^{I, J}$ is not a rectangular S-tensor.

Some necessary and/or sufficient conditions for a rectangular tensor to be a rectangular S-tensor are presented as follows.

Theorem 8. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_p j_1 j_2 \dots j_q}) \in \mathbb{R}^{[p;q;m;n]}$ be a rectangular S-tensor. Then, there exists $i \in [m]$ and $j \in [n]$ such that

$$a_{i, \dots, i, j, \dots, j} + \sum_{\substack{a_{i_2, \dots, i_p j_1, \dots, j_q} > 0, \\ (i_2, \dots, i_p j_1, \dots, j_q) \neq (i, \dots, i, j, \dots, j)}} a_{i_2, \dots, i_p j_1, \dots, j_q} > 0. \quad (52)$$

Proof. Since $\mathcal{A}$ is a rectangular S-tensor, then there exists $\mathbf{0} < \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{0} < \mathbf{y} \in \mathbb{R}^n$ such that

$$(\mathcal{A}\mathbf{x}^{p-1}\mathbf{y}^q)_i > 0, \quad i \in [m]. \quad (53)$$

Let $x_t = \max_{i \in [m]} x_i$ and $y_s = \max_{j \in [n]} y_j$; then, $x_t > 0$ and $y_s > 0$, and

$$\begin{aligned}
0 < (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_t &= \sum_{i_2, \dots, i_p=1}^m \sum_{j_1, \dots, j_q=1}^n a_{ti_2, \dots, i_p j_1, \dots, j_q} x_{i_2}, \dots, x_{i_p} y_{j_1}, \dots, y_{j_q} \\
&= a_{t, \dots, t s, \dots, s} x_t^{p-1} y_s^q + \sum_{\substack{a_{ti_2, \dots, i_p j_1, \dots, j_q} > 0, \\ (ti_2, \dots, i_p j_1, \dots, j_q) \neq (t, \dots, t s, \dots, s)}} a_{ti_2, \dots, i_p j_1, \dots, j_q} x_{i_2}, \dots, x_{i_p} y_{j_1}, \dots, y_{j_q} \\
&\quad + \sum_{\substack{a_{ti_2, \dots, i_p j_1, \dots, j_q} < 0, \\ (ti_2, \dots, i_p j_1, \dots, j_q) \neq (t, \dots, t s, \dots, s)}} a_{ti_2, \dots, i_p j_1, \dots, j_q} x_{i_2}, \dots, x_{i_p} y_{j_1}, \dots, y_{j_q} \leq a_{t, \dots, t s, \dots, s} x_t^{p-1} y_s^q \\
&\quad + \sum_{\substack{a_{ti_2, \dots, i_p j_1, \dots, j_q} > 0, \\ (ti_2, \dots, i_p j_1, \dots, j_q) \neq (t, \dots, t s, \dots, s)}} a_{ti_2, \dots, i_p j_1, \dots, j_q} x_{i_2}, \dots, x_{i_p} y_{j_1}, \dots, y_{j_q} \\
&\leq \left(a_{t, \dots, t s, \dots, s} + \sum_{\substack{a_{ti_2, \dots, i_p j_1, \dots, j_q} > 0, \\ (ti_2, \dots, i_p j_1, \dots, j_q) \neq (t, \dots, t s, \dots, s)}} a_{ti_2, \dots, i_p j_1, \dots, j_q} x_t^{p-1} y_s^q \right)
\end{aligned} \tag{54}$$

which means

$$a_{t, \dots, t s, \dots, s} + \sum_{\substack{a_{ti_2, \dots, i_p j_1, \dots, j_q} > 0, \\ (ti_2, \dots, i_p j_1, \dots, j_q) \neq (t, \dots, t s, \dots, s)}} a_{ti_2, \dots, i_p j_1, \dots, j_q} > 0. \tag{55}$$

□

Theorem 9. Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_p j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$. If there exists a principal rectangular subtensor $\mathcal{A}_{r_m, r_n}^{I, J}$ of $\mathcal{A}$ is a rectangular S-tensor, $a_{ii_2, \dots, i_p j_1, \dots, j_q} > 0$ for all $i_2, \dots, i_p \in I$, $j_1, \dots, j_q \in J$, $i \notin I$ and $a_{i_1, \dots, i_p j j_2, \dots, j_q} > 0$ for all $i_1, \dots, i_p \in I$ and $j_2, \dots, j_q \in J$, $j \notin J$; then, $\mathcal{A}$ is a rectangular S-tensor.

Proof. Since $\mathcal{A}_{r_m, r_n}^{I, J}$ is a rectangular S-tensor, then there exists $\mathbf{0} < \hat{\mathbf{x}} \in \mathbb{R}^{r_m}$ and $\mathbf{0} < \hat{\mathbf{y}} \in \mathbb{R}^{r_n}$ such that

$$\begin{aligned}
(\mathcal{A} \hat{\mathbf{x}}^{p-1} \hat{\mathbf{y}}^q)_i &> 0, \quad i \in I, \\
(\mathcal{A} \hat{\mathbf{x}}^p \hat{\mathbf{y}}^{q-1})_j &> 0, \quad j \in J.
\end{aligned} \tag{56}$$

Let $\mathbf{0} \leq \tilde{\mathbf{x}} \in \mathbb{R}^m / \{\mathbf{0}\}$ and $\mathbf{0} \leq \tilde{\mathbf{y}} \in \mathbb{R}^n / \{\mathbf{0}\}$ with

$$\begin{cases} x_i = \tilde{x}_i, & \text{if } i \in I, \\ \tilde{x}_i = 0, & \text{if } i \notin I, \\ \tilde{y}_j = \hat{y}_j, & \text{if } j \in J, \\ \tilde{y}_j = 0, & \text{if } j \notin J. \end{cases} \tag{57}$$

Then, for any $i \in I$ and $j \in J$, we have

$$\begin{aligned}
(\mathcal{A} \tilde{\mathbf{x}}^{p-1} \tilde{\mathbf{y}}^q)_i &= \sum_{i_2, \dots, i_p=1}^m \sum_{j_1, \dots, j_q=1}^n a_{ii_2, \dots, i_p j_1, \dots, j_q} \tilde{x}_{i_2}, \dots, \tilde{x}_{i_p} \tilde{y}_{j_1}, \dots, \tilde{y}_{j_q} \\
&= \sum_{i_t \in I, t \in \{2, \dots, m\}} \sum_{j_s \in J, s \in [n]} a_{ii_2, \dots, i_p j_1, \dots, j_q} \tilde{x}_{i_2}, \dots, \tilde{x}_{i_p} \tilde{y}_{j_1}, \dots, \tilde{y}_{j_q}, \\
(\mathcal{A} \tilde{\mathbf{x}}^p \tilde{\mathbf{y}}^{q-1})_j &= \sum_{i_1, \dots, i_p=1}^m \sum_{j_2, \dots, j_q=1}^n a_{i_1, \dots, i_p j j_2, \dots, j_q} x_{i_1}, \dots, x_{i_p} y_{j_2}, \dots, y_{j_q} \\
&= \sum_{i_t \in I, t \in [m]} \sum_{j_s \in J, s \in \{2, \dots, n\}} a_{i_1, \dots, i_p j j_2, \dots, j_q} \tilde{x}_{i_1}, \dots, \tilde{x}_{i_p} \tilde{y}_{j_2}, \dots, \tilde{y}_{j_q}.
\end{aligned} \tag{58}$$

Therefore, if $i \in I$, we obtain

$$(\mathcal{A} \tilde{\mathbf{x}}^{p-1} \tilde{\mathbf{y}}^q)_i = (\mathcal{A}_{r_m, r_n}^{I, J} \tilde{\mathbf{x}}^{p-1} \tilde{\mathbf{y}}^q)_i > 0, \tag{59}$$

if $i \notin I$, we obtain

$$\begin{aligned}
(\mathcal{A} \tilde{\mathbf{x}}^{p-1} \tilde{\mathbf{y}}^q)_i &= \sum_{i_t \in I, t \in \{2, \dots, m\}} \sum_{j_s \in J, s \in [n]} a_{ii_2, \dots, i_p j_1, \dots, j_q} \tilde{x}_{i_2}, \dots, \tilde{x}_{i_p} \tilde{y}_{j_1} \\
&\quad, \dots, \tilde{y}_{j_q} > 0,
\end{aligned} \tag{60}$$

which implies that $(\mathcal{A} \tilde{\mathbf{x}}^{p-1} \tilde{\mathbf{y}}^q)_i > 0$. Similarly, we have $(\mathcal{A} \tilde{\mathbf{x}}^p \tilde{\mathbf{y}}^{q-1})_j > 0$. Then, $\mathcal{A}$ is a rectangular S-tensor.

A sufficient and necessary condition for a rectangular tensor to be a rectangular S-tensor is given as follows. □

Theorem 10. Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_p j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$. Then, $\mathcal{A}$ is a rectangular S-tensor if and only if there exists

$\mathbf{0} < \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{0} < \mathbf{y} \in \mathbb{R}^n$, for any nonzero nonnegative diagonal matrices D_x and D_y such that

$$\begin{aligned} \mathbf{x}^T D_x (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q) &> 0, \\ \mathbf{y}^T D_y (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1}) &> 0. \end{aligned} \quad (61)$$

Proof. If $\mathcal{A}$ is a rectangular S-tensor and $\mathbf{0} < \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{0} < \mathbf{y} \in \mathbb{R}^n$, then for any $k \in [m]$ and $l \in [n]$ such that

$$\begin{aligned} x_k (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_k &> 0, \\ y_l (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_l &> 0. \end{aligned} \quad (62)$$

Then, for any $\mu, \nu \geq 0$, we have

$$\begin{aligned} x_k (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_k + \mu \left(\sum_{i=1, i \neq k}^m x_i (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_i \right) &> 0, \\ y_l (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_l + \nu \left(\sum_{j=1, j \neq l}^n y_j (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_j \right) &> 0. \end{aligned} \quad (63)$$

Therefore, we have

$$\begin{aligned} \mathbf{x}^T D_x (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q) &> 0, \\ \mathbf{y}^T D_y (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1}) &> 0, \end{aligned} \quad (64)$$

where $D_x = \text{diag}(d_1, d_2, \dots, d_m)$ with $d_k = 1$ and $d_i = \mu$ for $i \neq k$ and $D_y = \text{diag}(e_1, e_2, \dots, e_n)$ with $e_l = 1$ and $e_j = \nu$ for $j \neq l$.

On the contrary, for any nonzero nonnegative diagonal matrices

$$D_x = \text{diag}(d_1, d_2, \dots, d_m), D_y = \text{diag}(e_1, e_2, \dots, e_n), \quad (65)$$

such that

$$\mathbf{x}^T D_x (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q) > 0, \mathbf{y}^T D_y (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1}) > 0. \quad (66)$$

For any $i \in [m]$ and $j \in [n]$, let $D_x = \text{diag}(d_1, d_2, \dots, d_m)$ with $d_i = 1$ and $d_k = 0$ for $k \neq i$ and $D_y = \text{diag}(e_1, e_2, \dots, e_n)$ with $e_j = 1$ and $e_l = 0$ for $l \neq j$, and we obtain

$$\begin{aligned} \mathbf{x}^T D_x (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q) &= x_i (\mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q)_i > 0, \\ \mathbf{y}^T D_y (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1}) &= y_j (\mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1})_j > 0. \end{aligned} \quad (67)$$

By the conditions $\mathbf{x} \in \mathbb{R}^m > \mathbf{0}$ and $\mathbf{y} \in \mathbb{R}^n > \mathbf{0}$, we have

$$\begin{aligned} \mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q &> \mathbf{0}, \\ \mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1} &> \mathbf{0}, \end{aligned} \quad (68)$$

which means that $\mathcal{A}$ is a rectangular S-tensor. $\square$

5. Rectangular Tensor Complementarity Problems

Converting a bimatrix game $F(A, A)$ to a linear complementarity problem, we have the following LCP [22]:

$$\begin{cases} \mathbf{u} = \mathbf{q}_m + A\mathbf{y} \geq 0, \mathbf{x} \geq 0, \mathbf{x}^T \mathbf{u} = 0, \\ \mathbf{v} = \mathbf{q}_n + A^T \mathbf{x} \geq 0, \mathbf{y} \geq 0, \mathbf{y}^T \mathbf{v} = 0. \end{cases} \quad (69)$$

In this section, we study the rectangular tensor complementarity problem (RTCP), which can be viewed as the generalization of the linear complementarity problem (69) to the tensor case.

Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_p j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$, $\mathbf{q}_m \in \mathbb{R}^m$ and $\mathbf{q}_n \in \mathbb{R}^n$. The rectangular tensor complementarity problem, denoted by RTCP $(\mathcal{A}, \mathbf{q}_m, \mathbf{q}_n)$, is to find vectors $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$ such that

$$\begin{aligned} \mathbf{q}_m + \mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q &\geq \mathbf{0}, \mathbf{x} \geq \mathbf{0}, \mathbf{x}^T (\mathbf{q}_m + \mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q) = 0, \\ \mathbf{q}_n + \mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1} &\geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{y}^T (\mathbf{q}_n + \mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1}) = 0. \end{aligned} \quad (70)$$

Vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be feasible iff $\mathbf{x}$ and $\mathbf{y}$ satisfy the following inequalities:

$$\begin{aligned} \mathbf{q}_m + \mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q &\geq \mathbf{0}, \mathbf{x} \geq \mathbf{0}, \\ \mathbf{q}_n + \mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1} &\geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}. \end{aligned} \quad (71)$$

The following equivalent definition of rectangular S-tensor can be given by means of the solution of the RTCP $(\mathcal{A}, \mathbf{q}_m, \mathbf{q}_n)$.

Theorem 11. Let $\mathcal{A} = (a_{i_1 i_2, \dots, i_p j_1 j_2, \dots, j_q}) \in \mathbb{R}^{[p; q; m; n]}$. Then, $\mathcal{A}$ is a rectangular S-tensor if and only if the RTCP $(\mathcal{A}, \mathbf{q}_m, \mathbf{q}_n)$ is feasible for all $\mathbf{q}_m \in \mathbb{R}^m$ and $\mathbf{q}_n \in \mathbb{R}^n$.

Proof. If $\mathcal{A}$ is a rectangular S-tensor and $\mathbf{0} < \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{0} < \mathbf{y} \in \mathbb{R}^n$, then

$$\begin{aligned} \mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q &> \mathbf{0}, \\ \mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1} &> \mathbf{0}. \end{aligned} \quad (72)$$

Then, for each $\mathbf{q}_m \in \mathbb{R}^m$ and $\mathbf{q}_n \in \mathbb{R}^n$, there exists some scalar $t > 0$ such that

$$\begin{aligned} \mathcal{A} (t^{(1/p-1)} \mathbf{x})^{p-1} \mathbf{y}^q &= t \mathcal{A} \mathbf{x}^{p-1} \mathbf{y}^q \geq -\mathbf{q}_m, \\ \mathcal{A} (t^{(1/p-1)} \mathbf{x})^p \mathbf{y}^{q-1} &= t^{(p/p-1)} \mathcal{A} \mathbf{x}^p \mathbf{y}^{q-1} \geq -\mathbf{q}_n, \end{aligned} \quad (73)$$

which means that $t^{(1/p-1)} \mathbf{x}$ and $\mathbf{y}$ are the feasible vectors of the RTCP $(\mathcal{A}, \mathbf{q}_m, \mathbf{q}_n)$.

On the contrary, if the RTCP $(\mathcal{A}, \mathbf{q}_m, \mathbf{q}_n)$ is feasible for all $\mathbf{q}_m \in \mathbb{R}^m$ and $\mathbf{q}_n \in \mathbb{R}^n$, assume that $\mathbf{q}_m < \mathbf{0}$, $\mathbf{q}_n < \mathbf{0}$, and $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are the feasible solution of the RTCP $(\mathcal{A}, \mathbf{q}_m, \mathbf{q}_n)$. Then,

$$\begin{aligned} \mathbf{q}_m + \mathcal{A}\bar{\mathbf{x}}^{p-1}\bar{\mathbf{y}}^q &\geq \mathbf{0}, \bar{\mathbf{x}} \geq \mathbf{0}, \\ \mathbf{q}_n + \mathcal{A}\bar{\mathbf{x}}^p\bar{\mathbf{y}}^{q-1} &\geq \mathbf{0}, \bar{\mathbf{y}} \geq \mathbf{0}. \end{aligned} \quad (74)$$

Therefore,

$$\begin{cases} \mathcal{A}\bar{\mathbf{x}}^{p-1}\bar{\mathbf{y}}^q \geq -\mathbf{q}_m > \mathbf{0}, \\ \mathcal{A}\bar{\mathbf{x}}^p\bar{\mathbf{y}}^{q-1} \geq -\mathbf{q}_n > \mathbf{0}. \end{cases} \quad (75)$$

Then, $\mathcal{A}$ is a rectangular S-tensor by Theorem 7.

In the end of this section, we propose some relationships among these structured rectangular tensors as follows. $\square$

Theorem 12

- (a) A positive definite rectangular tensor is a rectangular P-tensor and a rectangular P-tensor is a rectangular S-tensor. The inverse implications are not true.
- (b) A positive definite rectangular tensor is a strictly copositive rectangular tensor, and a rectangular S-tensor is a strictly copositive rectangular tensor. The inverse implications are not true.

Proof. If $\mathcal{A}$ is a positive definite rectangular tensor, which means that $\mathcal{A}\mathbf{x}^p\mathbf{y}^q > 0$ for all $\mathbf{x} \in \mathbb{R}^m/\{\mathbf{0}\}$, $\mathbf{y} \in \mathbb{R}^n/\{\mathbf{0}\}$, and p and q are even, then

$$\begin{aligned} \sum_{i=1}^m x_i (\mathcal{A}\mathbf{x}^{p-1}\mathbf{y}^q)_i &> 0, \\ \sum_{j=1}^n y_j (\mathcal{A}\mathbf{x}^p\mathbf{y}^{q-1})_j &> 0, \end{aligned} \quad (76)$$

then,

$$\begin{aligned} \sum_{i=1}^m x_i^{p-1} (\mathcal{A}\mathbf{x}^{p-1}\mathbf{y}^q)_i &> 0, \\ \sum_{j=1}^n y_j^{q-1} (\mathcal{A}\mathbf{x}^p\mathbf{y}^{q-1})_j &> 0, \end{aligned} \quad (77)$$

therefore, there exists some indices $i_0 \in [m]$ and $j_0 \in [n]$ such that

$$\begin{aligned} x_{i_0}^{p-1} (\mathcal{A}\mathbf{x}^{p-1}\mathbf{y}^q)_{i_0} &> 0, \\ y_{j_0}^{q-1} (\mathcal{A}\mathbf{x}^p\mathbf{y}^{q-1})_{j_0} &> 0, \end{aligned} \quad (78)$$

then $\mathcal{A}$ is a rectangular P-tensor.

If $\mathcal{A}$ is a rectangular P-tensor, by definitions of rectangular P-tensors and rectangular S-tensors, $\mathcal{A}$ is a rectangular S-tensor. The conclusion of (b) can be obtained similarly by the definitions of positive definite rectangular tensors, copositive rectangular tensors, and rectangular S-tensors. $\square$

6. Conclusions

In this paper, based on the definition of V-singular value for rectangular tensors, we extend the concept of P-tensors and

P₀-tensors to rectangular P-tensors and rectangular P₀-tensors. It is shown that all the V-singular values of rectangular P-tensors are positive. Some properties of quantities for rectangular P-tensors are given, and a necessary and sufficient condition for a rectangular tensor to be a rectangular P-tensor is also obtained. The rectangular S-tensor can be viewed as a generalization of S-tensors, and an example is constructed to illustrate that the principal rectangular subtensor of a rectangular S-tensor is not always a rectangular S-tensor. Finally, we introduced the rectangular tensor complementarity problem, an equivalent definition of rectangular S-tensors is given by means of the solution of the rectangular tensor complementarity problem.

By the definition of H-singular value for rectangular tensors, another definitions of rectangular P-tensors and rectangular P₀-tensors can be given as follows.

Definition 7. A tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_p j_1 j_2 \dots j_q}) \in \mathbb{R}^{[p; q; m; n]}$ is called a rectangular HP-tensor, if for each $\mathbf{x} \in \mathbb{R}^m/\{\mathbf{0}\}$, and $\mathbf{y} \in \mathbb{R}^n/\{\mathbf{0}\}$, there exists some indices $i \in [m]$, $j \in [n]$ such that

$$\begin{aligned} x_i^{M-1} (\mathcal{A}\mathbf{x}^{p-1}\mathbf{y}^q)_i &> 0, \\ y_j^{M-1} (\mathcal{A}\mathbf{x}^p\mathbf{y}^{q-1})_j &> 0. \end{aligned} \quad (79)$$

A tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_p j_1 j_2 \dots j_q}) \in \mathbb{R}^{[p; q; m; n]}$ is called a rectangular HP₀-tensor, if for each $\mathbf{x} \in \mathbb{R}^m/\{\mathbf{0}\}$ and $\mathbf{y} \in \mathbb{R}^n/\{\mathbf{0}\}$, there exists some indices $i \in [m]$ and $j \in [n]$ such that

$$\begin{aligned} x_i^{M-1} (\mathcal{A}\mathbf{x}^{p-1}\mathbf{y}^q)_i &\geq 0, \\ y_j^{M-1} (\mathcal{A}\mathbf{x}^p\mathbf{y}^{q-1})_j &\geq 0. \end{aligned} \quad (80)$$

Then, all the results for rectangular P-tensors and rectangular P₀-tensors can be extended to rectangular HP-tensors and rectangular HP₀-tensors except Theorem 5, since the rectangular identity tensor for the definition of H-singular value is hard to define. Similarly, we can get that a positive definite rectangular tensor is a rectangular HP-tensor and a rectangular HP-tensor is a rectangular S-tensor.

Data Availability

No additional data are available for this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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