Research Article

Almost Sure Limit Theorems for Multivariate General Standard Normal Sequences and Applications

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Under suitable conditions, the almost sure central limit theorems for the maximum of general standard normal sequences of random vectors are proved. The simulation of the almost sure convergence for the maximum is firstly performed, which helps to visually understand the theorems by applying to two new examples.

1. Introduction

The classical extreme value theory studies extreme phenomena in nature and human society, such as floods, hurricane, stock market crash, megaseism, and so on [1–3]. Almost sure (a.s.) convergence is a nice behavior of the various ways of convergence [4, 5]. The almost sure central limit theorem (ASCLT) on partial sums is initiated by Brosamler and Schatte for independent identically distributed (i.i.d.) random variables [6, 7]. Let $X_1, X_2, \dots$ be i.i.d. random variables with $E(X_n) = 0$, $\text{Var}(X_n) = 1$ and $T_n = \sum_{k=1}^n X_k$. Under some regularity conditions, we have

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} I\left(\frac{T_k}{\sqrt{k}} \leq x\right) = \Phi(x), \quad \text{a.s.}, \quad (1)$$

for any x , where I , $\log n$, and $\Phi(x)$ denote the indicator function, the natural logarithm, and the standard normal distribution function. Later, Lacey and Philipp prove that equation (1) holds when f is a bound Lipschitz function [8]. The almost sure central limit theorem on maximum of i.i.d random variables is firstly discovered by Fahrnar and Stadtmüller and Cheng et al., respectively [5, 9]. Berkes et al. consider the general nonlinear functionals case [10]. Csáki and Gondigdanzan investigate ASCLT for the

maximum of a stationary Gaussian sequences [11]. Chen and Lin extend to the nonstationary Gaussian sequences [12]. Chen and Peng provide an ASCLT for the maxima of multivariate stationary Gaussian sequences under some mild conditions [13, 14].

In recent years, the almost sure convergence of the joint distribution is explored. The almost sure limit theorems related to the maxima of the complete and incomplete samples for stationary sequences are considered [15]. Peng et al. extend the result to the maxima and minima of the complete and incomplete samples for weakly dependent stationary Gaussian sequences [16]. The ASCLT for the maxima and sums of i.i.d. random variables is investigated by Zang et al. [17]. Tan and Wang consider ASCLT for the maxima and sums of standardized stationary Gaussian sequences under some conditions [18]. The almost sure convergence of the maxima and minima for stationary sequences holds under some dependence conditions [19]. The behavior of almost sure convergence of the maxima and the minima for a strongly dependent stationary Gaussian sequence is extended to multivariate vectors [20, 21].

In this paper, we study ASCLT of the multivariate general standard normal sequences under some suitable conditions. Throughout this paper, $\{X_1, X_2, \dots\}$ is a standardized normal sequence of d -dimensional random vector, i.e., each

component of the random vector has a zero mean and a unit standard deviation. Set

$$\begin{aligned} r_{ij}(p) &= \text{Cov}(X_i(p), X_j(p)), \\ r_{ij}(p, q) &= \text{Cov}(X_i(p), X_j(q)), \quad 1 \leq p \neq q \leq d, \\ M_{k,n} &= (M_{k,n}(1), \dots, M_{k,n}(d)), \\ M_n &= M_{0,n}, \\ M_{k,n}(p) &= \max_{k+1 \leq i \leq n} X_i(p), \\ M_n(p) &= M_{0,n}(p), \quad p = 1, \dots, d, \end{aligned} \quad (2)$$

and $u_n = (u_n(1), \dots, u_n(d))$ and $v_n = (v_n(1), \dots, v_n(d))$ are two real vectors. $u_n > v_n$ implies $u_n(p) > v_n(p)$ for all $p = 1, \dots, d$, and $a \ll b$ stands for $a = O(b)$. Finally, write $a_n = (2 \log n)^{1/2}$ and $b_n = a_n - (1/2)a_n^{-1} \log(4\pi \log n)$. In this paper, the following results are obtained (proofs are given in Section 2).

2. Results

2.1. Theorems

Theorem 1. Let $\{X_n\}_{n=1}^\infty$ be a standardized normal sequence of d -dimensional random vector, satisfying $|r_{ij}(p)| \leq \rho_{|j-i|}(p)$, $|r_{ij}(p, q)| \leq \rho_{|j-i|}(p, q)$ for $i, j = 1, 2, \dots$ and

$$\begin{aligned} \sup_{\substack{n \geq 1 \\ 1 \leq p \leq d}} \rho_n(p) &< 1, \\ \sup_{\substack{n \geq 0 \\ 1 \leq p \neq q \leq d}} \rho_n(p, q) &< 1, \end{aligned} \quad (3)$$

as $n \rightarrow \infty$ and

$$\begin{aligned} \rho_n(p) \log n (\log \log n)^{(1+\varepsilon)} &= O(1), \\ \rho_n(p, q) \log n (\log \log n)^{(1+\varepsilon)} &= O(1). \end{aligned} \quad (4)$$

Let the levels $\{u_{ni}(p)\}$ be such that $n(1 - \Phi(\lambda_n(p)))$ is bounded and $\lambda_n(p) = \min_{1 \leq i \leq n} u_{ni}(p) \geq c(\log n)^{1/2}$ for some $c > 0$. If $\sum_{i=1}^n (1 - \Phi(u_{ni}(p))) \rightarrow \tau_p$ for some $\tau_p \geq 0$ and $p = 1, \dots, d$, then

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} I\left(\bigcap_{i=1}^k (X_i \leq u_{ki})\right) = \prod_{p=1}^d \exp(-\tau_p), \quad \text{a.s.} \quad (5)$$

In particular, let $u_n(p) = a_n^{-1}x_p + b_n$, $x_p \in \mathbb{R}$ for $p = 1, \dots, d$, and then

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} I(M_k \leq u_k) = \prod_{p=1}^d \exp(-\exp(-x_p)), \quad \text{a.s.} \quad (6)$$

Theorem 2. Let $\{X_n\}_{n=1}^\infty$ be a standardized normal d -dimensional random vector sequence satisfying

- (a) $\delta = \max_{p \neq q} (\sup_{n \geq 1} |r_n(p)|, \sup_{n \geq 0} \rho_n(p, q)) < 1$
- (b) there exists $\gamma \geq ((2(1 + \delta))/(1 - \delta))$, such that

$$\begin{aligned} \frac{1}{n^2} \sum_{p=1}^d \sum_{1 \leq i < j \leq n} |r_{ij}(p)| \exp(\gamma |r_{ij}(p)| \log(j-i)) \\ \ll (\log \log n)^{-(1+\varepsilon)}, \\ \frac{1}{n^2} \sum_{1 \leq p \neq q \leq n} \sum_{1 \leq i < j \leq n} |r_{ij}(p, q)| \exp(\gamma |r_{ij}(p, q)| \log(j-i)) \\ \ll (\log \log n)^{-(1+\varepsilon)}, \end{aligned} \quad (7)$$

where $\varepsilon > 0$.

If $n(1 - \Phi(u_n(p))) \rightarrow \tau_p$, $0 \leq \tau_p < \infty$, for $p = 1, 2, \dots$, d , then

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} I(M_k \leq U_k) = \prod_{p=1}^d \exp(-\tau_p), \quad \text{a.s.} \quad (8)$$

Let $u_n(p) = a_n^{-1}x_p + b_n$, where x_p are real numbers for $p = 1, \dots, d$, and then

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} I(M_k \leq u_k) = \prod_{p=1}^d \exp(-\exp(-x_p)), \quad \text{a.s.} \quad (9)$$

We shall be concerned with the maxima $M_n^* = \max_{1 \leq i \leq n} Y_i$ for a normal sequence $\{Y_n\}_{n=1}^\infty$ given by $Y_n = X_n + m_n$ where $\{X_n\}_{n=1}^\infty$ is a standardized normal sequence of d -dimensional vector with each component having a zero mean and unit standard deviation, and $\{m_n\}$ are added deterministic components. Define $m_n = (m_n(1), \dots, m_n(d))$, and we shall assume that the constant $m_n(p)$ is such that

$$\begin{aligned} \beta_n(p) &= \max_{1 \leq i \leq n} |m_i(p)| = o(\sqrt{\log n}), \\ p &= 1, \dots, d, \text{ as } n \rightarrow \infty. \end{aligned} \quad (10)$$

Hence, the mean and the deviation of each component of $\{Y_n\}$ are $m_n(p)$ and 1, and

$$\begin{aligned} \text{Cov}(Y_i(p), Y_j(p)) &= r_{ij}(p), \\ \text{Cov}(Y_i(p), Y_j(q)) &= r_{ij}(p, q), \\ 1 &\leq p \neq q \leq d. \end{aligned} \quad (11)$$

Let $m_n^* = (m_n^*(1), \dots, m_n^*(d))$ be chosen such that $|m_n^*(p)| \leq \beta_n(p)$ and

$$a_n \left(\max_{1 \leq i \leq n} m_i(p) - m_i^*(p) \right) \leq D, \quad p = 1, \dots, d, \quad (12)$$

where D is a constant.

Theorem 3. Let $\{Y_n\}_{n=1}^\infty$ be a normal sequence of d -dimensional vector defined above with $\{X_n\}_{n=1}^\infty$ satisfying the conditions of Theorem 1. Then,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} I(a_k(M_k^* - b_k^* - m_k^*) \leq x) \\ = \prod_{p=1}^d \exp(-e^{-x_p}), \quad \text{a.s.}, \end{aligned} \quad (13)$$

where $M_n^* = (M_n^*(1), \dots, M_n^*(d))$ with $M_n^*(p) = \max_{1 \leq i \leq n} Y_i(p)$ and $b_n^* = b_n I_d$ with $I_d = \frac{(1, \dots, 1)}{d}$ and $x = (x_1, \dots, x_d)$.

2.2. Proofs of the Main Results. We need the following lemmas for the proofs of the main results.

Lemma 1. Let $\{\xi_n\}_{n=1}^\infty$ and $\{\eta_n\}_{n=1}^\infty$ be standardized normal sequences of d -dimensional random vector with $r_{ij}^0(p) = \text{Cov}(\xi_i(p), \xi_j(p))$, $r_{ij}^0(p, q) = \text{Cov}(\xi_i(p), \xi_j(q))$ and $r_{ij}'(p) = \text{Cov}(\eta_i(p), \eta_j(p))$, $r_{ij}'(p, q) = \text{Cov}(\eta_i(p), \eta_j(q))$. Let $\rho_{ij}(p) = \max(|r_{ij}^0(p)|, |r_{ij}'(p)|)$, $\rho_{ij}(p, q) = \max(|r_{ij}^0(p, q)|, |r_{ij}'(p, q)|)$ and $u_1, \dots, u_n$ be real vectors. Then,

$$\begin{aligned} & |P(\xi_j \leq u_{nj}, j = 1, \dots, n) - P(\eta_j \leq u_{nj}, j = 1, \dots, n)| \\ & \leq K_1 \sum_{p=1}^d \sum_{1 \leq i < j \leq n} |r_{ij}^0(p) - r_{ij}'(p)| (1 - \rho_{ij}^2(p))^{-(1/2)} \exp\left(\frac{u_{ni}^2(p) + u_{nj}^2(p)}{2(1 + \rho_{ij}(p))}\right) \\ & + K_2 \sum_{1 \leq p \neq q \leq d} \sum_{1 \leq i \leq j \leq n} |r_{ij}^0(p, q) - r_{ij}'(p, q)| (1 - \rho_{ij}^2(p, q))^{-(1/2)} \exp\left(\frac{u_{ni}^2(p) + u_{nj}^2(q)}{2(1 + \rho_{ij}(p, q))}\right), \end{aligned} \quad (14)$$

for constants K_1, K_2 .

Proof. It follows from Theorem 4.2.1 in [22].

Lemma 2. Let $\{X_n\}_{n=1}^\infty$ be a standardized normal sequence of d -dimensional random vector satisfying equations (3) and (4), and $n(1 - \Phi(\lambda_n(p)))$ be bounded for $p = 1, \dots, d$, then

$$\begin{aligned} S_n^* &:= \sum_{p=1}^d \sum_{1 \leq i < j \leq n} |r_{ij}(p)| \exp\left(\frac{u_{ni}^2(p) + u_{nj}^2(p)}{2(1 + |r_{ij}(p)|)}\right) \\ &\ll (\log \log n)^{-(1+\epsilon)}, \end{aligned} \quad (15)$$

$$\begin{aligned} S_n &:= \sum_{1 \leq p \neq q \leq d} \sum_{1 \leq i \leq j \leq n} |r_{ij}(p, q)| \exp\left(\frac{u_{ni}^2(p) + u_{nj}^2(q)}{2(1 + |r_{ij}(p, q)|)}\right) \\ &\ll (\log \log n)^{-(1+\epsilon)}. \end{aligned} \quad (16)$$

Proof. By equation (3), we have $\sup_{1 \leq i \leq j \leq n, 1 \leq p \neq q \leq d} r_{ij}(p, q) = \delta < 1$. Define $\theta_n = e^{\eta(\lambda_n(p))^2}$ for $\eta < \eta_0 = ((1 - \delta)/(2(1 + \delta)))$. Split S_n into two parts, then

$$\begin{aligned} S_n &= \sum_{1 \leq p \neq q \leq d} \sum_{\substack{j-i \leq \theta_n \\ 1 \leq i \leq j \leq n}} |r_{ij}(p, q)| \exp\left(\frac{u_{ni}^2(p) + u_{nj}^2(q)}{2(1 + |r_{ij}(p, q)|)}\right) \\ &+ \sum_{1 \leq p \neq q \leq d} \sum_{\substack{j-i > \theta_n \\ 1 \leq i \leq j \leq n}} |r_{ij}(p, q)| \exp\left(\frac{u_{ni}^2(p) + u_{nj}^2(q)}{2(1 + |r_{ij}(p, q)|)}\right) \\ &:= S_n^{(1)} + S_n^{(2)}. \end{aligned} \quad (17)$$

Notice that $n(1 - \Phi(\lambda_n(p)))$ is bounded for $p = 1, \dots, d$, which implies that

$$\begin{aligned} \exp\left(\frac{\lambda_n^2(p)}{2}\right) &\sim \frac{C\sqrt{2\pi}\lambda_n(p)}{n}, \\ \lambda_n(p) &\sim \sqrt{2\log n}, \end{aligned} \quad (18)$$

for some constant $C > 0$, $p = 1, \dots, d$, and then

$$\begin{aligned} S_n^{(1)} &\leq \sum_{1 \leq p \neq q \leq d} \delta \sum_{\substack{j-i \leq \theta_n \\ 1 \leq i \leq j \leq n}} \exp\left(\frac{u_{ni}^2(p) + u_{nj}^2(q)}{2(1 + \delta)}\right) \\ &\leq \sum_{1 \leq p \neq q \leq d} \delta \theta_n n \exp\left(\frac{\lambda_n^2(p) + \lambda_n^2(q)}{2(1 + \delta)}\right) \\ &\ll \sum_{1 \leq p \neq q \leq d} n^{2(\eta - \eta_0)} (\log n)^{(1/(1+\delta)) - \eta} \\ &\ll (\log \log n)^{-(1+\epsilon)}, \end{aligned} \quad (19)$$

for $\eta_0 > \eta$.

Now, consider $S_n^{(2)}$. Let $\delta_t = \sup_{j \geq t} \rho_j(p, q)$, $1 \leq p \neq q \leq d$ and $t = n^\alpha$. Since $n(1 - \Phi(\lambda_n(p)))$ is bounded as $n \rightarrow \infty$

such that $\lambda_n^2(p) > c \log n$ for some $c > 0$ and $\theta_n = \exp(\eta \lambda_n^2(p)) \geq \exp(c^2 \eta \log n) = n^\alpha$, $\alpha = c^2 \eta > 0$, and then

$$\begin{aligned}
 S_n^{(2)} &\leq \sum_{1 \leq p \neq q \leq d} \sum_{\substack{j-i > \theta_n \\ 1 \leq i \leq j \leq n}} \delta_t \exp\left(-\frac{\lambda_n^2(p) + \lambda_n^2(q)}{2(1 + \delta_t)}\right) \\
 &\leq \sum_{1 \leq p \neq q \leq d} \delta_t n^2 \exp\left(-\frac{\lambda_n^2(p) + \lambda_n^2(q)}{2(1 + \delta_t)}\right) \\
 &\ll \sum_{1 \leq p \neq q \leq d} \delta_t (n(1 - \Phi(\lambda_n(p)))) (n(1 - \Phi(\lambda_n(q)))) \lambda_n(p) \exp\left(\frac{\lambda_n^2(p)}{2} \delta_t\right) \lambda_n(q) \exp\left(\frac{\lambda_n^2(q)}{2} \delta_t\right) \\
 &\ll \sum_{1 \leq p \neq q \leq d} \delta_t \lambda_n(p) \exp\left(\frac{\lambda_n^2(p)}{2} \delta_t\right) \lambda_n(q) \exp\left(\frac{\lambda_n^2(q)}{2} \delta_t\right) \\
 &\ll \sum_{1 \leq p \neq q \leq d} \sqrt{\delta_t \lambda_n^2(p)} \sqrt{\delta_t \lambda_n^2(q)} \exp\left(\frac{\lambda_n^2(p)}{2} \delta_t\right) \exp\left(\frac{\lambda_n^2(q)}{2} \delta_t\right) \\
 &\ll \sum_{1 \leq p \neq q \leq d} \delta_t \log t \exp\left(\frac{1}{\alpha} \delta_t \log t\right).
 \end{aligned} \tag{20}$$

As

$$\delta_t \log t (\log \log t)^{-(1+\varepsilon)} \leq \sup_{j \geq t} \rho_j(p, q) \log j (\log \log j)^{-(1+\varepsilon)} = O(1), \tag{21}$$

we have

$$\delta_t \log t \ll (\log \log n)^{-(1+\varepsilon)} \rightarrow 0, \tag{22}$$

and $\exp((\lambda_n^2(p)/2)\delta_t)$ is bounded. Then,

$$S_n^{(2)} \ll (\log \log n)^{-(1+\varepsilon)}. \tag{23}$$

Lemma 3. Let $\{X_n\}_{n=1}^\infty$ be a standardized normal d -dimensional random vector sequence satisfying the conditions (a) and (b) of Theorem 2, and then equations (15) and (16) hold.

Proof. The proof of Lemma 3 is similar to that of Lemma 2.

Lemma 4. Let $\{X_n\}_{n=1}^\infty$ be a standardized normal sequence of d -dimensional random vector satisfying Lemma 2 (Lemma 3), then

$$\left| \text{Cov}\left(I\left(\bigcap_{i=1}^k (X_i \leq u_{ki})\right), I\left(\bigcap_{i=n+1}^n (X_i \leq u_{ki})\right)\right) \right| \ll (\log \log n)^{-(1+\varepsilon)}. \tag{24}$$

Proof. By Lemma 1,

$$\begin{aligned}
 &\left| \text{Cov}\left(I\left(\bigcap_{i=1}^k (X_i \leq u_{ki})\right), I\left(\bigcap_{i=k+1}^n (X_i \leq u_{ni})\right)\right) \right| \\
 &= \left| P\left(\left(\bigcap_{i=1}^k (X_i \leq u_{ki})\right), \left(\bigcap_{i=k+1}^n (X_i \leq u_{ni})\right)\right) - P\left(\bigcap_{i=1}^k (X_i \leq u_{ki})\right) P\left(\bigcap_{i=k+1}^n (X_i \leq u_{ni})\right) \right| \\
 &\leq \sum_{p=1}^d \sum_{i=1}^k \sum_{j=k+1}^n |r_{ij}(p)| \exp\left(-\frac{u_{ki}^2(p) + u_{nj}^2(p)}{2(1 + |r_{ij}(p)|)}\right) \\
 &\quad + \sum_{1 \leq p \neq q \leq d} \sum_{i=1}^k \sum_{j=k+1}^n |r_{ij}(p, q)| \exp\left(-\frac{u_{ki}^2(p) + u_{nj}^2(q)}{2(1 + |r_{ij}(p, q)|)}\right) \\
 &:= D_n + E_n.
 \end{aligned} \tag{25}$$

Both D_n and E_n are of order $(\log \log n)^{-(1+\varepsilon)}$, but we will only prove $E_n \ll (\log \log n)^{-(1+\varepsilon)}$. In fact,

$$\begin{aligned}
 E_n &= \sum_{1 \leq p \neq q \leq d} \sum_{i=1}^k \sum_{j=k+1}^n |r_{ij}(p, q)| \exp \left(-\frac{u_{ki}^2(p) + u_{nj}^2(q)}{2(1 + |r_{ij}(p, q)|)} \right) \\
 &\leq \sum_{1 \leq p \neq q \leq d} \sum_{\substack{j-i < \theta_n \\ 1 \leq i \leq k \\ k+1 \leq j \leq n}} |r_{ij}(p, q)| \exp \left(-\frac{u_{ki}^2(p) + u_{nj}^2(q)}{2(1 + |r_{ij}(p, q)|)} \right) \\
 &\quad + \sum_{1 \leq p \neq q \leq d} \sum_{\substack{j-i < \theta_n \\ 1 \leq i \leq k \\ k+1 \leq j \leq n}} |r_{ij}(p, q)| \exp \left(-\frac{u_{ki}^2(p) + u_{nj}^2(q)}{2(1 + |r_{ij}(p, q)|)} \right) \\
 &:= E_n^1 + E_n^2.
 \end{aligned} \tag{26}$$

We have

$$\begin{aligned}
 E_n^1 &\leq \sum_{1 \leq p \neq q \leq d} \delta \sum_{\substack{j-i < \theta_n \\ 1 \leq i \leq k \\ k+1 \leq j \leq n}} \exp \left(-\frac{\lambda_k^2(p) + \lambda_n^2(q)}{2(1 + \delta)} \right) \\
 &\leq \sum_{1 \leq p \neq q \leq d} \delta k \theta_n \exp \left(-\frac{\lambda_k^2(p) + \lambda_n^2(q)}{2(1 + \delta)} \right) \\
 &\leq \sum_{1 \leq p \neq q \leq d} \delta k \theta_n \left(\frac{\lambda_k(p)}{k} \right)^{1/(1+\delta)} \left(\frac{\lambda_n(q)}{n} \right)^{1/(1+\delta)} \\
 &\leq \sum_{1 \leq p \neq q \leq d} \delta k^{1-(1/(1+\delta))} (\log k)^{1/2(1+\delta)} n^{2\eta-(1/(1+\delta))} (\log n)^{(1/2(1+\delta))-\eta} \\
 &\leq \sum_{1 \leq p \neq q \leq d} \delta n^{-2(\eta_0-\eta)} (\log n)^{(1/(1+\delta))-\eta} \\
 &\ll (\log \log n)^{-(1+\varepsilon)}, \\
 E_n^2 &\leq \sum_{1 \leq p \neq q \leq d} \sum_{\substack{j-i \geq \theta_n \\ 1 \leq i \leq k \\ k+1 \leq j \leq n}} \rho_{ij}(p, q) \exp \left(-\frac{u_{ki}^2(p) + u_{nj}^2(q)}{2(1 + \rho_{ij}(p, q))} \right) \\
 &\leq \sum_{1 \leq p \neq q \leq d} \delta_t \sum_{i=1}^k \exp \left(-\frac{u_{ki}^2(p)}{2(1 + \delta_t)} \right) \sum_{i=1}^n \exp \left(-\frac{u_{ni}^2(p)}{2(1 + \delta_t)} \right) \\
 &\leq \sum_{1 \leq p \neq q \leq d} \delta_t k \exp \left(-\frac{\lambda_k^2}{2} \right) \exp \left(\frac{1}{2} \lambda_k^2 \delta_t \right) n \exp \left(-\frac{\lambda_n^2}{2} \right) \exp \left(\frac{1}{2} \lambda_n^2 \delta_t \right) \\
 &\ll \sum_{1 \leq p \neq q \leq d} \delta_t \log n \exp(\lambda_n^2 \delta_t) \\
 &\ll (\log \log n)^{-(1+\varepsilon)}.
 \end{aligned} \tag{27}$$

Remark 1. Under the conditions of Theorem 2, Lemma 4 can be proved by Lemma 1 and Lemma 3.

Lemma 5. Let $\{X_n\}_{n=1}^\infty$ be a standardized normal sequence of d -dimensional random vector satisfying Lemma 2 (Lemma 3), then

$$E \left| I \left(\bigcap_{i=1}^n (X_i \leq u_{ni}) \right) - I \left(\bigcap_{i=k+1}^n (X_i \leq u_{ni}) \right) \right| \ll \frac{k}{n} + (\log \log n)^{-(1+\varepsilon)}, \quad \text{As } \sum_{i=1}^n (1 - \Phi(u_{ni}(p))) \longrightarrow \tau_p \text{ for } \tau_p \geq 0 \text{ and all } p = 1, \dots, d, \text{ then} \quad (28)$$

where E denotes the mean.

Proof.

$$\begin{aligned} & E \left| I \left(\bigcap_{i=1}^n (X_i \leq u_{ni}) \right) - I \left(\bigcap_{i=k+1}^n (X_i \leq u_{ni}) \right) \right| \\ &= P \left(\bigcap_{i=1}^n (X_i \leq u_{ni}) \right) - P \left(\bigcap_{i=k+1}^n (X_i \leq u_{ni}) \right) \\ &\leq \left| P \left(\bigcap_{i=k+1}^n (X_i \leq u_{ni}) \right) - \prod_{p=1}^d \prod_{j=k+1}^n \Phi(u_{nj}(p)) \right| \\ &\quad + \left| P \left(\bigcap_{i=1}^n (X_i \leq u_{ni}) \right) - \prod_{p=1}^d \prod_{j=1}^n \Phi(u_{nj}(p)) \right| \\ &\quad + \left| \prod_{p=1}^d \prod_{j=k+1}^n \Phi(u_{nj}(p)) - \prod_{p=1}^d \prod_{j=1}^n \Phi(u_{nj}(p)) \right| \\ &\triangleq A + B + C. \end{aligned} \quad (29)$$

By Lemma 1 and Lemma 2, we have

$$\begin{aligned} A &\ll (\log \log n)^{-(1+\varepsilon)}, \\ B &\ll (\log \log n)^{-(1+\varepsilon)}. \end{aligned} \quad (30)$$

Meanwhile,

$$\begin{aligned} 0 \leq C &= \prod_{j=k+1}^n \prod_{p=1}^d \Phi(u_{nj}(p)) - \prod_{j=1}^n \prod_{p=1}^d \Phi(u_{nj}(p)) \\ &\leq 1 - \prod_{j=1}^k \prod_{p=1}^d \Phi(u_{nj}(p)) \\ &\leq \sum_{p=1}^d \sum_{j=1}^k (1 - \Phi(u_{nj}(p))) \\ &\leq \sum_{p=1}^d k(1 - \Phi(\lambda_n(p))) \\ &\leq \sum_{p=1}^d \frac{k}{n} [n(1 - \Phi(\lambda_n(p)))] \\ &\ll \frac{k}{n}, \end{aligned} \quad (31)$$

from which the lemma follows.

Remark 2. Under the conditions of Theorem 2, Lemma 5 also can be proved using Lemma 1 and Lemma 3.

Lemma 6. Suppose that $\{X_n\}_{n=1}^\infty$ is a standardized normal sequence of d -dimensional random vector satisfying equation (3) of Theorem 1, and

$$\begin{aligned} \rho_n(p) \log n &\longrightarrow 0, \\ \rho_n(p, q) \log n &\longrightarrow 0. \end{aligned} \quad (32)$$

$$\lim_{k \rightarrow \infty} P \left(\bigcap_{i=1}^k (X_i \leq u_{ki}) \right) = \prod_{p=1}^d \exp(-\tau_p). \quad (33)$$

Especially, let $u_n(p) = (1/a_n)x_p + b_n$ with $x_p \in R$ for all $p = 1, \dots, d$; then,

$$\lim_{k \rightarrow \infty} P(M_k \leq u_k) = \prod_{p=1}^d \exp(-e^{-x_p}). \quad (34)$$

Proof. By Lemma 1, equation (32) can be obtained under the condition of equation (31), and the process of proof is similar to that of Lemma 2.

Lemma 7. Let $\{X_n\}_{n=1}^\infty$ be a standardized normal sequence of d -dimensional random vector satisfying (a) of Theorem 3.

(c) There exists $\gamma \geq (2(1+\delta))/(1-\delta)$, as $n \rightarrow \infty$

$$\frac{1}{n^2} \sum_{p=1}^d \sum_{1 \leq i < j \leq n} |r_{ij}(p)| \exp(\gamma |r_{ij}(p)| \log(j-i)) \longrightarrow 0, \quad (35)$$

$$\frac{1}{n^2} \sum_{1 \leq p \neq q \leq d} \sum_{1 \leq i \leq j \leq n} |r_{ij}(p, q)| \exp(\gamma |r_{ij}(p, q)| \log(j-i)) \longrightarrow 0, \quad (36)$$

where $\varepsilon > 0$.

If $n(1 - \Phi(u_n(p))) \longrightarrow \tau_p$, $0 \leq \tau_p < \infty$, for $p = 1, 2, \dots, d$, then

$$\lim_{k \rightarrow \infty} P(M_k \leq U_k) = \prod_{p=1}^d \exp(-\tau_p). \quad (37)$$

Let $u_n(p) = (1/a_n)x_p + b_n$ with $x_p \in R$ for all $p = 1, \dots, d$, then

$$\lim_{k \rightarrow \infty} P(M_k \leq u_k) = \prod_{p=1}^d \exp(-e^{-x_p}). \quad (38)$$

Proof. By Lemma 1, equation (36) can be obtained using equations (34) and (35), the method of proof is similar to that of Lemma 6

Lemma 8. Let $\{Y_n\}_{n=1}^\infty$ be a sequence of bounded random variables. If

$$\text{Var} \left(\sum_{k=1}^n \frac{1}{k} Y_k \right) \ll (\log n)^2 (\log \log n)^{-(1+\varepsilon)}, \quad (39)$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} (Y_k - EY_k) = 0, \quad \text{a.s.} \quad (40)$$

Proof. See Lemma 3.1 in [11].

Proof of Theorem. 1. Let $\alpha_k = I(\cap_{i=1}^n (X_i \leq u_{ki}))$, and

$$\begin{aligned} \text{Var} \left(\sum_{k=1}^n \frac{1}{k} \alpha_k \right) &= \sum_{k=1}^n \frac{1}{k^2} \text{Var}(\alpha_k) + 2 \sum_{1 \leq k < l \leq n} \frac{\text{Cov}(\alpha_k, \alpha_l)}{kl} \\ &\leq \sum_{k=1}^n \frac{1}{k^2} + 2 \sum_{1 \leq k < l \leq n} \frac{\text{Cov}(\alpha_k, \alpha_l)}{kl}. \end{aligned} \quad (41)$$

By Lemma 4 and Lemma 5,

$$\begin{aligned} |\text{Cov}(\alpha_k, \alpha_l)| &= \left| \text{Cov} \left(I \left(\bigcap_{i=1}^k (X_i \leq u_{ki}) \right), I \left(\bigcap_{i=1}^l (X_i \leq u_{li}) \right) \right) \right| \\ &\leq \left| \text{Cov} \left(I \left(\bigcap_{i=1}^k (X_i \leq u_{ki}) \right), \right. \right. \\ &\quad \left. \left. I \left(\bigcap_{i=1}^l (X_i \leq u_{li}) \right) - I \left(\bigcap_{i=k+1}^l (X_i \leq u_{li}) \right) \right) \right| \\ &\quad + \left| \text{Cov} \left(I \left(\bigcap_{i=1}^k (X_i \leq u_{ki}) \right), I \left(\bigcap_{i=k+1}^l (X_i \leq u_{li}) \right) \right) \right| \\ &\leq 2E \left| I \left(\bigcap_{i=1}^k (X_i \leq u_{ki}) \right) - I \left(\bigcap_{i=k+1}^l (X_i \leq u_{li}) \right) \right| \\ &\quad + \left| \text{Cov} \left(I \left(\bigcap_{i=1}^k (X_i \leq u_{ki}) \right), I \left(\bigcap_{i=k+1}^l (X_i \leq u_{li}) \right) \right) \right| \\ &\ll \frac{k}{l} + (\log \log n)^{-(1+\varepsilon)}, \end{aligned}$$

$$\begin{aligned} \text{Var} \left(\sum_{k=1}^n \frac{1}{k} \alpha_k \right) &\leq \sum_{k=1}^n \frac{1}{k^2} + 2 \sum_{1 \leq k < l \leq n} \frac{1}{kl} \left(\frac{k}{l} + (\log \log l)^{-(1+\varepsilon)} \right) \\ &= \sum_{k=1}^n \frac{1}{k^2} + 2 \sum_{1 \leq k < l \leq n} \frac{1}{l^2} + 2 \sum_{1 \leq k < l \leq n} \frac{1}{kl} (\log \log l)^{-(1+\varepsilon)} \\ &\ll \log n + 2 \sum_{l=3}^n \frac{1}{l (\log \log l)^{(1+\varepsilon)}} \sum_{k=1}^{l-1} \frac{1}{k} \\ &\ll \log n + 2 \sum_{l=3}^n \frac{\log l}{l (\log \log l)^{(1+\varepsilon)}} \\ &\ll (\log n)^2 (\log \log n)^{-(1+\varepsilon)}. \end{aligned} \quad (42)$$

By Lemma 6 and Lemma 8, the proof of the theorem is completed.

Proof of Theorem. 2. Under the conditions of Theorem 2, using Lemma 4 and Lemma 5, we have

$$\text{Var} \left(\sum_{k=1}^n \frac{1}{k} \alpha_k \right) \ll (\log n)^2 (\log \log n)^{-(1+\varepsilon)}. \quad (43)$$

By Lemma 7 and Lemma 8, Theorem 2 is obtained.

Proof of Theorem. 3. Write $u_{ni}(p) = u_n(p) + m_n^*(p) - m_i(p)$ where $u_n(p) = (x_p/a_n) + b_n$, $p = 1, \dots, d$.

$$\begin{aligned} P(a_n(M_n^* - b_n^* - m_n^*) \leq x) &= P \left(M_n^* = \frac{1}{a_n} x + b_n^* + m_n^* \right) \\ &= P(M_n^* \leq u_n + m_n^*) \\ &= P(M_n \leq u_{ni}). \end{aligned} \quad (44)$$

By Theorem 1, it is needed to check that $n(1 - \Phi(\lambda_n(p)))$ is bounded with $\lambda_n(p) = \min_{1 \leq i \leq n} u_{ni}(p) = u_n(p) + m_n^*(p) - \max_{1 \leq i \leq n} m_i(p)$. Notice that $\beta_n(p) = \max_{1 \leq i \leq n} |m_i(p)| = o(\sqrt{\log n})$ as $n \rightarrow \infty$, $|m_n^*(p)| \leq \beta_n(p)$ and $a_n(\max_{1 \leq i \leq n} m_i(p) - m_i^*(p)) \leq D$, $p = 1, \dots, d$, and

$$\begin{aligned} u_n(p) \left(\max_{1 \leq i \leq n} m_i(p) - m_n^*(p) \right) &- \frac{1}{2} \left(\max_{1 \leq i \leq n} m_i(p) - m_n^*(p) \right)^2 \\ &= a_n \left(\max_{1 \leq i \leq n} m_i(p) - m_n^*(p) \right) \\ &\quad \cdot \left(1 - \frac{\log(4\pi \log n)}{a_n} + \frac{x_p}{a_n^2} - \frac{\max_{1 \leq i \leq n} m_i(p) - m_n^*(p)}{2a_n} \right) \\ &\leq D(1 + o(1)), \end{aligned} \quad (45)$$

for large n . Hence,

$$\begin{aligned} n(1 - \Phi(\lambda_n(p))) &\sim n\lambda_n^{-1}(p) \exp \left(-\frac{1}{2} \left(u_n(p) + m_n^*(p) - \max_{1 \leq i \leq n} m_i(p) \right)^2 \right) \\ &\sim n(1 - \Phi(u_n(p))) \exp \left(-u_n(p) \left(m_n^*(p) - \max_{1 \leq i \leq n} m_i(p) \right) \right. \\ &\quad \left. - \frac{1}{2} \left(m_n^*(p) - \max_{1 \leq i \leq n} m_i(p) \right)^2 \right) \\ &\leq D'. \end{aligned} \quad (46)$$

$n(1 - \Phi(\lambda_n(p)))$ is bounded and satisfies the conditions of Theorem 1. As $u_n(p) = (x_p/a_n) + b_n$, Theorem 3 can be proved.

TABLE 1: The rate of convergence.

N	10^3	10^4	10^5	10^6	10^7	10^8	10^9
Value	1.0836	1.0627	1.0501	1.0408	1.0358	1.0318	1.0279

^a N denotes the number of random variables.

FIGURE 1: Simulating almost sure central limit theorem for one-dimensional random variable sequence.

2.3. Simulation. The almost sure central limit theorems for the maximum of the multivariate normal sequences are complete. Simulating the almost sure convergence for the maxima, which visually describe the theorems, is not performed in previous papers [12, 13, 15, 16, 23–26]. To construct a suitable standardized nonstationary normal sequence is the key to the simulation.

Let $\{Z_1, Z_2, \dots\}$ be an independent random sequence from the standardized normal distribution $N(0, 1)$. Set $X_i = (1/\sqrt{i})Z_1 + ((\sqrt{i-1})/\sqrt{i})Z_i$, then $E(X_i) = 0$ and $\text{Var}(X_i) = 1$, that is, $X_i \sim N(0, 1)$, $i = 1, 2, \dots$. As the correlation coefficient of X_i and X_j , $r_{ij} = \text{corr}(X_i, X_j) = (1/\sqrt{ij})^{j=i+n} (1/(\sqrt{i(n+i)}))$, $\{X_1, X_2, \dots\}$ be a standardized nonstationary normal random variable sequence. For $r_{ij} \leq (1/\sqrt{n})$, the sequence satisfies equations (3) and (4). When $\{X_1, X_2, \dots\}$ are one-dimensional variables and $\lambda_n = a_n^{-1}x + b_n$, equation (6) of Theorem 2 holds.

Theorem 2 is simulated based on the standardized nonstationary normal random variable sequence $\{X_1, X_2, \dots\}$ by Matlab. N denotes the number of random variable sequence $\{X_1, X_2, \dots\}$ and $M_k = \max_{1 \leq i \leq k} X_i$. $a_n = (2 \log n)^{(1/2)}$, $b_n = a_n - (1/2)a_n^{-1} \log(4\pi \log n)$, especially $a_1 = 1$, $b_1 = 0$ and $\lambda_n = a_n^{-1}x + b_n$. We note that $(1/(\log n)) \sum_{k=1}^n (1/k)$ is greater than 1 and converges to 1, but the rate of convergence is slow. The value $(1/(\log n)) \sum_{k=1}^n 1/k$ and the number of the random variables N are displayed in Table 1. The fitting effect gets better with the number of random variable N increasing. The detailed simulation is shown in Figure 1.

FIGURE 2: Simulating almost sure central limit theorem for two-dimensional random variable sequence.

Let $Y_i = -(1/\sqrt{i})Z_1 + ((\sqrt{i-1})/\sqrt{i})Z_i$, then $E(Y_i) = 0$, and $\text{Var}(Y_i) = 1$, $Y_i \sim N(0, 1)$, $i = 1, 2, \dots$. $\xi_i = (X_i, Y_i)$ is a two-dimensional random vector.

$$r_{ij}(2) = \text{corr}(Y_i, Y_j) = \frac{1}{\sqrt{ij}} (i \neq j),$$

$$r_{ij}(1, 2) = \text{corr}(X_i, Y_j) = \begin{cases} \frac{1}{\sqrt{ij}}, & i \neq j, \\ \frac{i-2}{i}, & i = j. \end{cases} \quad (47)$$

$\{\xi_1, \xi_2, \dots\}$ is a standardized nonstationary normal sequence of two-dimensional random vector satisfying equations (3) and (4), and then Theorem 2 holds as $\{\xi_1, \xi_2, \dots\}$ is a 2-dimensional random vector. The detailed simulation is shown in Figure 2.

3. Conclusions

The extreme value theory deals with extreme phenomena which are less likely to occur, but more harmful [1–3]. Almost sure limit theorems, nice behavior of convergence, for the maxima of normal vector sequences are put forward and proved. Figures 1 and 2 display good simulation for two examples and show the rates of the convergence visually, which promote the intuitive understanding about the almost sure central limit theorems for the maximum of nonstationary normal sequences of random vectors.

Data Availability

All data generated or analyzed during this study are included within this manuscript. If necessary, other relevant data can be made available from the corresponding author via mail upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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