Research Article

Vertex-Disjoint Paths in a 3-Ary n -Cube with Faulty Vertices

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The construction of vertex-disjoint paths (disjoint paths) is an important research topic in various kinds of interconnection networks, which can improve the transmission rate and reliability. The k -ary n -cube is a family of popular networks. In this paper, we determine that there are m ($2 \leq m \leq n$) disjoint paths in 3-ary n -cube covering $Q_n^3 - F$ from S to T (many-to-many) with $|F| \leq 2n - 2m$ and from s to T (one-to-many) with $|F| \leq 2n - m - 1$ where s is in a fault-free cycle of length three.

1. Introduction

Vertex-disjoint paths (disjoint paths for short) are a set of paths in a graph that they do not share any vertices. A disjoint path cover of a graph is a disjoint path between two different vertices, and it covers all vertices of the graph. The disjoint path cover problem has been applied in many fields such as software testing, database design, and code optimization. Actually, vertex-disjoint paths can accelerate data transmission by providing parallel communication paths to avoid communication congestion. Besides, disjoint paths enhance the robustness of vertex failure and load balancing capability [1] and accelerate the transmission of large amounts of data by splitting data into disjoint communication paths of multiple vertices. The problem of many-to-many disjoint paths is an important problem in networks. It has attracted much attention because of its application in fault-tolerant routings for high-performance interconnection network.

Depending on the number of sources or sinks, there are one-to-one, one-to-many, and many-to-many disjoint paths problems. Among them, the one-to-one disjoint path is what we usually call the Hamilton path problem. A nonbipartite graph G is one-to-one m -disjoint path coverable if there is an m -disjoint path cover between any two vertices of G . A bipartite graph is one-to-one m -disjoint path coverable if there is an m -disjoint path cover between any two vertices in

different partite sets. Shih and Kao [1] constructed the one-to-one m -disjoint path cover of k -ary n -cube for $1 \leq m \leq 2n$, $n \geq 2$ and $k \geq 3$. Li et al. [2] studied many-to-many k -disjoint paths in hypercubes with $f \leq 2n - 2k - 2$, $1 \leq k \leq n - 1$ and each fault-free vertex has at least two fault-free neighbors. Chen in [3] considered many-to-many k disjoint (S, T) -paths in the hypercube with $f_v + f_e \leq n - k - 1$, $1 \leq k \leq n - 1$ and any two sets S and T of k fault-free vertices in different parts. Zhang and Wang [4] proved that many-to-many m disjoint paths cover in k -ary n -cubes with even $k \geq 4$ and $1 \leq m \leq 2n - 1$. So, we consider some disjoint path problems about 3-ary n -cubes. There is also research on other graphs. Such as one-to-many disjoint paths of (n, k) -star graph [5] and hyper-star networks [6] and one-to-one disjoint path covers on alternating group graphs [7].

For a set $S = \{s_1, s_2, \dots, s_k\}$ of k sources and a set $T = \{t_1, t_2, \dots, t_k\}$ of k sinks in $V(G)$ with $S \cap T = \emptyset$, the many-to-many k -disjoint path problem is to determine whether there exist k disjoint paths each joining a source and a sink. In other words, we need to find k disjoint paths $P_1, P_2, \dots, P_k$ in G joining S and T , which cover all the vertices in the graph; that is, $\bigcup_{1 \leq i \leq k} V(P_i) = V(G)$ and $V(P_i) \cap V(P_j) = \emptyset$ for $i \neq j$ and $1 \leq i, j \leq k$. In addition, this problem is also subclassified into paired and unpaired types. In paired type, each source should be joined to a specific sink; that is, s_i should be joined to t_i . Otherwise, each source can be joined to an arbitrary sink; that is, s_i could be joined to

t_j for $i \neq j$. We study the unpaired type in this paper. Specially, if $S = \{s\}$, then it is called one-to-many disjoint paths; that is, $V(P_i) \cap V(P_j) = \{s\}$ for all $i \neq j$ and $1 \leq i, j \leq k$.

With the development of science and the expansion of the scale of computer system, people have more higher requirements for the reliability of network. However, computer failure is inevitable in a system. So, the disjoint path cover problem is also extended to a graph with some faulty elements F (vertices and/or edges). A graph G is called f -fault hamiltonian-connected if each pair of vertices are joined by a hamiltonian path in $G - F$ for any faulty elements $|F| \leq f$.

In this paper, we study m disjoint paths of 3-ary n -cube covering $Q_n^3 - F$ with faulty vertices $|F| \leq 2n - 2m$ from S to T for $|S| = |T| = m$ in Theorem 1. Besides, we also consider m disjoint paths of 3-ary n -cube covering $Q_n^3 - F$ with faulty vertices $|F| \leq 2n - m - 1$ from s to T where $|T| = m$ and s is in a fault-free cycle of length three in Theorem 2.

2. Preliminaries

2.1. Notations. An interconnection network can be represented by a graph $G = (V, E)$, where V represents the vertex set and E represents the edge set. Let G be a simple undirected graph. For a vertex v , $N_G(v)$ is the set of vertices adjacent to v in G . The path $P = \langle v_1, v_2, \dots, v_{n-1}, v_n \rangle$ is a sequence of adjacent vertices where all the vertices are distinct, which is denoted by $P = (v_1, v_n)$. If the path P satisfies $v_1 = v_n$, it is called a cycle. In addition, $P^{-1} = \langle v_n, v_{n-1}, \dots, v_2, v_1 \rangle$. For $1 \leq i < j \leq n$, $P[v_i, v_j]$ denotes the section from v_i to v_j in P . For $uv_1, v_n w \in E$, $\{uv_1\} \cup P \cup \{v_n w\} = \langle u, v_1, \dots, v_{n-1}, v_n, w \rangle$. A path or cycle which contains every vertex of a graph is called a Hamilton path (hamiltonian path) or Hamilton cycle (hamiltonian cycle) of the graph. A graph is hamiltonian if it contains a Hamilton cycle. A graph is hamiltonian-connected if there exists a Hamilton path between any two distinct vertices of the graph. For graph-theoretical terminology and notation not defined here, we follow [8].

2.2. The k -Ary n -Cube Q_n^k

Definition 1. The k -ary n -cube Q_n^k ($n \geq 1$) is a graph with k^n vertices, each of which has the form $u = u_{n-1}u_{n-2} \dots u_0$, where $0 \leq u_i \leq k-1$ for every $0 \leq i \leq n-1$. Two vertices $u = u_{n-1}u_{n-2} \dots u_0$ and $v = v_{n-1}v_{n-2} \dots v_0$ in Q_n^k are adjacent if and only if there exists an integer j , $0 \leq j \leq n-1$, such that $u_j = v_j \pm 1 \pmod{k}$ and $u_i = v_i$ for every $i \in \{0, 1, \dots, n-1\} \setminus \{j\}$. For clarity of presentation, we omit writing “ $\pmod{k}$ ” in similar expressions for the remainder of the paper.

We discuss the 3-ary n -cube Q_n^3 with $2n$ regular. Obviously, Q_1^3 is a cycle of length 3, and Q_2^3 is a 3×3 wrap-around mesh. We can partition Q_n^3 along j -dimension, $0 \leq j \leq n-1$, by deleting all the j -dimension edges, into three disjoint subcubes, $Q_n^3[0]$, $Q_n^3[1]$ and $Q_n^3[2]$ (abbreviated as $Q[0]$, $Q[1]$ and $Q[2]$, if there are no ambiguities). $Q[j]$ is

isomorphic to Q_{n-1}^3 for $0 \leq j \leq 2$. For $0 \leq i, j \leq 2$ and $i \neq j$, the edge set between $Q[i]$ and $Q[j]$ is a perfect matching.

Lemma 1 (see [9]). *The k -ary n -cube is hamiltonian-connected when k is odd.*

Lemma 2 (see [10]). *Assume k is odd with $k \geq 3$. Let F be a set of faulty vertices and/or edges in a k -ary n -cube Q_n^k .*

- (1) *If $|F| \leq 2n - 2$, then the graph $Q_n^k - F$ is hamiltonian.*
- (2) *If $|F| \leq 2n - 3$, then the graph $Q_n^k - F$ is hamiltonian-connected.*

Lemma 3 (see [11]). *Let $n \geq 2$, $F \subset V(Q_n^3)$, $|F| \leq 2n - 4$. If s_1, t_1, s_2, t_2 are four vertices in $Q_n^3 - F$, then there are two vertex-disjoint paths $P_1 = (s_1, t_1)$ and $P_2 = (s_2, t_2)$ such that $V(P_1) \cup V(P_2) = V(Q_n^3 - F)$.*

3. Many-to-Many Disjoint Paths

Theorem 1. *Let Q_n^3 be a 3-ary n -cube for $n \geq 2$. Let F be a vertex faulty set of Q_n^3 . Let $S, T \subset V(Q_n^3)$ with $2 \leq |S| = |T| = m \leq n$. If $|F| \leq 2n - 2m$, then there are m disjoint paths from S to T covering $Q_n^3 - F$.*

Proof. We prove the theorem by induction on m . When $m = 2$, $|S| = |T| = 2$ and $|F| \leq 2n - 4$. By Lemma 3, the theorem holds. Assume the theorem holds on $m - 1$. We prove the theorem holds on m for $n \geq m \geq 3$.

Suppose there is $s_i t_j \in E(Q_n^3)$ for $1 \leq i, j \leq m$. Note $|F \cup \{s_i, t_j\}| \leq 2n - 2m + 2 = 2n - 2(m - 1)$. By the induction hypothesis, there are $(m - 1)$ disjoint paths P_k ($1 \leq k \leq m - 1$) from $S \setminus \{s_i\}$ to $T \setminus \{t_j\}$ covering $Q_n^3 - F - \{s_i, t_j\}$. Let $P_m = \langle s_i, t_j \rangle$. Then, the theorem holds. Thus, we suppose $s_i t_j \notin E(Q_n^3)$ for all $1 \leq i, j \leq m$.

We decompose Q_n^3 into three isomorphic subgraphs $Q[0], Q[1], Q[2]$ along one dimension. Let $F_i = F \cap V(Q[i])$, $S_i = S \cap V(Q[i])$ and $T_i = T \cap V(Q[i])$ for $0 \leq i \leq 2$. Let $S = \{s_1, \dots, s_m\}$ and $T = \{t_1, \dots, t_m\}$. If $s_i, t_i \in V(Q[j])$ for $1 \leq i \leq m$ and $0 \leq j \leq 2$, then let $s_i^k = N(s_i) \cap V(Q[k])$ and $t_i^k = N(t_i) \cap V(Q[k])$ for $k \in (\{0, 1, 2\} \setminus \{j\})$.

Case 1. $|S_i| = |T_i|$ for all $0 \leq i \leq 2$.

Case 1.1. $|S_i| = |T_i| = m$ for some $0 \leq i \leq 2$.

Without loss of generality, let $|S_0| = |T_0| = m$. Then, $|S_1| = |S_2| = |T_1| = |T_2| = 0$. Let $S_0 = \{s_1, \dots, s_m\}$ and $T_0 = \{t_1, \dots, t_m\}$.

Case 1.1.1. $|F_0| = 2n - 2m$.

Then, $|F_1| = |F_2| = 0$. Let $S'_0 = S_0 \setminus \{s_m\}$ and $T'_0 = T_0 \setminus \{t_m\}$. Then, $|S'_0| = |T'_0| = m - 1$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $R_i = (s_i, t_i)$ ($1 \leq i \leq m - 1$) from S'_0 to T'_0 covering $Q[0] - F_0 - s_m - t_m$. Then, $s_m, t_m \in \cup_{i=1}^{m-1} V(R_i)$.

Case 1.1.1.1. $s_m, t_m \in V(R_i)$ for some $1 \leq i \leq m - 1$.

Without loss of generality, let $s_m, t_m \in V(R_1)$. Suppose s_m is before t_m in R_1 . Then, $R_1 = R_1[s_1, s_m] \cup R_1[s_m, t_m] \cup R_1[t_m, t_1]$. Let u (resp. v) be the before

vertex (resp. later vertex) of s_m (resp. t_m) in R_1 . Let $u_1 = N(u) \cap V(Q[1])$ and $v_1 = N(v) \cap V(Q[1])$. By Lemma 1, there is a Hamilton path $L = (u_1, v_1)$ in $Q[1]$. Let $xy \in E(L)$, $x_2 = N(x) \cap V(Q[2])$ and $y_2 = N(y) \cap V(Q[2])$. By Lemma 1, there is a Hamilton path $M = (x_2, y_2)$ in $Q[2]$. Let $P_1 = R_1[s_1, u] \cup \{uu_1, vv_1\} \cup (L - xy) \cup \{xx_2, yy_2\} \cup M \cup R_1[v, t_1]$, $P_i = R_i$ ($2 \leq i \leq m-1$) and $P_m = R_1[s_m, t_m]$. Similarly, t_m is before s_m in R_1 .

Case 1.1.1.2. $s_m \in V(R_i)$, $t_m \in V(R_j)$ for $1 \leq i, j \leq m-1$ and $i \neq j$.

Without loss of generality, let $s_m \in V(R_1)$ and $t_m \in V(R_2)$. Let u and v be the before and later vertex of s_m in R_1 , respectively. Let $u_1 = N(u) \cap V(Q[1])$ and $v_1 = N(v) \cap V(Q[1])$. Let x and y be the before and later vertex of t_m in R_2 , respectively. Let $x_1 = N(x) \cap V(Q[1])$ and $y_1 = N(y) \cap V(Q[1])$. Note $|F_1| = 0 \leq 2n-6$ for $n \geq 3$. By Lemma 3, there are two disjoint paths $L_1 = (u_1, v_1)$ and $L_2 = (x_1, y_1)$ covering $Q[1] - F_1$. By Lemma 1, there is a Hamilton path $M = (s_m^2, t_m^2)$ in $Q[2]$. Let $P_1 = (R_1 - s_m) \cup \{uu_1, vv_1\} \cup L_1$, $P_2 = (R_2 - t_m) \cup \{xx_1, yy_1\} \cup L_2$, $P_i = R_i$ ($3 \leq i \leq m-1$) and $P_m = \{s_m s_m^2, t_m t_m^2\} \cup M$.

Case 1.1.2. $|F_0| = 2n - 2m - 1$.

Then, $|F_1 \cup F_2| \leq 1$. Without loss of generality, let $|F_1| = 0$ and $|F_2| \leq 1$. Let $S'_0 = S_0 \setminus \{s_m\}$ and $T'_0 = T_0 \setminus \{t_m\}$. Then, $|S'_0| = |T'_0| = m-1$. Note $|F_0 \cup \{t_m\}| = 2n - 2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $R_i = (s_i, t_i)$ ($1 \leq i \leq m-1$) from S'_0 to T'_0 covering $Q[0] - F_0 - t_m$. Then, $s_m \in V(R_i)$ for some $1 \leq i \leq m-1$. Without loss of generality, let $s_m \in V(R_1)$. Then, $R_1 = R_1[s_1, s_m] \cup R_1[s_m, t_1]$. Let u and v be the before and later vertex of s_m in R_1 , respectively. Let $u_i = N(u) \cap V(Q[i])$ and $v_i = N(v) \cap V(Q[i])$ for $1 \leq i \leq 2$.

If $u_2 \in F_2$ or $v_2 \in F_2$, then $s_m^2 \notin F_2$ and $t_m^2 \notin F_2$. By Lemma 2, there is a path $L_1 = (u_1, v_1)$ and $M_1 = (s_m^2, t_m^2)$ covering $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_1 = (R_1 - s_m) \cup \{uu_1, vv_1\} \cup L_1$, $P_i = R_i$ ($2 \leq i \leq m-1$) and $P_m = \{s_m s_m^2, t_m t_m^2\} \cup M_1$.

Suppose $u_2 \notin F_2$ and $v_2 \notin F_2$. By Lemma 2, there is a path $L_2 = (s_m^1, t_m^1)$ and $M_2 = (u_2, v_2)$ covering $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_1 = (R_1 - s_m) \cup \{uu_2, vv_2\} \cup M_2$, $P_i = R_i$ ($2 \leq i \leq m-1$) and $P_m = \{s_m s_m^1, t_m t_m^1\} \cup L_2$.

Case 1.1.3. $2n - 2m - 3 \leq |F_0| \leq 2n - 2m - 2$.

In this case, $2n - 2m - 2 \geq 0$. We get $n \geq m+1 \geq 4$ for $m \geq 3$. We have $|F_1 \cup F_2| \leq 3$. Without loss of generality, let $|F_1| \leq \lfloor 3/2 \rfloor = 1$. Since $|S| = |T| = m \geq 3 > 1 \geq |F_1|$, there are two vertices $s_i^1, t_i^1 \notin F_1$ for some $1 \leq i \leq m$. Without loss of generality, let $s_m^1, t_m^1 \notin F_1$. Let $S'_0 = S_0 \setminus \{s_m\}$ and $T'_0 = T_0 \setminus \{t_m\}$. Then, $|S'_0| = |T'_0| = m-1$. Note $|F_0 \cup \{s_m, t_m\}| \leq 2n - 2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $P_i = (s_i, t_i)$ ($1 \leq i \leq m-1$) from S'_0 to T'_0 covering $Q[0] - F_0 - s_m - t_m$. Since $|V(Q[1])| - |F_1 \cup F_2| -$

$| \{s_m^1, t_m^1 \} | \geq 3^{n-1} - 3 - 2 \geq 4$ for $n \geq 3$, there are two edges $x_i x_2$ and $y_1 y_2$ such that $\{x_i, y_i\} \subset V(Q[i] - F_i - \{s_m^1, t_m^1\})$ for $1 \leq i \leq 2$. Note $|F_1| \leq 1 \leq 2n - 6$ for $n \geq 4$. By Lemma 3, there are two disjoint paths $L_1 = (s_m^1, x_1)$ and $L_2 = (y_1, t_m^1)$ covering $Q[1] - F_1$. Note $|F_2| \leq 3 \leq 2n - 5$ for $n \geq 4$. By Lemma 2, there is a path $M = (x_2, y_2)$ covering $Q[2] - F_2$. Let $P_m = \{s_m s_m^1\} \cup L_1 \cup \{x_1 x_2\} \cup M \cup \{y_2 y_1\} \cup L_2 \cup \{t_m^1 t_m\}$.

Case 1.1.4. $|F_0| \leq 2n - 2m - 4$.

Let $N_{Q[0]}(s_m) = \{u_1, \dots, u_{2n-2}\}$ and $u_i^j = N(u_i) \cap V(Q[j])$ for $1 \leq i \leq 2n-2$ and $j = 1, 2$. Since $|N_{Q[0]}(s_m)| - |S \setminus \{s_m\}| = 2n - 2 - (m-1) > 2n - 2m \geq |F|$ for $m \geq 3$, there is an edge $u_i u_i^1$ such that $u_i \in V(Q[0] - F_0 - S)$ and $u_i^1 \in V(Q[1] - F_1)$ for $1 \leq i \leq 2n-2$. Without loss of generality, let $u_1 u_1^1$ be satisfied. Let $N_{Q[0]}(t_m) = \{v_1, \dots, v_{2n-2}\}$ and $v_i^j = N(v_i) \cap V(Q[j])$ for $1 \leq i \leq 2n-2$ and $j = 1, 2$. Since $|N_{Q[0]}(t_m)| - |T \setminus \{t_m\}| - |\{u_i\}| \geq 2n - 2 - (m-1) - 1 > 2n - 2m \geq |F|$ for $m \geq 3$, there is an edge $v_i v_i^1$ such that $v_i \in V(Q[0] - F_0 - u_1)$ and $v_i^1 \in V(Q[1] - F_1 - u_1^1)$ for $1 \leq i \leq 2n-2$. Without loss of generality, let $v_1 v_1^1$ be satisfied.

Note $|F_0 \cup \{s_m, t_m, u_1, v_1\}| \leq 2n - 2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $P_i = (s_i, t_i)$ ($1 \leq i \leq m-1$) covering. Since $|V(Q[1])| - |F| - |\{u_1^1, v_1^1\}| \geq 3^{n-1} - (2n - 2m) - 2 \geq 7$ for $3 \leq m \leq n$, there are two edges $x_i x_2, y_1 y_2$ such that $\{x_i, y_i\} \subset V(Q[i] - F_i - \{u_1^1, v_1^1\})$ for $i = 1, 2$. Note $|F_1| \leq 2n - 2m \leq 2n - 6$ for $m \geq 3$. By Lemma 3, there are two disjoint paths $L_1 = (u_1^1, x_1)$ and $L_2 = (y_1, v_1^1)$ covering $Q[1] - F_1$. Note $|F_2| \leq 2n - 2m < 2n - 5$ for $m \geq 3$. By Lemma 2, there is a path $M = (x_2, y_2)$ covering $Q[2] - F_2$. Let $P_m = \{s_m u_1, u_1 u_1^1, t_m v_1, v_1 v_1^1\} \cup L_1 \cup L_2 \cup \{x_1 x_2, y_1 y_2\} \cup M$.

Case 1.2. $|S_i| = |T_i| = m-1$ for some $0 \leq i \leq 2$.

Without loss of generality, let $|S_0| = |T_0| = 1$, $|S_1| = |T_1| = m-1$ and $|S_2| = |T_2| = 0$. Let $S_0 = \{s_1\}$, $T_0 = \{t_1\}$, $S_1 = \{s_2, \dots, s_m\}$ and $T_1 = \{t_2, \dots, t_m\}$. By the induction hypothesis, there are $(m-1)$ disjoint paths $P_i = (s_i, t_i)$ ($2 \leq i \leq m$) covering $Q[1] - F_1$. Since $|V(Q[i])| - |F| - |S_0| - |T_0| \geq 3^{n-1} - (2n - 2m) - 2 \geq 7$ for $3 \leq m \leq n$, there are two edges $x_1 x_2, y_1 y_2$ such that $\{x_i, y_i\} \subset V(Q[i] - F_i - S_0 - T_0)$ for $i = 0, 2$. Note $|F_0| \leq 2n - 2m \leq 2n - 6$ for $m \geq 3$. By Lemma 3, there are two disjoint paths $R_1 = (s_1, x_0)$ and $R_2 = (y_0, t_1)$ covering $Q[0] - F_0$. By Lemma 2, there is a path $M = (x_2, y_2)$ covering $Q[2] - F_2$. Let $P_1 = R_1 \cup \{x_0 x_2\} \cup M \cup \{y_0 y_2\} \cup R_2$.

Case 1.3. $1 \leq |S_i| = |T_i| \leq m-2$ for all $0 \leq i \leq 2$.

Let $|S_i| = |T_i| = p_i \leq m-2$ for $0 \leq i \leq 2$. Then, $p_0 + p_1 + p_2 = m$. Note $|F_i| \leq 2n - 2m \leq 2(n-1) - 2p_i$. By the induction hypothesis, there are p_i disjoint paths covering $Q[i] - F_i$ for $0 \leq i \leq 2$.

Case 1.4. $|S_i| = |T_i| = 0$ for only one $0 \leq i \leq 2$.

Let $|S_0| = |T_0| = 0$. If there is $|S_i| = |T_i| = m-1$ or $|S_i| = |T_i| = 1$ for some $1 \leq i \leq 2$, then it is similar to Case 1.2.

Thus, suppose $2 \leq |S_i| = |T_i| \leq m-2$ for $1 \leq i \leq 2$. Let $|S_1| = |T_1| = p$. Then, $|S_2| = |T_2| = m-p$. Let $S_1 = \{s_1, \dots, s_p\}$, $T_1 = \{t_1, \dots, t_p\}$, $S_2 = \{s_{p+1}, \dots, s_m\}$ and $T_2 = \{t_{p+1}, \dots, t_m\}$. Note $|S_1| = |T_1| = p \leq m-2$. By the induction hypothesis, there are p disjoint paths $P_i = (s_i, t_i)$ ($1 \leq i \leq p$) covering $Q[1] - F_1$.

Since $|V(Q[i])| - |F| - |S| - |T| \geq 3^{n-1} - (2n-2m) - 2m \geq 3$ for $n \geq 3$, there are two edges $u_0 u_2, v_0 v_2$ such that $u_i, v_i \in V(Q[i] - F - S - T)$ for $i = 0, 2$. Note $|S_2 \cup \{v_2\}| = |T_2 \cup \{u_2\}| = m-p+1 \leq m-1$. By the induction hypothesis, there are $(m-p+1)$ disjoint paths $P_i = (s_i, t_i)$ ($p+1 \leq i \leq m-1$), $M_1 = (s_m, u_2)$ and $M_2 = (v_2, t_m)$ covering $Q[2] - F_2$. Note $|F_0| \leq 2n-2m < 2n-5$ for $m \geq 3$. By Lemma 2, there is a path $R = (u_0, v_0)$ covering $Q[0] - F_0$. Let $P_m = M_1 \cup \{u_2 u_0\} \cup R \cup \{v_0 v_2\} \cup M_2$.

Case 2. $|S_i| = |T_i|$ for only one $0 \leq i \leq 2$.

Without loss of generality, let $|S_0| = |T_0|$. Then, $|S_1| \neq |T_1|$, $|S_2| \neq |T_2|$ and $|S_1| + |S_2| = |T_1| + |T_2|$. Without loss of generality, let $|S_1| > |T_1|$. Then, $|S_2| < |T_2|$ and $|S_1| - |T_1| = |T_2| - |S_2|$. Since $|V(Q[i])| - |F| - |S| - |T| \geq 3^{n-1} - (2n-2m) - 2m \geq m$ for $3 \leq m \leq n$, there are m cycles of length three $C_i = \langle w_i^0, w_i^1, w_i^2, w_i^0 \rangle$ for $1 \leq i \leq m$ such that $w_i^j \in V(Q[j] - F_j - S - T)$ for $0 \leq j \leq 2$.

Case 2.1. $|S_0| = |T_0| = 0$.

Then, $|S_1| + |S_2| = |T_1| + |T_2| = m$.

Case 2.1.1. $|S_1| = m$ and $|T_1| = 0$.

Then, $|S_2| = 0$ and $|T_2| = m$. Let $S_1 = \{s_1, \dots, s_m\}$, $T_2 = \{t_1, \dots, t_m\}$.

Case 2.1.1.1. $|F_0| = 0$.

Let $S'_0 = \{s_i^0: 1 \leq i \leq m-1\}$, $T'_0 = \{t_i^0: 1 \leq i \leq m-1\}$. Then, $|S'_0| = |T'_0| = m-1$. By the induction hypothesis, there are $(m-1)$ disjoint paths $R_i = (s_i^0, t_i^0)$ ($1 \leq i \leq m-1$) from S'_0 to T'_0 covering $Q[0] - F_0$. Let $P_i = \{s_i s_i^0\} \cup R_i \cup \{t_i^0 t_i\}$ ($1 \leq i \leq m-1$).

Let $W_1 = \{w_1^1\}$ and $W_2 = \{w_1^2\}$. Note $|S'_1| = |S_1| \setminus \{s_m\} = m-1 = |T_2| \setminus \{t_m\} = |T'_2|$ and $|F_1| + |S'_1| \leq 2n-2m + (m-1) = 2n-m-1 \leq 2n-5$ for $m \geq 4$. By Lemma 2, there is a path $L = (s_m, w_1^1)$ covering $Q[1] - F_1$. Similarly, there is a path $M = (w_1^2, t_m)$ covering $Q[2] - F_2$. Let $P_m = L \cup \{w_1^1 w_1^2\} \cup M$.

Suppose $m = 3$. Then, $|F| \leq 2n-2m = 2n-6$. Let $W_1 = \{w_1^1, w_2^1\}$ and $W_2 = \{w_1^2, w_2^2\}$. By Lemma 2, there are two paths $L_1 = (s_1, w_1^1)$ and $L_2 = (s_2, w_2^1)$ (resp. $M_1 = (w_1^2, t_1)$ and $M_2 = (w_2^2, t_2)$) covering $Q[1] - F_1$ (resp. $Q[2] - F_2$). Let $P'_i = L_i \cup \{w_1^1 w_i^1\} \cup M_i$ for $i = 1, 2$. Then, $s_3, t_3 \in P'_i$ for $1 \leq i \leq 2$.

Suppose $s_3, t_3 \in P'_1$. Let $P_3 = P'_1[s_3, t_3]$. Let u and v be the before and later vertex of s_3 and t_3 , respectively. Let $u_0 = N(u) \cap V(Q[0])$ and $v_0 = N(v) \cap V(Q[0])$.

If $u_0 = v_0$, then let $P_1 = P'_1[s_1, u] \cup \{uu_0, v_0 v\} \cup P'_1[v, t_1]$. Let $wz \in P'_2$, $w_0 = N(w) \cap V(Q[0])$ and $z_0 = N(z) \cap V(Q[0])$. Note $|F_0 \cup \{u_0\}| = 1 \leq 2n-5$ for

$n \geq 3$. By Lemma 2, there is a path $R = (w_0, z_0)$ covering $Q[0] - F_0 - u_0$. Let $P_2 = (P'_2 - wz) \cup \{ww_0, zz_0\} \cup R$.

Suppose $u_0 \neq v_0$. Note $|F_0| = 0$. By Lemma 1, there is a Hamilton path $R = (u_0, v_0)$ in $Q[0]$. Let $P_1 = P'_1[s_1, u] \cup \{uu_0, v_0 v\} \cup R \cup P'_1[v, t_1]$ and $P_2 = P'_2$.

Suppose $s_3 \in P'_1$ and $t_3 \in P'_2$. Let u and v be the before and later vertex of s_3 and t_3 in P'_1 and P'_2 , respectively. Let $u_0 = N(u) \cap V(Q[0])$ and $v_0 = N(v) \cap V(Q[0])$.

If $u_0 = v_0$, then let $P_1 = P'_1[s_1, u] \cup \{uu_0, v_0 v\} \cup P'_2[v, t_2]$. Let $wz \in L_2$, $w_0 = N(w) \cap V(Q[0])$ and $z_0 = N(z) \cap V(Q[0])$. Note $|F_0 \cup \{u_0\}| = 1 \leq 2n-5$ for $n \geq 3$. By Lemma 2, there is a path $R = (w_0, z_0)$ covering $Q[0] - F_0 - u_0$. Let $P_2 = (P'_2[s_2, t_3] - wz) \cup \{ww_0, zz_0\} \cup R$ and $P_3 = P'_1[s_3, t_1]$.

Suppose $u_0 \neq v_0$. Note $|F_0| = 0$. By Lemma 1, there is a Hamilton path $R = (u_0, v_0)$ in $Q[0]$. Let $P_1 = P'_1[s_1, u] \cup \{uu_0, v_0 v\} \cup R \cup P'_2[v, t_2]$, $P_2 = P'_2[s_2, t_3]$ and $P_3 = P'_1[s_3, t_1]$.

Case 2.1.1.2. $|F_0| = 1$.

There is at most one vertex $s_i^0 \in F_0$ or $t_i^0 \in F_0$ for $1 \leq i \leq m$. Without loss of generality, let $s_i^0, t_i^0 \notin F_0$ for all $1 \leq i \leq m-1$. Let $S'_0 = \{s_i^0: 1 \leq i \leq m-2\}$, $T'_0 = \{t_i^0: 1 \leq i \leq m-2\}$. Then, $|S'_0| = |T'_0| = m-2$. Note $|F_0| = 1 \leq 2(n-1) - 2(m-2)$. By the induction hypothesis, there are $(m-2)$ disjoint paths $R_i = (s_i^0, t_i^0)$ ($1 \leq i \leq m-2$) from S'_0 to T'_0 covering $Q[0] - F_0$. Let $P_i = \{s_i s_i^0\} \cup R_i \cup \{t_i^0 t_i\}$ ($1 \leq i \leq m-2$).

Let $W_1 = \{w_1^1, w_2^1\}$ and $W_2 = \{w_1^2, w_2^2\}$. Note $|S'_1| = |S_1| \setminus \{s_{m-1}, s_m\} = m-2 = |T_2| \setminus \{t_{m-1}, t_m\} = |T'_2|$ and $|F_1| + |S'_1| \leq 2n-2m-1 + (m-2) = 2n-m-3 \leq 2n-6$ for $m \geq 3$. By Lemma 3, there are two disjoint paths $L_1 = (s_{m-1}, w_1^1)$ and $L_2 = (s_m, w_2^1)$ covering $Q[1] - F_1$. Similarly, there are two disjoint paths $M_1 = (w_1^2, t_{m-1})$ and $M_2 = (w_2^2, t_m)$ covering $Q[2] - F_2$. Let $P_{m-1} = L_1 \cup \{w_1^1 w_1^2\} \cup M_1$ and $P_m = L_2 \cup \{w_2^2 w_2^1\} \cup M_2$.

Case 2.1.1.3. $|F_0| \geq 2$.

Then, $|F_i| \leq 2n-2m-2$ for $i = 1, 2$. Let $N_{Q[1]}(s_m) = \{u_1, \dots, u_{2n-2}\}$ and $u_i^0 = N(u_i) \cap V(Q[0])$ for $1 \leq i \leq 2n-2$. Since $|N_{Q[1]}(s_m)| - |S \setminus \{s_m\}| = 2n-2 - (m-1) > 2n-2m \geq |F|$ for $m \geq 3$, there is an edge $u_i u_i^0$ ($1 \leq i \leq 2n-2$) such that $u_i \in V(Q[1] - F_1 - S_1)$ and $u_i^0 \in V(Q[0] - F_0)$. Without loss of generality, let $u_1 u_1^0$ be satisfied. Let $N_{Q[2]}(t_m) = \{v_1, \dots, v_{2n-2}\}$ and $v_i^0 = N(v_i) \cap V(Q[0])$ for $1 \leq i \leq 2n-2$. Since $|N_{Q[2]}(t_m)| - |T \setminus \{t_m\}| = 2n-2 - (m-1) > 2n-2m + 1 \geq |F| + |\{u_1^0\}|$ for $m \geq 3$, there is an edge $v_i v_i^0$ such that $v_i \in V(Q[2] - F_2 - T_2)$ and $v_i^0 \in V(Q[0] - F_0 - \{u_1^0\})$. Without loss of generality, let $v_1 v_1^0$ be satisfied. Let $S'_1 = S_1 \setminus \{s_m\}$ and $T'_2 = T_2 \setminus \{t_m\}$. Then, $|S'_1| = |T'_2| = m-1$. Let $W_1 = \{w_i^1: 1 \leq i \leq m-1\}$ and $W_2 = \{w_i^2: 1 \leq i \leq m-1\}$. Note $|F_1 \cup \{s_m, u_1\}| \leq 2n-2m$ and $|F_2 \cup \{t_m, v_1\}| \leq 2n-2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $L_i = (s_i, w_i^1)$ (resp. $M_i = (w_i^2, t_i)$) ($1 \leq i \leq m-1$) from S'_1 to W_1 (resp. from W_2 to T'_2) covering $Q[1] - F_1 - s_m - u_1$ (resp. $Q[2] - F_2 - t_m -$

v_1). Note $|F_0| \leq 2n - 2m < 2n - 5$ for $m \geq 3$. By Lemma 2, there is a path $R = (u_1^0, v_1^0)$ covering $Q[0] - F_0$. Let $P_i = L_i \cup \{w_i^1 w_i^2\} \cup M_i$ ($1 \leq i \leq m-1$) and $P_m = \{s_m u_1, u_1 u_1^0, t_m v_1, v_1 v_1^0\} \cup R$.

Case 2.1.2. $|S_1| = m$ and $1 \leq |T_1| \leq m-1$.

Let $S_1 = \{s_1, \dots, s_m\}$. Let $|T_1| = p \leq m-1$. Then, $|T_2| = m - p \leq m-1$. Let $T_1 = \{t_1, \dots, t_p\}$ and $T_2 = \{t_{p+1}, \dots, t_m\}$. Let $W_2 = \{w_i^1: p+1 \leq i \leq m\}$. Note $|W_2| = |T_2| = m - p \leq m-1$. By the induction hypothesis, there are $(m-p)$ disjoint paths $M_i = (w_i^2, t_i)$ ($p+1 \leq i \leq m$) covering $Q[2] - F_2$.

Case 2.1.2.1. $|F_0| = 0$.

Let $W_1 = \{w_i^1: p+1 \leq i \leq m-1\}$ for $p \leq m-2$, $W_1 = \emptyset$ for $p = m-1$. Then, $|W_1| = m - p - 1$. Note $|T_1| + |W_1| = p + (m - p - 1) = m - 1 = |S_1| \setminus \{s_m\}$. By the induction hypothesis, there are $(m-1)$ disjoint paths $L_i = (s_i, t_i)$ ($1 \leq i \leq p$) and $L_i = (s_i, w_i^1)$ ($p+1 \leq i \leq m-1$) covering $Q[1] - F_1$. Let $P'_i = L_i$ ($1 \leq i \leq p$), $P'_i = L_i \cup \{w_i^1 w_i^2\} \cup M_i$ ($p+1 \leq i \leq m-1$). Then, $s_m \in V(P'_j)$ for some $1 \leq j \leq m-1$. We get $P'_j = P'_j[s_j, s_m] \cup P'_j[s_m, t_j]$. Let u and v be the before and later vertex of s_m in P'_j , respectively. Let $u_0 = N(u) \cap V(Q[0])$, $v_0 = N(v) \cap V(Q[0])$. Let $W_0 = \{w_0^0\}$.

Suppose $w_m^0 \neq u_0$ and $w_m^0 \neq v_0$. Note $|F_0| = 0$. By Lemma 3, there are two disjoint paths $R_1 = (u_0, v_0)$ and $R_2 = (s_m^0, w_m^0)$ covering $Q[0] - F_0$. Let $P_j = (P'_j - s_m) \cup \{uu_0, vv_0\} \cup R_1$, $P_i = P'_i$ ($i \in \{1, \dots, m-1\} \setminus \{j\}$) and $P_m = \{s_m s_m^0, w_m^0 w_m^1\} \cup R_2 \cup M_m$.

Suppose $w_m^0 = u_0$. Let z be the before vertex of u in P'_j and $z_0 = N(z) \cap V(Q[0])$. Note $|F_0 \cup \{u_0\}| = 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a path $R = (z_0, v_0)$ covering $Q[0] - F_0 - u_0$. Let $P_j = P'_j[s_j, z] \cup \{zz_0, vv_0\} \cup R \cup P'_j[v, t_j]$, $P_i = P'_i$ ($i \in \{1, \dots, m-1\} \setminus \{j\}$) and $P_m = \{s_m u, uu_0, w_m^0 w_m^1\} \cup M_m$. Similar to $w_m^0 = v_0$.

Case 2.1.2.2. $|F_0| = 1$.

Then, $|F_i| \leq 2n - 2m - 1$ for $i = 1, 2$. Since $|S_1| = m > 1 = |F_0|$, there is a vertex $s_i \in S_1$ such that $s_i^0 \notin F_0$ for $1 \leq i \leq m$. Without loss of generality, let $s_m^0 \notin F_0$. Let $W_1 = \{w_i^1: p+1 \leq i \leq m-1\}$ for $p \leq m-2$, $W_1 = \emptyset$ for $p = m-1$. Then, $|W_1| = m - p - 1$. Note $|T_1| + |W_1| = p + (m - p - 1) = m - 1 = |S_1| \setminus \{s_m\}$ and $|F_1 \cup \{s_m\}| \leq 2n - 2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $L_i = (s_i, t_i)$ ($1 \leq i \leq p$) and $L_i = (s_i, w_i^1)$ ($p+1 \leq i \leq m-1$) covering $Q[1] - F_1 - s_m$. Let $P_i = L_i$ ($1 \leq i \leq p$) and $P_i = L_i \cup \{w_i^1 w_i^2\} \cup M_i$ for $(p+1 \leq i \leq m-1)$.

Let $W_0 = \{w_0^0\}$. Note $|F_0| = 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a path $R = (s_m^0, w_m^0)$ covering $Q[0] - F_0$. Let $P_m = \{s_m s_m^0\} \cup R \cup \{w_m^0 w_m^1\} \cup M_m$.

Case 2.1.2.3. $|F_0| \geq 2$.

Then, $|F_i| \leq 2n - 2m - 2$ for $i = 1, 2$. Let $N_{Q[1]}(s_m) = \{u_1, \dots, u_{2n-2}\}$ and $u_i^0 = N(u_i) \cap V(Q[0])$ for $1 \leq i \leq 2n-2$. Since $|N_{Q[1]}(s_m)| - |S_1 \setminus \{s_m\}| = 2n - 2 - (m-1) > 2n - 2m + 1 \geq |F \cup \{w_m^0\}|$ for $m \geq 3$, there is an edge

$u_i u_i^0$ ($1 \leq i \leq 2n-2$) such that $u_i \in V(Q[1] - F_1 - S_1)$ and $u_i^0 \in V(Q[0] - F_0 - \{w_m^0\})$. Without loss of generality, let $u_1 u_1^0$ be satisfied.

Let $W_1 = \{w_i^1: p+1 \leq i \leq m-1\}$ for $p \leq m-2$, $W_1 = \emptyset$ for $p = m-1$. Then, $|W_1| = m - p - 1$. Note $|T_1| + |W_1| = p + (m - p - 1) = m - 1 = |S_1| \setminus \{s_m\}$ and $|F_1 \cup \{s_m, u_1\}| \leq 2n - 2m$. By the induction hypothesis, there are $(m-1)$ disjoint paths $L_i = (s_i, t_i)$ ($1 \leq i \leq p$) and $L_i = (s_i, w_i^1)$ ($p+1 \leq i \leq m-1$) covering $Q[1] - F_1 - s_m - u_1$. Let $P_i = L_i$ ($1 \leq i \leq p$) and $P_i = L_i \cup \{w_i^1 w_i^2\} \cup M_i$ ($p+1 \leq i \leq m-1$).

Let $W_0 = \{w_0^0\}$. Note $|F_0| \leq 2n - 2m < 2n - 5$ for $m \geq 3$. By Lemma 2, there is a path $R = (u_1^0, w_m^0)$ covering $Q[0] - F_0$. Let $P_m = \{s_m u_1, u_1 u_1^0\} \cup R \cup \{w_m^0 w_m^1\} \cup M_m$.

Case 2.1.3. $1 \leq |S_1| \leq m-1$.

Let $|S_1| = p$. Then, $|S_2| = m - p$. Let $S_1 = \{s_1, \dots, s_p\}$ and $S_2 = \{s_{p+1}, \dots, s_m\}$. If $|T_1| = m$ or $|T_2| = m$, then it is similar to Case 2.1.2. Thus, $1 \leq |T_i| \leq m-1$ for $i = 1, 2$. Let $1 \leq |T_1| = q < p = |S_1| \leq m-1$. Then, $1 \leq |T_2| = m - q \leq m-1$. Let $T_1 = \{t_1, \dots, t_q\}$ and $T_2 = \{t_{q+1}, \dots, t_m\}$.

Let $W_j = \{w_i^j: q+1 \leq i \leq p-1\}$ for $p-q \geq 2$ and $W_j = \emptyset$ for $p-q = 1$ where $j = 1, 2$. Then, $|W_1| = |W_2| = p - q - 1 \leq m - 3$. Since $|V(Q[i])| - |F| - |S| - |T| - |W_i| \geq 3^{n-1} - (2n - 2m) - 2m - (m - 3) \geq 3$ for $3 \leq m \leq n$, there are two edges $u_0 u_1, v_0 v_2$ such that $\{u_i, v_i\} \subset V(Q[i] - F - S - T - W_i)$ for $0 \leq i \leq 2$. Note $|F_0| = 2n - 2m < 2n - 5$ for $m \geq 3$. By Lemma 2, there is a path $R = (u_0, v_0)$ covering $Q[0] - F_0$.

Note $|W_2| + |\{v_2\}| + |S_2| = (p - q - 1) + 1 + (m - p) = m - q = |T_2|$. By the induction hypothesis, there are $(m-q)$ disjoint paths $M_i = (w_i^2, t_i)$ ($q+1 \leq i \leq p-1$), $M_p = (v_2, t_p)$ and $M_i = (s_i, t_i)$ ($p+1 \leq i \leq m$) from $S_2 \cup W_2 \cup \{v_2\}$ to T_2 covering $Q[2] - F_2$.

Note $|S_1| = p = q + (p - q - 1) + 1 = |T_1| + |W_1| + |\{u_1\}|$. By the induction hypothesis, there are p disjoint paths $L_i = (s_i, t_i)$ ($1 \leq i \leq q$), $L_i = (s_i, w_i^1)$ ($q+1 \leq i \leq p-1$), $L_p = (s_p, u_1)$ from S_1 to $T_1 \cup W_1 \cup \{u_1\}$ covering $Q[1] - F_1$.

Let $P_i = L_i$ ($1 \leq i \leq q$), $P_i = L_i \cup \{w_i^1 w_i^2\} \cup M_i$ ($q+1 \leq i \leq p-1$), $P_p = L_p \cup \{u_1 u_0\} \cup R \cup \{v_0 v_2\} \cup M_p$ and $P_i = M_i$ ($p+1 \leq i \leq m$).

Case 2.2. $1 \leq |S_0| = |T_0| \leq m-1$.

By the induction hypothesis, there are $|S_0|$ disjoint paths from S_0 to T_0 covering $Q[0] - F_0$. Then, $1 \leq |S_1| - |T_1| = x = |T_2| - |S_2| \leq m-1$. Let $W_j = \{w_i^j: 1 \leq i \leq x\}$ for $j = 1, 2$. Then, $|T_1| + |W_1| = |S_1| \leq m-1$ and $|S_2| + |W_2| = |T_2| \leq m-1$. By the induction hypothesis, there are $|S_1|$ (resp. $|T_2|$) disjoint paths from S_1 to $T_1 \cup W_1$ (resp. $S_2 \cup W_2$ to T_2) covering $Q[1] - F_1$ (resp. $Q[2] - F_2$).

Case 3. $|S_i| \neq |T_i|$ for all $0 \leq i \leq 2$.

By Case 2, there are m cycles of length three $C_i = \langle w_i^0, w_i^1, w_i^2 \rangle$ for $1 \leq i \leq m$ such that $w_i^j \in V(Q[j] - F_j - S - T)$ for $0 \leq j \leq 2$.

Case 3.1. $|S_i| > |T_i|$ for only one $0 \leq i \leq 2$.

Without loss of generality, let $|S_0| > |T_0|$. Then, $|S_1| < |T_1|$, $|S_2| < |T_2|$. We get $1 \leq |S_0| - |T_0| = (|T_1| - |S_1|) + (|T_2| - |S_2|) \leq m$.

Case 3.1.1. $|S_0| = m$ and $|T_0| = 0$.

Then, $|S_1| = |S_2| = 0$ and $1 \leq |T_1|, |T_2| \leq m - 1$. Let $|T_1| = p$ and $|T_2| = m - p$. Let $S_0 = \{s_1, \dots, s_m\}$, $T_1 = \{t_1, \dots, t_p\}$ and $T_2 = \{t_{p+1}, \dots, t_m\}$. Without loss of generality, let $|T_1| = p \geq m - p = |T_2|$. Then, $2p \geq m \geq 3$. We get $p \geq 2$ and $m - p \leq m - 2$. Let $W_2 = \{w_i^1: p + 1 \leq i \leq m\}$. By the induction hypothesis, there are $(m - p)$ disjoint paths $M_i = (w_i^2, t_i) (p + 1 \leq i \leq m)$ from W_2 to T_2 covering $Q[2] - F_2$.

Case 3.1.1.1. $|F_0| = 2n - 2m$.

Let $S'_0 = S_0 \setminus \{s_1\}$. Then, $|S'_0| = m - 1$. Let $W_0 = \{w_i^0: 2 \leq i \leq m\}$. Note $|W_0| = m - 1 = |S'_0|$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $R_i = (s_i, w_i) (2 \leq i \leq m)$ from S'_0 to W_0 covering $Q[0] - F_0$. Then, $s_1 \in R_i$ for some $2 \leq i \leq m$. Without loss of generality, let $s_1 \in R_2$. Let u be the before vertex of s_1 in R_2 and $u_1 = N(u) \cap V(Q[1])$.

Suppose $u_1 \in T_1$. Without loss of generality, let $u_1 = t_1$. Let $W_1 = \{w_i^1: 2 \leq i \leq p\}$. Note $|W_1| = p - 1 = |T_1 \setminus \{t_1\}|$ and $|F_1 \cup \{t_1\}| = 1 \leq 2(n - 1) - 2(m - 2) \leq 2(n - 1) - 2(p - 1)$. By the induction hypothesis, there are $(p - 1)$ disjoint paths $L_i = (w_i^1, t_i) (2 \leq i \leq p)$ covering $Q[1] - F_1 - t_1$. Let $P_1 = R_2[s_1, w_2^0] \cup \{w_2^0 w_2^1\} \cup L_2$, $P_2 = R_2[s_2, u] \cup \{ut_1\}$, $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i (3 \leq i \leq p)$ and $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (p + 1 \leq i \leq m)$.

Suppose $u_1 \notin T_1$. Note $|W_1 \cup \{u_1\}| = |T_1| = p$. By the induction hypothesis, there are p disjoint paths $L_1 = (u_1, t_1)$, $L_i = (w_i^1, t_i) (2 \leq i \leq p)$ from $W_1 \cup \{u_1\}$ to T_1 covering $Q[1] - F_1$. Let $P_1 = R_2[s_1, w_2^0] \cup \{w_2^0 w_2^1\} \cup L_2$, $P_2 = R_2[s_2, u] \cup \{uu_1\} \cup L_1$, $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i (3 \leq i \leq p)$ and $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (p + 1 \leq i \leq m)$.

Case 3.1.1.2. $|F_0| = 2n - 2m - 1$.

Then, $|F_1 \cup F_2| \leq 1$. Since $|S_0| = m > 1 \geq |F_1|$, there is a vertex s_i such that $s_i^1 \notin F_1$ for $1 \leq i \leq m$. Without loss of generality, let $s_1^1 \notin F_1$. Let $W_0 = \{w_i^0: 2 \leq i \leq m\}$. Note $|S_0 \setminus \{s_1\}| = m - 1 = |W_0|$ and $|F_0 \cup \{s_1\}| = 2n - 2m$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $R_i = (s_i, w_i) (2 \leq i \leq m)$ from $S_0 \setminus \{s_1\}$ to W_0 covering $Q[0] - F_0 - s_1$. Let $P_i = R_i \cup \{w_i^0 w_i^1\} \cup M_i (p + 1 \leq i \leq m)$.

Let $W_1 = \{w_i^1: 2 \leq i \leq p\}$. Note $|W_1| + |\{s_1^1\}| = p = |T_1|$. By the induction hypothesis, there are p disjoint paths $L_1 = (s_1^1, t_1)$, $L_i = (w_i^1, t_i) (2 \leq i \leq p)$ from $\{s_1^1\} \cup W_1$ to T_1 covering $Q[1] - F_1$. Let $P_1 = \{s_1 s_1^1\} \cup L_1$ and $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i (2 \leq i \leq p)$.

Case 3.1.1.3. $|F_0| \leq 2n - 2m - 2$.

If $s_1^1 \notin F$, then it is similar to Case 3.1.1.2. Suppose $s_1^1 \in F$. Let $N_{Q[0]}(s_1) = \{u_1, \dots, u_{2n-2}\}$ and $u_i^1 = N(u_i) \cap V(Q[1])$ for $1 \leq i \leq 2n - 2$. Since $|N_{Q[0]}(s_1)| - |S \setminus \{s_1\}| = 2n - 2 - (m - 1) > 2n - 2m - 1 + p \geq |F \setminus \{s_1^1\}| + |T_1|$ for $m > p$, there is a vertex $u_i \in V(Q[0] - F_0 - S)$ such that $u_i^1 \in V(Q[1] - F_1 - T_1)$. Without loss of generality, let u_1, u_1^1 be satisfied.

Let $W_0 = \{w_i^0: 2 \leq i \leq m\}$ and $u_1 \notin W_0$. Then, $|F_0 \cup \{s_1, u_1\}| \leq 2n - 2m$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $R_i = (s_i, w_i) (2 \leq i \leq m)$ from $S_0 \setminus \{s_1\}$ to W_0 covering $Q[0] - F_0 - s_1 - u_1$. Let $P_i = R_i \cup \{w_i^0 w_i^1\} \cup M_i (p + 1 \leq i \leq m)$.

Let $W_1 = \{w_i^1: 2 \leq i \leq p\}$. Note $|W_1| + |\{u_1^1\}| = p = |T_1|$. By the induction hypothesis, there are p disjoint paths $L_1 = (u_1^1, t_1)$, $L_i = (w_i^1, t_i) (2 \leq i \leq p)$ from $\{u_1^1\} \cup W_1$ to T_1 covering $Q[1] - F_1$. Let $P_1 = \{s_1 u_1, u_1 u_1^1\} \cup L_1$ and $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i (2 \leq i \leq p)$.

Case 3.1.2. $|S_0| = m$ and $|T_0| \geq 1$.

Let $S_0 = \{s_1, \dots, s_m\}$. Since $|S_i| < |T_i|$ for $i = 1, 2$, $1 \leq |T_i| \leq m - 2$ for $i = 0, 1, 2$. Let $1 \leq |T_0| = m - p - q \leq m - 2$, $1 \leq |T_1| = p \leq m - 2$ and $1 \leq |T_2| = q \leq m - 2$. Let $T_0 = \{t_1, \dots, t_{m-p-q}\}$, $T_1 = \{t_{m-p-q+1}, \dots, t_{m-q}\}$ and $T_2 = \{t_{m-q+1}, \dots, t_m\}$.

Case 3.1.2.1. $|F_0| = 2n - 2m$.

Then, $|F_1| = |F_2| = 0$. Let $S'_1 = \{s_i^1: m - p - q + 1 \leq i \leq m - q\}$ and $S'_2 = \{s_i^2: m - q + 1 \leq i \leq m\}$. Note $|S'_1| = p = |T_1|$ and $|S'_2| = q = |T_2|$. By the induction hypothesis, there are p (resp. q) disjoint paths $L_i = (s_i^1, t_i) (m - p - q + 1 \leq i \leq m - q)$ (resp. $M_i = (s_i^2, t_i) (m - q + 1 \leq i \leq m)$) from S'_1 to T_1 (resp. S'_2 to T_2) covering $Q[1] - F_1$ (resp. $Q[2] - F_2$). Let $P_i = \{s_i s_i^1\} \cup L_i (m - p - q + 1 \leq i \leq m - q)$ and $P_i = \{s_i s_i^2\} \cup M_i (m - q + 1 \leq i \leq m)$.

Let $S'_0 = \{s_i: 1 \leq i \leq m - p - q\}$. Note $|S'_0| = m - p - q = |T_0|$ and $|F_0 \cup \{S'_0 \setminus S'_0\}| = (2n - 2m) + (p + q) \leq 2(n - 1) - 2(m - p - q)$ for $p + q \geq 2$. By the induction hypothesis, there are $(m - p - q)$ disjoint paths $P_i = (s_i, t_i) (1 \leq i \leq m - p - q)$ from S'_0 to T_0 covering $Q[0] - F_0$.

Case 3.1.2.2. $|F_0| = 2n - 2m - 1$.

Since $|F_1 \cup F_2| \leq 1 < m = |S_0|$, there is a vertex s_i such that $s_i^2 \notin F_2$ for $1 \leq i \leq m$. Without loss of generality, let $s_m^2 \notin F_2$. Let $W_0 = \{w_i^0: m - p - q + 1 \leq i \leq m - 1\}$ for $p + q \geq 2$. Note $|W_0| + |T_0| = (p + q - 1) + (m - p - q) = m - 1 = |S_0 \setminus \{s_m\}|$ and $|F_0 \cup \{s_m\}| = 2n - 2m$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $R_i = (s_i, t_i) (1 \leq i \leq m - p - q)$ and $R_i = (s_i, w_i) (m - p - q + 1 \leq i \leq m - 1)$ from $S'_0 \setminus \{s_m\}$ to $T_0 \cup W_0$ covering $Q[0] - F_0 - s_m$. Let $P_i = R_i (1 \leq i \leq m - p - q)$.

Let $W_1 = \{w_i^1: m - p - q + 1 \leq i \leq m - q\}$. Note $|T_1| = |W_1| = p \leq m - 2$ and $|F_1| \leq 1 \leq 2(n - 1) - 2(m - 2) \leq 2(n - 1) - 2p$. By the induction hypothesis, there are p disjoint paths $L_i = (w_i^1, t_i) (m - p - q + 1 \leq$

$i \leq m - q$ from W_1 to T_1 covering $Q[1] - F_1$. Let $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i (m - p - q + 1 \leq i \leq m - q)$.

Let $W_2 = \{w_i^2: m - q + 1 \leq i \leq m - 1\}$ for $q \geq 2$ and $W_2 = \emptyset$ for $q = 1$. Since $w_i^2 \in V(Q[2] - F - S - T)$, $s_m^2 \notin W_2$. Note $|T_2| = |W_2 \cup \{s_m^2\}| = q \leq m - 2$ and $|F_2| \leq 1 \leq 2(n - 1) - 2(m - 2) \leq 2(n - 1) - 2q$. By the induction hypothesis, there are q disjoint paths $M_i = (w_i^2, t_i) (m - q + 1 \leq i \leq m - 1)$ and $M_m = (s_m^2, t_m)$ from $W_2 \cup \{s_m^2\}$ to T_2 covering $Q[2] - F_2$. Let $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (m - q + 1 \leq i \leq m - 1)$ and $P_m = \{s_m s_m^2\} \cup M_m$.

Case 3.1.2.3. $|F_0| \leq 2n - 2m - 2$.

Let $N_{Q[0]}(s_m) = \{u_1, \dots, u_{2n-2}\}$ and $u_i^2 = N(u_i) \cap V(Q[2])$ for $1 \leq i \leq 2n - 2$. Since $|N_{Q[0]}(s_m)| - |S \setminus \{s_m\}| = 2n - 2 - (m - 1) > (2n - 2m) + q \geq |F| + |T_2|$ for $m - 1 > q$, there is a vertex $u_i \in V(Q[0] - F_0 - T_0)$ such that $u_i^2 \in V(Q[2] - F_2 - T_2)$. Without loss of generality, let u_1 be satisfied.

Let $W_0 = \{w_i^0: m - p - q + 1 \leq i \leq m - 1\}$ and $u_1 \notin W_0$. Note $|S_0| = |S \setminus \{s_m\}| = m - 1 = |T_0| + |W_0|$ and $|F_0 \cup \{s_m, u_1\}| = 2n - 2m$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $R_i = (s_i, t_i) (1 \leq i \leq m - p - q)$ and $R_i = (s_i, w_i) (m - p - q + 1 \leq i \leq m - 1)$ from S_0' to $T_0 \cup W_0$ covering $Q[0] - F_0 - s_m - u_1$. Let $P_i = R_i (1 \leq i \leq m - p - q)$.

Let $W_1 = \{w_i^1: m - p - q + 1 \leq i \leq m - q\}$. Note $|T_1| = |W_1| = p \leq m - 2$. By the induction hypothesis, there are p disjoint paths $L_i = (w_i^1, t_i) (m - p - q + 1 \leq i \leq m - q)$ from W_1 to T_1 covering $Q[1] - F_1$. Let $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i (m - p - q + 1 \leq i \leq m - q)$.

Let $W_2 = \{w_i^2: m - q + 1 \leq i \leq m - 1\}$ for $q \geq 2$ and $W_2 = \emptyset$ for $q = 1$. Note $|T_2| = |W_2 \cup \{u_1^2\}| = q \leq m - 2$. By the induction hypothesis, there are q disjoint paths $M_i = (w_i^2, t_i) (m - q + 1 \leq i \leq m - 1)$ and $M_m = (u_1^2, t_m)$ from $W_2 \cup \{u_1^2\}$ to T_2 covering $Q[2] - F_2$. Let $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i (m - q + 1 \leq i \leq m - 1)$ and $P_m = \{s_m u_1, u_1 u_1^2\} \cup M_m$.

Case 3.1.3. $1 \leq |S_0| \leq m - 1$.

If $|T_i| = m$ for some $1 \leq i \leq 2$, then it is similar to Case 3.1.1 or 3.1.2. Since $|T_i| > |S_i| \geq 0$ for $1 \leq i \leq 2$, $1 \leq |T_i| \leq m - 1$ for $1 \leq i \leq 2$. Let $W_0 = \{w_i^0: 1 \leq i \leq |S_0| - |T_0|\}$, $W_1 = \{w_i^1: 1 \leq i \leq |T_1| - |S_1|\}$ and $W_2 = \{w_i^2: 1 \leq i \leq |T_2| - |S_2|\}$. Thus, $|W_0| = |S_0| - |T_0| = (|T_1| - |S_1|) + (|T_2| - |S_2|) = |W_1| + |W_2|$. Note $|S_0| = |T_0| + |W_0| \leq m - 1$, $|T_1| = |S_1| + |W_1| \leq m - 1$ and $|T_2| = |S_2| + |W_2| \leq m - 1$. By the induction hypothesis, there are $|S_0|$, $|T_1|$ and $|T_2|$ disjoint paths from S_0 to $W_0 \cup T_0$, from $S_1 \cup W_1$ to T_1 and from $S_2 \cup W_2$ to T_2 covering $Q[0] - F_0$, $Q[1] - F_1$ and $Q[2] - F_2$, respectively.

Case 3.2. $|S_i| > |T_i|$ for two $0 \leq i \leq 2$.

There is only one i such that $|S_i| < |T_i|$ for $0 \leq i \leq 2$. It is similar to Case 3.1. $\square$

4. One-to-Many Disjoint Paths

Lemma 4. Let $0 \leq i \leq 2$ be a 3-ary n -cube for $n \geq 2$. Let F be a faulty set of $V(Q_n^3)$. Let $\{s, t_1, t_2\}$ be the subset of $V(Q_n^3 - F)$. If $|F| \leq 2n - 3$, then there exist two disjoint paths from s to $t_i (i = 1, 2)$ covering $Q_n^3 - F$.

Proof. Note $|F| \leq 2n - 3$ for $n \geq 2$. By Lemma 2, there is a path $P = (t_1, t_2)$ covering $Q_n^3 - F$. Then, $s \in V(P)$. Thus, there are two disjoint paths $P^{-1}[s, t_1]$ and $P[s, t_2]$ covering $Q_n^3 - F$. $\square$

Lemma 5. Let Q_n^3 be a 3-ary n -cube for $n \geq 3$. Let F be a faulty set of $V(Q_n^3)$. Let $C = \langle s_0, s_1, s_2, s_0 \rangle$ be a fault-free cycle of length three, $s \in V(C)$ and T be the subset of $V(Q_n^3 - F - V(C))$ with $|T| = 3$. If $|F| \leq 2n - 4$, then there are three disjoint paths from s to T covering $Q_n^3 - F$.

Proof. Let $T = \{t_0, t_1, t_2\}$. We decompose Q_n^3 into three isomorphic subgraphs $Q[0], Q[1], Q[2]$ along one dimension such that $s_i \in V(Q[i])$ for $i \in \{0, 1, 2\}$. Suppose there is $st_i \in E(Q_n^3)$ for some $0 \leq i \leq 2$. Let $P_i = (s, t_i)$. Note $|F \cup \{t_i\}| \leq 2n - 3$. By Lemma 4, there are two disjoint paths $P_j = (s, t_j)$ for $j \in \{0, 1, 2\} \setminus \{i\}$. Thus, the lemma holds. So, suppose $st_i \notin E(Q_n^3)$ for all $0 \leq i \leq 2$. Let $T_i = T \cap V(Q[i])$ and $F_i = F \cap V(Q[i])$ for $0 \leq i \leq 2$. Without loss of generality, let $|T_0| > 0$ and $s = s_0$.

Case 1. $|T_0| = 1$, $|T_1| = 2$ and $|T_2| = 0$.

Let $T_0 = \{t_0\}$ and $T_1 = \{t_1, t_2\}$.

Case 1.1. $|F_i| \leq 2n - 5$ for $0 \leq i \leq 2$.

By Lemma 2, there is a Hamilton path $L = (t_1, t_2)$ in $Q[1] - F_1$. Since $|V(L)| - 2|F| \geq 3^{n-1} - 2(2n - 4) > 0$ for $n \geq 3$, there is an edge $xy \in R$ such that $x_i, y_i \in V(Q[i] - F_i)$ for $i = 0, 2$ where $x_i = N(x) \cap V(Q[i])$ and $y_i = N(y) \cap V(Q[i])$. By Lemma 2, there is a Hamilton path $M = (s_2, y_2)$ in $Q[2] - F_2$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, x_0)$ covering $Q[0] - F_0$. Let $P_0 = R_1$, $P_1 = R_2 \cup \{x_0 x\} \cup L^{-1}[x, t_1]$ and $P_2 = \{ss_2\} \cup M \cup \{y_2 y\} \cup L[y, t_2]$.

Case 1.2. $|F_i| = 2n - 4$ for some $0 \leq i \leq 2$.

Case 1.2.1. $|F_0| = 2n - 4$.

Let $f \in F_0$ and $F_0' = F_0 \setminus \{f\}$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, f)$ covering $Q[0] - F_0$. Let u be the before vertex of f in R_2 and $u_1 = N(u) \cap V(Q[1])$.

$u_1 = t_1$ or $u_1 = t_2$. Without loss of generality, let $u_1 = t_1$. Note $|F_1 \cup \{t_1\}| = 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $L = (s_1, t_2)$ in $Q[1] - F_1 - t_1$. Let $xy \in L$, $x_2 = N(x) \cap V(Q[2])$ and $y_2 = N(y) \cap V(Q[2])$. By Lemma 2, there is a Hamilton path $M = (x_2, y_2)$ in $Q[2] - F_2$. Let $P_0 = R_1$, $P_1 =$

$R_2[s, u] \cup \{ut_1\}$ and $P_2 = \{ss_1\} \cup (L - xy) \cup \{xx_2, yy_2\} \cup M$.

$u_1 \neq t_1$ and $u_1 \neq t_2$. Let $t_2' = N(t_2) \cap V(Q[2])$. Note $|F_1 \cup \{t_2\}| = 1 \leq 2n - 5$ and $|F_2| = 0$. By Lemma 2, there is a Hamilton path $L = (u_1, t_1)$ and $M = (s_2, t_2')$ in $Q[1] - F_1 - t_2$ and $Q[2] - F_2$, respectively. Let $P_0 = R_1$, $P_1 = R_2[s, u] \cup \{uu_1\} \cup L$ and $P_2 = \{ss_2\} \cup M \cup \{t_2't_2\}$.

Case 1.2.2. $|F_1| = 2n - 4$.

Let $f \in F_1$ and $F_1' = F_1 \setminus \{f\}$. By Lemma 4, there are two disjoint paths $L_1 = (f, t_1)$ and $L_2 = (f, t_2)$ covering $Q[1] - F_1'$. Let z_i be the next vertex of f in L_i for $i = 1, 2$ and $z_i' = N(u) \cap V(Q[i])$ for $i = 0, 2$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, z_1')$ covering $Q[0] - F_0$. By Lemma 2, there is a Hamilton path $M = (s_2, z_2')$ in $Q[2] - F_2$. Let $P_0 = R_1$, $P_1 = R_2 \cup \{z_1'z_1\} \cup L_1[z_1, t_1]$ and $P_2 = \{ss_2\} \cup M \cup \{z_2'z_2\} \cup L_2[z_2, t_2]$.

Case 1.2.3. $|F_2| = 2n - 4$.

By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0] - F_0$. Let $f \in F_2$ and $F_2' = F_2 \setminus \{f\}$. By Lemma 2, there is a Hamilton path $M = (s_2, f)$ in $Q[2] - F_2'$. Let u be the before vertex of f in M and $u_1 = N(u) \cap V(Q[1])$.

$u_1 = t_1$ or $u_1 = t_2$. Without loss of generality, let $u_1 = t_1$. Note $|F_1 \cup \{t_1\}| = 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $L = (s_1, t_2)$ in $Q[1] - F_1 - t_1$. Let $P_0 = R$, $P_1 = \{ss_2\} \cup M[s_2, u] \cup \{ut_1\}$ and $P_2 = \{ss_1\} \cup L$.

$u_1 \neq t_1$ and $u_1 \neq t_2$. Since $u \neq s_2$, $u_1 \neq s_1$. By Lemma 3, there are two disjoint paths $L_1 = (u_1, t_1)$ and $L_2 = (s_1, t_2)$ covering $Q[1] - F_1$. Let $P_0 = R$, $P_1 = \{ss_2\} \cup M[s_2, u] \cup \{uu_1\} \cup L_1$ and $P_2 = \{ss_1\} \cup L_2$.

Case 2. $|T_0| = 1$, $|T_1| = 1$ and $|T_2| = 1$.

Let $T_0 = \{t_0\}$, $T_1 = \{t_1\}$ and $T_2 = \{t_2\}$.

Case 2.1. $|F_i| \leq 2n - 5$ for $0 \leq i \leq 2$.

By Lemma 2, there is a Hamilton path $R = (s, t_0)$, $L = (s_1, t_1)$ and $M = (s_2, t_2)$ in $Q[0] - F_0$, $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_0 = R$, $P_1 = \{ss_1\} \cup L$ and $P_2 = \{ss_2\} \cup M$.

Case 2.2. $|F_i| = 2n - 4$ for some $0 \leq i \leq 2$.

Case 2.2.1. $|F_0| = 2n - 4$.

Then, $|F_1| = |F_2| = 0$. Since $|F_0| = 2n - 4 \geq 2$ for $n \geq 3$, let $f \in F_0$ and $F_0' = F_0 \setminus \{f\}$. Then, $|F_0'| = 2n - 5$. By Lemma 4, there are two disjoint paths $P = (s_0, t_0)$ and $R = (s_0, f)$ covering $Q[0] - F_0'$. Then, $|V(R)| \geq 2$.

Case 2.2.1.1. $|V(R)| = 2$.

Then, $R = \langle s_0, f \rangle$. By Lemma 2, there is a Hamilton path $P_i' = (s_i, t_i)$ in $Q[i]$ for $i = 1, 2$. Let $P_0 = P$, $P_1 = \{ss_1\} \cup P_1'$ and $P_2 = \{ss_2\} \cup P_2'$.

Case 2.2.1.2. $|V(R)| \geq 3$.

Let u be the before vertex of f in R . Let $u_i = N(u) \cap V(Q[i])$ for $i \in \{1, 2\}$. By the definition of Q_n^3 , $u_i \in V(Q[i] - s_i)$ for $i \in \{1, 2\}$.

Case 2.2.1.2.1. $u_1 = t_1$ or $u_2 = t_2$.

Without loss of generality, let $u_1 = t_1$. Since $|V(Q[i])| - |F_i| - |\{s_i, t_1, t_2\}| = 3^{n-1} - 3 > 1$ for $i = 1, 2$, there is an edge x_1x_2 such that $x_i \in V(Q[i] - F_i - \{s_i, t_1, t_2\})$ for $i = 1, 2$. Note $|F_1 \cup \{t_1\}| = 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $L = (s_1, x_1)$ in $Q[1] - F_1 - t_1$. Note $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (x_2, t_2)$ in $Q[2]$. Let $P_0 = P$, $P_1 = R[s, u] \cup \{ut_1\}$, and $P_2 = \{ss_1\} \cup L \cup \{x_1x_2\} \cup M$.

Case 2.2.1.2.2. $u_1 \neq t_1$ and $u_2 \neq t_2$.

By Lemma 2, there is a Hamilton path $L = (u_1, t_1)$ and $M = (s_2, t_2)$ in $Q[1]$ and $Q[2]$, respectively. Let $P_0 = P$, $P_1 = R[s, u] \cup \{uu_1\} \cup L$, $P_2 = \{ss_2\} \cup M$.

Case 2.2.2. $|F_1| = 2n - 4$ or $|F_2| = 2n - 4$.

Let $|F_1| = 2n - 4$. Then, $|F_0| = |F_2| = 0$. Note $|F_1| = 2n - 4 \geq 2$ for $n \geq 3$. Let $f \in F_1$. Then, $|F_1 \setminus \{f\}| = 2n - 5$. By Lemma 4, there are two disjoint paths $L_1 = (s_1, t_1)$ and $L_2 = (s_1, f)$ covering $Q[1] - F_1 - f$.

Case 2.2.2.1. $|V(L_2)| = 2$.

Then, $L_2 = \langle s_1, f \rangle$. By Lemma 2, there is a Hamilton path $R = (s, t_0)$ and $M = (s_2, t_2)$ in $Q[0]$ and $Q[2]$, respectively. Let $P_0 = R$, $P_1 = \{ss_1\} \cup L_1$ and $P_2 = \{ss_2\} \cup M$.

Case 2.2.2.2. $|V(L_2)| = 3$.

Let $L_2 = \langle s_1, u, f \rangle$. Let $u_2 = N(u) \cap V(Q[2])$. Let w be the later vertex of s_1 in L_1 and $w_i = N(w) \cap V(Q[i])$ for $i = 0, 2$.

Suppose $u_2 = t_2$. Note $|F_2 \cup \{t_2\}| = 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $M = (s_2, w_2)$ in $Q[2] - F_2 - t_2$. By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0]$. Let $P_0 = R$, $P_1 = \{ss_2\} \cup M \cup \{w_2w\} \cup L_1[w, t_1]$ and $P_2 = \langle s, s_1, u, t_2 \rangle$.

Suppose $u_2 \neq t_2$. By Lemma 2, there is a Hamilton path $M = (u_2, t_2)$ in $Q[2]$. By the definition of Q_n^3 , $w_0 \in N(s)$. Since $st_i \notin E(Q_n^3)$, $w_0 \neq t_0$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, w_0)$ covering $Q[0] - F_0$. Let $P_0 = R_1$, $P_1 = R_2 \cup \{w_0w\} \cup L_1[w, t_1]$ and $P_2 = \langle s, s_1, u, u_2 \rangle \cup M$.

Case 2.2.2.3. $|V(L_2)| \geq 4$.

By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0]$. Let u and v be the later and before vertex of s_1 and f in L_2 , respectively. Let $u_2 = N(u) \cap V(Q[2])$ and $v_2 = N(v) \cap V(Q[2])$.

Suppose $t_2 = u_2$ or $t_2 = v_2$. Without loss of generality, let $t_2 = u_2$. Note $|F_2 \cup \{t_2\}| = 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $M = (s_2, v_2)$ in $Q[2] - F_2 - t_2$. Let $P_0 = R$, $P_1 = \{ss_1\} \cup L_1$ and $P_2 = \{ss_2\} \cup M \cup \{v_2v\} \cup L_2^{-1}[v, u] \cup \{ut_2\}$.

Suppose $t_2 \neq u_2$ and $t_2 \neq v_2$. By Lemma 3, there are two disjoint paths $M_1 = (s_2, u_2)$ and $M_2 = (v_2, t_2)$ covering $Q[2] - F_2$. Let $P_0 = R$, $P_1 = \{ss_1\} \cup L_1$ and $P_2 = \{ss_2\} \cup M_1 \cup \{u_2u\} \cup L_2[u, v] \cup \{vv_2\} \cup M_2$.

Case 3. $|T_0| = 2$, $|T_1| = 1$ and $|T_2| = 0$.

Let $T_0 = \{t_0, t_1\}$ and $T_1 = \{t_2\}$.

Case 3.1. $|F_i| \leq 2n - 6$ for all $0 \leq i \leq 2$.

Since $|V(Q[i])| - |F| - |T| - |\{s_i\}| \geq 3^{n-1} - (2n - 4) - 3 - 1 > 1$ for $n \geq 3$, there is an edge x_0x_2 such that $x_i \in V(Q[i] - F_i - T_i - s_i)$ for $i = 0, 2$. By Lemma 2, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (x_0, t_1)$ covering $Q[0] - F_0$. By Lemma 3, there is a Hamilton path $L = (s_1, t_2)$ and $M = (s_2, x_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_0 = R_1$, $P_1 = \{ss_2\} \cup M \cup \{x_2x_0\} \cup R_2$ and $P_2 = \{ss_1\} \cup L$.

Case 3.2. $|F_i| = 2n - 5$ for some $0 \leq i \leq 2$.

We get $|F_j| \leq 1 \leq 2n - 5$ for $n \geq 3$ and $j \in \{0, 1, 2\} \setminus \{i\}$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, t_1)$ covering $Q[0] - F_0$. Let $P_0 = R_1$, $P_1 = R_2$. Since $|V(Q[i])| - |F| - |T| - |\{s_i\}| \geq 3^{n-1} - (2n - 4) - 3 - 1 > 1$, there is an edge x_1x_2 such that $x_i \in V(Q[i] - F_i - T_i - s_i)$ for $i = 1, 2$. Note $|F_i| \leq 2n - 5$ for $i = 1, 2$. By Lemma 2, there is a Hamilton path $L = (x_1, t_2)$ and $M = (s_2, x_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_2 = \{ss_2\} \cup M \cup \{x_2x_1\} \cup L$.

Case 3.3. $|F_i| = 2n - 4$ for some $0 \leq i \leq 2$.

Case 3.3.1. $|F_0| = 2n - 4$.

Then, $|F_1| = |F_2| = 0$. By Lemma 2, there is a Hamilton path $L = (s_1, t_2)$ in $Q[1]$. Let $P_2 = \{ss_1\} \cup L$. Note $|F_0| = 2n - 4 \geq 2$ for $n \geq 3$. Let $f \in F_0$. Then, $|F'_0| = |F_0 \setminus \{f\}| = 2n - 5$. By Lemma 4, there are two disjoint paths $P = (s, t_0)$ and $R = (s, t_2)$ covering $Q[0] - F'_0$. Without loss of generality, let $f \in V(R)$. Then, $|V(R)| \geq 3$.

Case 3.3.1.1. $sf \in E(R)$ and $ft_1 \in E(R)$.

Then, $R = \langle s_0, f, t_1 \rangle$. Let $t_1^2 = N(t_1) \cap V(Q[2])$. Note that $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, t_1^2)$ in $Q[2]$. Let $P_0 = P$ and $P_1 = \{ss_2\} \cup M \cup \{t_1^2t_1\}$.

Case 3.3.1.2. $sf \in E(R)$ and $ft_1 \notin E(R)$.

Let u be the later vertex of f in R and $u_2 = N(u) \cap V(Q[2])$. Note that $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, u_2)$ in $Q[2]$. Let $P_0 = P$ and $P_1 = \{ss_2\} \cup M \cup \{u_2u\} \cup R[u, t_1]$.

Case 3.3.1.3. $sf \notin E(R)$ and $ft_1 \in E(R)$.

Let w and u be the later and before vertex of s_0 and f in P and R , respectively. Let $u_2 = N(u) \cap V(Q[2])$, $w_2 = N(w) \cap V(Q[2])$ and $t_1^2 = N(t_1) \cap V(Q[2])$. By the definition of Q_n^3 , u_2, w_2, t_1^2 are different in $V(Q[2] - s_2)$. Note that $|F_2| = 0 \leq 2n - 6$ for $n \geq 3$. By Lemma 3, there are two disjoint paths $M_1 = (s_2, t_1^2)$ and $M_2 = (u_2, w_2)$ covering $Q[2] - F_2$. Let $P_0 = R[s, u] \cup \{uu_2\} \cup M_2 \cup \{w_2w\} \cup P[w, t_0]$ and $P_1 = \{ss_2\} \cup M_1 \cup \{t_1^2t_1\}$.

Case 3.3.1.4. $sf \notin E(R)$ and $ft_2 \notin E(R)$.

Let u and v be the before and later vertex of f in R , respectively. Let w be the later vertex of s_0 in P . Let $u_2 = N(u) \cap V(Q[2])$, $v_2 = N(v) \cap V(Q[2])$ and $w_2 = N(w) \cap V(Q[2])$. By the definition of Q_n^3 , u_2, v_2, w_2 are different in $V(Q[2] - s_2)$. Note that $|F_2| =$

$0 \leq 2n - 6$ for $n \geq 3$. By Lemma 3, there are two disjoint paths $M_1 = (s_2, v_2)$ and $M_2 = (u_2, w_2)$ covering $Q[2] - F_2$. Let $P_0 = R[s, u] \cup \{uu_2\} \cup M_2 \cup \{w_2w\} \cup P[w, t_0]$ and $P_1 = \{ss_2\} \cup M_1 \cup \{v_2v\} \cup R[v, t_1]$.

Case 3.3.2. $|F_1| = 2n - 4$.

Note $|F_1| = 2n - 4 \geq 2$ for $n \geq 3$. Let $f \in F_1$. Then, $|F'_1| = |F_1 \setminus \{f\}| = 2n - 5$. By Lemma 4, there are two disjoint paths $L_1 = (s_1, t_2)$ and $L_2 = (s_1, f)$ covering $Q[1] - F'_1$. Let $P_2 = \{ss_1\} \cup L_1$. Then, $|V(L_2)| \geq 2$.

Case 3.3.2.1. $|V(L_2)| = 2$.

Then, $L_2 = \langle s_1, f \rangle$. Let $t_1^2 = N(t_1) \cap V(Q[2])$. Note $|F_0 \cup \{t_1\}| = 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0] - F_0 - t_1$. Note $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, t_1^2)$ in $Q[2]$. Let $P_0 = R$ and $P_1 = \{ss_2\} \cup M \cup \{t_1^2t_1\}$.

Case 3.3.2.2. $|V(L_2)| = 3$.

Let $L_2 = \langle s_1, u, f \rangle$, $u_i = N(u) \cap V(Q[i])$ for $i = 0, 2$. Then, $u_i \in V(Q[i] - s_i)$ for $i = 0, 2$.

Suppose $u_0 = t_0$ or $u_0 = t_1$. Without loss of generality, let $u_0 = t_1$. Note $|F_0 \cup \{t_1\}| = 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0] - F_0 - t_1$. Note $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, u_2)$ in $Q[2]$. Let $P_0 = R$ and $P_1 = \{ss_2\} \cup M \cup \{u_2u, ut_1\}$.

Suppose $u_0 \neq t_0$ and $u_0 \neq t_1$. Note $|F_0| = 0 \leq 2n - 6$ for $n \geq 3$. By Lemma 3, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (u_0, t_1)$ covering $Q[0] - F_0$. Note $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, u_2)$ in $Q[2]$. Let $P_0 = R_1$ and $P_1 = \{ss_2\} \cup M \cup \{u_2u, uu_0\} \cup R_2$.

Case 3.3.2.3. $|V(L_2)| \geq 4$.

Let u and v be the later and before vertex of s_1 and f in L_2 , respectively. Let $u_2 = N(u) \cap V(Q[2])$ and $v_0 = N(v) \cap V(Q[0])$. Note $|F_2| = 0$. By Lemma 2, there is a Hamilton path $M = (s_2, u_2)$ in $Q[2]$.

Suppose $v_0 = t_0$ or $v_0 = t_1$. Without loss of generality, let $v_0 = t_1$. Note $|F_0 \cup \{t_1\}| = 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 3, there is a Hamilton path $R = (s, t_0)$ in $Q[0] - F_0 - t_1$. Let $P_0 = R$ and $P_1 = \{ss_2\} \cup M \cup \{u_2u\} \cup L_2[u, v] \cup \{vt_1\}$.

Suppose $v_0 \neq t_0$ and $v_0 \neq t_1$. Note $|F_0| = 0 \leq 2n - 6$ for $n \geq 3$. By Lemma 2, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (v_0, t_1)$ covering $Q[0] - F_0$. Let $P_0 = R_1$ and $P_1 = \{ss_1\} \cup M \cup \{u_2u\} \cup L_2[u, v] \cup \{vv_0\} \cup R_2$.

Case 3.3.3. $|F_2| = 2n - 4$.

Note $|F_1| = 0$. By Lemma 2, there is a Hamilton path $L = (s_1, t_2)$ in $Q[1]$. Let $P_2 = \{ss_1\} \cup L$. Note $|F_2| = 2n - 4 \geq 2$ for $n \geq 3$. Let $f \in F_2$. Then, $|F'_2| = |F_2 \setminus \{f\}| = 2n - 5$. By Lemma 2, there is a Hamilton path $M = (s_2, f)$ in $Q[2] - F'_2$. Let w be the before vertex of f in M . Let $w_0 = N(w) \cap V(Q[0])$. Then, $w_0 \in V(Q[0] - s_0)$.

Case 3.3.3.1. $w_0 = t_0$ or $w_0 = t_1$.

Without loss of generality, let $w_0 = t_1$. Note $|F_0 \cup \{t_1\}| = 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $R = (s, t_0)$ in $Q[0] - F_0 - t_1$. Let $P_0 = R$ and $P_1 = \{ss_2\} \cup M[s_2, w] \cup \{wt_1\}$.

Case 3.3.3.2. $w_0 \neq t_0$ and $w_0 \neq t_1$.

Note $|F_0| = 0 \leq 2n - 6$ for $n \geq 3$. By Lemma 3, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (w_0, t_1)$ covering $Q[0] - F_0$. Let $P_0 = R_1$ and $P_1 = \{ss_2\} \cup M[s_2, w] \cup \{ww_0\} \cup R_2$.

Case 4. $|T_0| = 3$, $|T_1| = 0$ and $|T_2| = 0$.

Let $T_0 = \{t_0, t_1, t_2\}$. Since $|V(Q[i])| - |F| - |T| - |\{s\}| \geq 3^{n-1} - (2n - 4) - 3 - 1 > 2$ for $n \geq 3$, there are two cycles of length three $C_i = \langle w_i^0, w_i^1, w_i^2, w_i^0 \rangle$ such that $w_i^j \in V(Q[j] - F_j - T_j - s_j)$ for $0 \leq j \leq 2$ and $i \in \{1, 2\}$.

Case 4.1. $|F_i| \leq 2n - 6$ for all $0 \leq i \leq 2$.

Let $t_i^j = N(t_i) \cap V(Q[j])$ for $0 \leq i \leq 2$ and $j \in \{1, 2\}$. Suppose $|F_i| \leq 2$ for $i \in \{1, 2\}$. There is $t_i^1 \in V(Q[1] - F_i)$ for $0 \leq i \leq 2$. Let t_2 be satisfied. By Lemma 2, there is a Hamilton path $L = (w_1^1, t_2^1)$ and $M = (s_2, w_1^2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Note $|F_0 \cup \{t_2\}| \leq 2n - 5$. By Lemma 4, there are two disjoint paths $R_1 = (s, t_0)$ and $R_2 = (s, t_1)$ covering $Q[0] - F_0$. Let $P_0 = R_1$, $P_1 = R_2$ and $P_2 = \{ss_2\} \cup M \cup \{w_1^2 w_1^1\} \cup L \cup \{t_2^1 t_2\}$.

Suppose $|F_i| \geq 3$ for $i \in \{1, 2\}$. Then, $|F_0| \leq 2n - 4 - 6 = 2n - 10$. By Theorem 1, there are three disjoint paths $R_0 = (s, t_0)$ and $R_i = (w_i^0, t_i)$ ($i = 1, 2$) covering $Q[0] - F_0$. By Lemma 2, there is a Hamilton path $L = (s_1, w_1^1)$ and $M = (s_2, w_2^2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_0 = R_0$, $P_1 = \{ss_1\} \cup L \cup \{w_1^1 w_1^0\} \cup R_1$ and $P_2 = \{ss_2\} \cup M \cup \{w_2^2 w_2^0\} \cup R_2$.

Case 4.2. $|F_i| = 2n - 5$ for some $0 \leq i \leq 2$.

Case 4.2.1. $|F_0| = 2n - 5$.

Then, $|F_1 \cup F_2| \leq 1$. Without loss of generality, let $|F_1| = 0$ and $|F_2| \leq 1$. By Lemma 4, there are two disjoint paths $R_0 = (s, t_0)$ and $R_1 = (s, t_1)$ covering $Q[0] - F_0$. Then, $t_2 \in R_i$ for some $0 \leq i \leq 1$. Without loss of generality, let $t_2 \in R_1$. Then, $R_1 = R_1[s, t_2] \cup R_1[t_2, t_1]$. Let z and u be the later vertex of s and t_2 in R_0 and R_1 , respectively. Let $u_i = N(u) \cap V(Q[i])$ and $z_i = N(z) \cap V(Q[i])$ for $1 \leq i \leq 2$.

Suppose $u_2 \in F_2$. Then, $z_2 \notin F_2$. By Lemma 2, there is a Hamilton path $L = (s_1, u_1)$ and $M = (s_2, z_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_0 = \{ss_2\} \cup M \cup \{z_2 z\} \cup R_0[z, t_0]$, $P_1 = \{ss_1\} \cup L \cup \{u_1 u\} \cup R_1[u, t_1]$ and $P_2 = R_0[s, t_2]$.

Suppose $u_2 \notin F_2$. By Lemma 2, there is a Hamilton path $L = (s_1, z_1)$ and $M = (s_2, u_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_0 = \{ss_1\} \cup L \cup \{z_1 z\} \cup R_0[z, t_0]$, $P_1 = \{ss_2\} \cup M \cup \{u_2 u\} \cup R_1[u, t_1]$ and $P_2 = R_0[s, t_2]$.

Case 4.2.2. $|F_1| = 2n - 5$ or $|F_2| = 2n - 5$.

Let $|F_1| = 2n - 5$. Then, $|F_0 \cup F_2| \leq 1$.

Case 4.2.2.1. $|F_0| = 0$.

Since $|F_2| \leq 1$, there is a vertex t_i such that $t_i^2 \notin F_2$ where $t_i^2 = N(t_i) \cap V(Q[2])$ for $0 \leq i \leq 2$. Let t_2 be satisfied. By Lemma 2, there is a Hamilton path $L = (s_1, w_1^1)$ and $M = (w_1^2, t_2^2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Note $|F_0 \cup \{t_2\}| = 1 \leq 2n - 5$. By Lemma 4, there are two disjoint paths $R_0 = (s, t_0)$ and $R_1 = (s_0, t_1)$ covering $Q[0] - F_0$. Let $P_0 = R_0$, $P_1 = R_1$ and $P_2 = \{ss_1\} \cup L \cup \{w_1^1 w_1^2\} \cup M \cup \{t_2^2 t_2\}$.

Case 4.2.2.2. $|F_2| = 0$.

Then, $|F_0| \leq 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 4, there are two disjoint paths $R_1 = (s_0, t_1)$ and $R_2 = (s_0, t_2)$ covering $Q[0] - F_0$. Then, $t_0 \in R_i$ for some $1 \leq i \leq 2$. Without loss of generality, let $t_0 \in R_1$. Then, $R_1 = R_1[s_0, t_0] \cup R_1[t_0, t_1]$. Let u and v be the later vertex of t_0 and s_0 in R_1 and R_2 , respectively. Let $u_2 = N(u) \cap V(Q[2])$ and $v_2 = N(v) \cap V(Q[2])$. Since $|V(Q[i])| - |F| - |\{s_i, u_i, v_i\}| \geq 3^{n-1} - (2n - 4) - 3 \geq 1$ for $n \geq 3$, there is an edge $z_1 z_2$ such that $z_i \in V(Q[i] - F - \{s_i, u_2, v_2\})$ for $1 \leq i \leq 2$.

Note $|F_1| = 2n - 5$. By Lemma 2, there is a Hamilton path $L = (s_1, z_1)$ in $Q[1] - F_1$. Note $|F_2| = 0 \leq 2n - 6$ for $n \geq 3$. By Lemma 3, there are two disjoint paths $M_1 = (z_2, u_2)$ and $M_2 = (s_2, v_2)$ covering $Q[2] - F_2$. Let $P_0 = R_1[s_0, t_0]$, $P_1 = \{ss_1\} \cup L \cup \{z_1 z_2\} \cup M_1 \cup \{u_2 u\} \cup R_1[u, t_1]$ and $P_2 = \{ss_2\} \cup M_2 \cup \{v_2 v\} \cup R_2[v, t_2]$.

Case 4.3. $|F_i| = 2n - 4$ for some $0 \leq i \leq 2$.

Case 4.3.1. $|F_0| = 2n - 4$.

Then, $|F_1| = |F_2| = 0$. Note $|F_0| = 2n - 4 \geq 2$ for $n \geq 3$. Let $f_1, f_2 \in F_0$. Let $F'_0 = F_0 \setminus \{f_1, f_2\} \cup \{t_2\}$. Then, $|F'_0| = 2n - 5$. By Lemma 4, there are two disjoint paths $R_0 = (s, t_0)$ and $R_1 = (s, t_1)$ covering $Q[0] - F'_0$. Then, $f_1, f_2 \in R_i$ for $0 \leq i \leq 1$. Let $f_1, f_2 \in R_1$. Let u and v be the before and later vertex of f_1 , $u_i = N(u) \cap V(Q[i])$ and $v_i = N(v) \cap V(Q[i])$ for $1 \leq i \leq 2$. Let x and y be the before and later vertex of f_2 , $x_i = N(x) \cap V(Q[i])$ and $y_i = N(y) \cap V(Q[i])$ for $1 \leq i \leq 2$. Note $|F_1| = |F_2| = 0$. By Lemma 2, there is a Hamilton path $L = (u_1, v_1)$ in $Q[1] - F_1$. Let $t_2^2 = N(t_2) \cap V(Q[2])$. By Lemma 3, there are two disjoint paths $M_1 = (x_2, y_2)$ and $M_2 = (s_2, t_2^2)$ covering $Q[2] - F_2$. Let $P_0 = R_0$, $P_1 = (R_1 - \{f_1, f_2\}) \cup L \cup M$ and $P_2 = \{ss_2\} \cup M_2 \cup \{t_2^2 t_2\}$. Similarly, $f_1 \in R_0$ and $f_2 \in R_1$.

Case 4.3.2. $|F_1| = 2n - 4$ or $|F_2| = 2n - 4$.

Let $|F_1| = 2n - 4$. Then, $|F_0| = |F_2| = 0$. Note $|F_1| = 2n - 4 \geq 2$ for $n \geq 3$. Let $f \in F_1$. Then, $|F'_1| = |F_1 \setminus \{f\}| = 2n - 5$. By Lemma 2, there is a Hamilton path $L = (s_1, f)$ in $Q[1] - F_1$. Let u be the before vertex of f in L and $u_2 = N(u) \cap V(Q[2])$. Since $|T| = 3 < 1$, there is a t_i such that $t_i^2 \neq u_2$ where $t_i^2 = N(t_i) \cap V(Q[2])$ for $0 \leq i \leq 2$. Let t_2 be satisfied. Note $|F_0 \cup \{t_2\}| = 1 \leq 2n - 5$. By Lemma 4, there are two disjoint paths $R_0 = (s, t_0)$ and $R_1 = (s, t_1)$ covering $Q[0] - F_0 - t_2$. By Lemma 2, there is a Hamilton path

$M = (u_2, t_2^2)$ in $Q[2] - F_2$. Let $P_0 = R_0$, $P_1 = R_1$ and $P_2 = \{ss_1\} \cup (L - f) \cup \{uu_2\} \cup M \cup \{t_2^2 t_2\}$. $\square$

Theorem 2. Let Q_n^3 be a 3-ary n -cube for $n \geq 2$. Let F be a vertex faulty set of Q_n^3 . Let C be a fault-free cycle of length three. Let $s \in V(C)$ and $T \in V(Q_n^3 - F - V(C))$ with $2 \leq |T| = m \leq n$. If $|F| \leq 2n - m - 1$, then there are m disjoint paths from s to T covering $Q_n^3 - F$.

Proof. We prove the theorem by induction on m . If $m = 2$, then $|T| = 2$ and $|F| \leq 2n - 3$. By Lemma 4, there are two disjoint paths from s to T covering $Q_n^3 - F$. If $m = 3$, then $|T| = 3$ and $|F| \leq 2n - 4$. By Lemma 5, there are three disjoint paths from s to T covering $Q_n^3 - F$.

Assume the theorem is true for $m - 1$. Now we prove the theorem holds on m for $4 \leq m \leq n$. Let $T = \{t_1, \dots, t_m\}$. Suppose there is $st_i \in E(Q_n^3)$ for some $1 \leq i \leq m$. Then, $|T \setminus \{t_i\}| = m - 1$ and $|F \cup \{t_i\}| \leq 2n - m = 2n - (m - 1) - 1$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $P_j = (s, t_j)$ ($j \in \{1, \dots, m\} \setminus \{i\}$) covering $Q_n^3 - F - t_i$. Let $P_i = (s, t_i)$. Thus, $P_1, \dots, P_m$ are m disjoint paths covering $Q_n^3 - F$. So, we suppose $st_i \notin E(Q_n^3)$ for all $1 \leq i \leq m$.

Let $C = \langle s_0, s_1, s_2, s_0 \rangle$. So, we can decompose Q_n^3 into three subgraphs $Q[0]$, $Q[1]$, $Q[2]$ along one dimension such that $s_i \in V(Q[i])$ ($0 \leq i \leq 2$). Let $T_i = T \cap V(Q[i])$ and $F_i = F \cap V(Q[i])$ for $0 \leq i \leq 2$. Without loss of generality, let $|T_0| \geq 1$ and $s = s_0$.

Case 1. $|T_0| = m$.

Let $T_0 = \{t_1, \dots, t_m\}$ and $t_i^j = N(t_i) \cap V(Q[j])$ for $1 \leq i \leq m$ and $j = 1, 2$. Then, $|T_1| = |T_2| = 0$.

Case 1.1. $2n - m - 2 \leq |F_0| \leq 2n - m - 1$.

Then, $|F_1 \cup F_2| \leq 1$. Without loss of generality, let $|F_1| = 0$ and $|F_2| \leq 1$. Let $T'_0 = T_0 \setminus \{t_{m-1}, t_m\}$. Then, $|T'_0| = m - 2$. Note $|F_0| \leq 2n - m - 1 = 2(n - 1) - (m - 2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i)$ ($1 \leq i \leq m - 2$) covering $Q[0] - F_0$. Then, $t_{m-1}, t_m \in \cup_{i=1}^{m-2} V(R_i - t_i - s)$.

Case 1.1.1. $t_{m-1}, t_m \in V(R_i)$ for some $1 \leq i \leq m - 2$.

Without loss of generality, let $t_{m-1}, t_m \in V(R_1)$. Let t_{m-1} is before t_m in R_1 . Then, $R_1 = R_1[s, t_{m-1}] \cup R_1[t_{m-1}, t_m] \cup R_1[t_m, t_1]$. Let u and v be the later vertex of t_{m-1} and t_m on R_1 , respectively. Let $u_i = N(u) \cap V(Q[i])$ $v_i = N(v) \cap V(Q[i])$ for $i = 1, 2$. Since $|F_2| \leq 1$, $u_2 \notin F_2$ or $v_2 \notin F_2$.

Suppose $u_2 \notin F_2$. Note $|F_2| \leq 1 \leq 2n - 5$ for $n \geq 3$. By Lemma 2, there is a Hamilton path $M = (s_2, u_2)$ in $Q[2] - F_2$. Note $|F_1| = 0$. By Lemma 2, there is a Hamilton path $L = (s_1, v_1)$ in $Q[1]$. Let $P_1 = \{ss_1\} \cup L \cup \{v_1 v\} \cup R_1[v, t_1]$, $P_i = R_i$ ($2 \leq i \leq m - 2$), $P_{m-1} = R_1[s, t_{m-1}]$ and $P_m = \{ss_2\} \cup M \cup \{u_2 u\} \cup R_1[u, t_m]$. Similar to $v_2 \notin F_2$.

Similarly, t_m is before t_{m-1} on R_1 .

Case 1.1.2. $t_{m-1} \in V(R_i)$, $t_m \in V(R_j)$ for $1 \leq i, j \leq m - 2$ and $i \neq j$.

Without loss of generality, let $t_{m-1} \in V(R_1)$ and $t_m \in V(R_2)$. Then, $R_1 = R_1[s, t_{m-1}] \cup R_1[t_{m-1}, t_1]$ and $R_2 = R_2[s, t_m] \cup R_2[t_m, t_2]$. Let u and v be the later vertex of t_{m-1} and t_m on R_1 and R_2 , respectively. Let $u_i = N(u) \cap V(Q[i])$ and $v_i = N(v) \cap V(Q[i])$ for $i = 1, 2$. Since $|F_2| \leq 1$, $u_2 \notin F_2$ or $v_2 \notin F_2$.

Suppose $u_2 \notin F_2$. By Lemma 2, there is a Hamilton path $L = (s_1, v_1)$ and $M = (s_2, u_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_1 = \{ss_2\} \cup M \cup \{u_2 u\} \cup R_1[u, t_1]$, $P_2 = \{ss_1\} \cup L \cup \{v_1 v\} \cup R_2[v, t_2]$, $P_i = R_i$ ($3 \leq i \leq m - 2$), $P_{m-1} = R_1[s, t_{m-1}]$ and $P_m = R_2[s, t_m]$. Similar to $v_2 \notin F_2$.

Case 1.2. $|F_0| = 2n - m - 3$.

Then, $|F_1 \cup F_2| \leq 2$. Without loss of generality, let $|F_1| \leq 1$. Since $|T_0| = m > 1 \geq |F_1|$, there is a vertex t_i such that $t_i^1 \notin F_1$ for $1 \leq i \leq m$. Without loss of generality, let $t_m^1 \notin F_1$. Note $|F_0 \cup \{t_m\}| = 2n - m - 2 = 2(n - 1) - (m - 1) - 1$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $P_i = (s, t_i)$ ($1 \leq i \leq m - 1$) covering $Q[0] - F_0 - t_m$.

Since $|V(Q[i])| - |F| - |\{s_i, t_m^i\}| \geq 3^{n-1} - (2n - m - 1) - 2 \geq 5$ for $n \geq 3$, there is an edge $w_1 w_2$ such that $w_i \in V(Q[i] - F - \{s_i, t_m^i\})$ for $i = 1, 2$. Note $|F_1 \cup F_2| \leq 2 \leq 2n - 5$ for $n \geq 4$. By Lemma 2, there is a Hamilton path $R = (w_1, t_m^1)$ and $L = (s_2, w_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_m = \{ss_2\} \cup L \cup \{w_2 w_1\} \cup R \cup \{t_m^1 t_m\}$.

Case 1.3. $|F_0| \leq 2n - m - 4$.

If $t_m^1 \notin F_1$, then it is similar to Case 1.2. Suppose $t_m^1 \in F_1$. Let $T'_0 = T_0 \setminus \{t_m\}$. Let $N_{Q[0]}(t_m) = \{u_1, \dots, u_{2n-2}\}$. Since $|N_{Q[0]}(t_m)| - |T'_0| = 2n - 2 - (m - 1) = 2n - m - 1 > 2n - m - 2 \geq |F| \setminus \{t_m^1\}$, there is a vertex $u_i \in V(Q[0] - F_0 - T_0 - s)$ such that $u_i^j \notin (F_j \setminus \{t_m^j\})$ where $u_i^j = N(u_i) \cap V(Q[j])$ for $1 \leq j \leq 2$ and $1 \leq i \leq 2n - 2$. Without loss of generality, let u_1 be satisfied. Note $|F_0| + |\{t_m, u_1\}| \leq 2n - m - 2 = 2(n - 1) - (m - 1) - 1$. By the induction hypothesis, there are $(m - 1)$ disjoint paths $P_i = (s, t_i)$ ($1 \leq i \leq m - 1$) covering $Q[0] - F_0 - \{t_m, u_1\}$. Since $|V(Q[i])| - |F| - |\{s_i, t_m^i, u_1^i\}| \geq 3^{n-1} - (2n - m - 1) - 3 > 1$ for $4 \leq m \leq n$, there is an edge $x_1 x_2$ such that $x_i \in V(Q[i] - F_i - \{s_i, t_m^i, u_1^i\})$ for $i = 1, 2$. Note $|F_i| \leq 2n - 5$ for $i = 1, 2$. By Lemma 2, there is a Hamilton path $L = (x_1, u_1^1)$ and $M = (s_2, x_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_m = \{ss_2\} \cup M \cup \{x_2 x_1\} \cup L \cup \{u_1^1 u_1, u_1 t_m\}$.

Case 2. $|T_0| = m - 1$.

Let $T_0 = \{t_1, \dots, t_{m-1}\}$, $T_1 = \{t_m\}$ and $|T_2| = 0$.

Case 2.1. $|F_0| = 2n - m - 1$.

Then, $|F_1| = |F_2| = 0$. Let $T'_0 = T_0 \setminus \{t_{m-1}\}$. Then, $|T'_0| = m - 2$. Note $|F_0| = 2n - m - 1 = 2(n - 1) - (m - 2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i)$ ($1 \leq i \leq m - 2$) covering $Q[0] - F_0$. Then, $t_{m-1} \in V(R_i - t_i - s)$ for some $1 \leq i \leq m - 2$. Without loss of generality, let $t_{m-1} \in R_1$. Let u be the later vertex of t_m on R_1 and $u_2 = N(u) \cap V(Q[2])$. By Lemma 2, there is a

Hamilton path $L = (s_1, t_m)$ and $M = (s_2, u_2)$ in $Q[1]$ and $Q[2]$, respectively. Let $P_1 = \{ss_2\} \cup M \cup \{u_2u\} \cup R_1[u, t_1]$, $P_i = R_i (2 \leq i \leq m-2)$, $P_{m-1} = R_1[s, t_{m-1}]$ and $P_m = \{ss_1\} \cup L$.

Case 2.2. $|F_0| \leq 2n - m - 2$.

Note $|F_0| \leq 2n - m - 2 = 2(n-1) - (m-1) - 1$. By the induction hypothesis, there are $(m-1)$ disjoint paths $P_i = (s, t_i) (1 \leq i \leq m-1)$ covering $Q[0] - F_0$. Since $|V(Q[i])| - |F| - |\{s_i, t_m\}| \geq 3^{n-1} - (2n - m - 1) - 2 > 1$ for $4 \leq m \leq n$, there is an edge x_1x_2 such that $x_i \in V(Q[i] - F_i - \{s_i, t_m\})$ for $i = 1, 2$. Note $|F_i| \leq 2n - 5$ for $i = 1, 2$. By Lemma 2, there is a Hamilton path $L = (x_1, t_m)$ and $M = (s_2, x_2)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Let $P_m = \{ss_2\} \cup M \cup \{x_2x_1\} \cup L$.

Case 3. $1 \leq |T_0| \leq m - 2$.

Let $|T_0| = p$. Then, $1 \leq p \leq m - 2$ and $2 \leq |T_1 \cup T_2| = m - p \leq m - 1$.

Case 3.1. $|T_1| = 0$ or $|T_2| = 0$.

Without loss of generality, let $|T_1| = 0$. Then, $2 \leq |T_2| = m - p \leq m - 1$. Let $T_0 = \{t_1, \dots, t_p\}$ and $T_2 = \{t_{p+1}, \dots, t_m\}$. Let $t_i^j = N(t_i) \cap V(Q[j])$ for $p + 1 \leq i \leq m$ and $j = 0, 1$. Since $|V(Q[i])| - |F| - |T| - |\{s_i\}| \geq 3^{n-1} - (2n - m - 1) - m - 1 > m - 1 \geq m - p$, there are $(m - p)$ cycles of length three $C_i = \langle w_i^0, w_i^1, w_i^2, w_i^0 \rangle$ such that $w_i^j \in V(Q[j] - F_j - T_j - s_j)$ for $0 \leq j \leq 2$ and $p + 1 \leq i \leq m$.

Case 3.1.1. $|T_0| > |T_2|$.

Since $|T_0| = p > m - p = |T_2|$, we get $m \leq 2p - 1$. Let $W_2 = \{w_i^2: p + 1 \leq i \leq m - 1\}$ for $m - p \geq 2$. Note $|W_2 \cup \{s_2\}| = m - p = |T_2|$ and $|F_2| \leq 2n - m - 1 \leq 2(n-1) - 2(m-p)$ for $m \leq 2p - 1$. By Theorem 1, there are $(m - p)$ disjoint paths $M_i = (w_i^2, t_i) (p + 1 \leq i \leq m - 1)$ and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T_2 covering $Q[2] - F_2$.

Let $W_1 = \{w_{p+1}^1\}$. Note $|F_1| \leq 2n - 5$. By Lemma 2, there is a Hamilton path $L = (s_1, w_{p+1}^1)$ in $Q[1] - F_1$.

Let $W_0 = \{w_i^0: p + 2 \leq i \leq m - 1\}$ for $m - p \geq 3$ and $W_0 = \emptyset$ for $m - p = 2$. Then, $|W_0| = m - p - 2$. Note $|T_0| + |W_0| = p + (m - p - 2) = m - 2$ and $|F_0| \leq 2n - m - 1 = 2(n-1) - (m-2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i) (1 \leq i \leq p)$ and $R_i = (s, w_i^0) (p + 2 \leq i \leq m - 1)$ from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i (1 \leq i \leq p)$, $P_{p+1} = \{ss_1\} \cup L \cup \{w_{p+1}^1w_{p+1}^2\} \cup R_{p+1}$, $P_i = R_i \cup \{w_i^0w_i^2\} \cup M_i (p + 2 \leq i \leq m - 1)$ and $P_m = \{ss_2\} \cup M_m$.

Case 3.1.2. $|T_0| = |T_2|$.

Since $|T_0| = p = m - p = |T_2|$, we get $m = 2p \geq 4$. Thus, $m - p = p \geq 2$.

Case 3.1.2.1. $|F_2| = 2n - m - 1$.

Let $T'_2 = T_2 \setminus \{t_{p+1}\}$. Then, $|T'_2| = p - 1$. Note $|F_1| = 0$. By Lemma 2, there is a Hamilton path $L = (s_1, t_{p+1}^1)$ in $Q[1]$.

Let $W_2 = \{w_i^2: p + 2 \leq i \leq m - 1\}$ for $m - p \geq 3$ and $W_2 = \emptyset$ for $m - p = 2$. Then, $|W_2| = m - p - 2 = p - 2$. Note $|W_2 \cup \{s_2\}| = |T'_2| = p - 1$ and $|F_2 \cup \{t_{p+1}\}| = 2n - m = 2(n-1) - 2(p-1)$ for $m = 2p$. By Theorem 1, there are $(p - 1)$ disjoint paths $M_i = (w_i^2, t_i) (p + 2 \leq i \leq m - 1)$ and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T'_2 covering $Q[2] - F_2 - t_{p+1}$.

Let $W_0 = \{w_i^0: p + 2 \leq i \leq m - 1\}$ for $m - p \geq 3$ and $W_0 = \emptyset$ for $m - p = 2$. Then, $|W_0| = m - p - 2$. Note $|T_0| + |W_0| = p + (m - p - 2) = m - 2$ and $|F_0| \leq 2n - m - 1 = 2(n-1) - (m-2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i) (1 \leq i \leq p)$ and $R_i = (s, w_i^0) (p + 2 \leq i \leq m - 1)$ from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i (1 \leq i \leq p)$, $P_{p+1} = \{ss_1\} \cup L \cup \{t_{p+1}^1t_{p+1}\}$, $P_i = R_i \cup \{w_i^0w_i^2\} \cup M_i (p + 2 \leq i \leq m - 1)$, and $P_m = \{ss_2\} \cup M_m$.

Case 3.1.2.2. $|F_2| \leq 2n - m - 2$.

Let $W_2 = \{w_i^2: p + 1 \leq i \leq m - 1\}$. Note $|\{s_2\} \cup W_2| = 1 + (m - p - 1) = m - p = p = |T_2|$ and $|F_2| \leq 2n - m - 2 = 2(n-1) - 2p$ for $m = 2p$. By Theorem 1, there are p disjoint paths $M_i = (w_i^2, t_i) (p + 1 \leq i \leq m - 1)$ and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T_2 covering $Q[2] - F_2$.

Let $W_1 = \{w_{p+1}^1\}$. Note $|F_1| \leq 2n - 5$. By Lemma 2, there is a Hamilton path $L = (s_1, w_{p+1}^1)$ in $Q[1] - F_1$.

Let $W_0 = \{w_i^0: p + 2 \leq i \leq m - 1\}$ for $m - p \geq 3$ and $W_0 = \emptyset$ for $m - p = 2$. Then, $|W_0| = m - p - 2$. Note $|T_0| + |W_0| = p + (m - p - 2) = m - 2$ and $|F_0| \leq 2n - m - 1 = 2(n-1) - (m-2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i) (1 \leq i \leq p)$ and $R_i = (s, w_i^0) (p + 2 \leq i \leq m - 1)$ from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i (1 \leq i \leq p)$, $P_{p+1} = \{ss_1\} \cup L \cup \{w_{p+1}^1w_{p+1}^2\} \cup M_{p+1}$, $P_i = R_i \cup \{w_i^0w_i^2\} \cup M_i (p + 2 \leq i \leq m - 1)$, and $P_m = \{ss_2\} \cup M_m$.

Case 3.1.3. $|T_0| < |T_2|$.

Since $|T_0| = p < m - p = |T_2|$, we get $m > 2p \geq 2$.

Case 3.1.3.1. $|F_2| \leq 2n - 2m + 2p - 2$.

Note $|F_2| \leq 2n - 2m + 2p - 2 = 2(n-1) - 2(m-p)$. Let $W_2 = \{w_i^2: p + 1 \leq i \leq m - 1\}$. Note $|\{s_2\} \cup W_2| = 1 + (m - p - 1) = m - p = |T_2|$. By Theorem 1, there are $(m - p)$ disjoint paths $M_i = (w_i^2, t_i) (p + 1 \leq i \leq m - 1)$ and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T_2 covering $Q[2] - F_2$.

Let $W_1 = \{w_{p+1}^1\}$. Note $|F_1| \leq 2n - m - 1 \leq 2n - 5$ for $m \geq 4$. By Lemma 2, there is a Hamilton path $L = (s_1, w_{p+1}^1)$ in $Q[1] - F_1$.

Let $W_0 = \{w_i^0: p + 2 \leq i \leq m - 1\}$ for $m - p \geq 3$ and $W_0 = \emptyset$ for $m - p = 2$. Then, $|W_0| = m - p - 2$. Note

$|T_0| + |W_0| = p + (m - p - 2) = m - 2$ and $|F_0| \leq 2n - m - 1 = 2(n - 1) - (m - 2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i)$ ($1 \leq i \leq p$) and $R_i = (s, w_i^0)$ ($p + 2 \leq i \leq m - 1$) from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i$ ($1 \leq i \leq p$), $P_{p+1} = \{ss_1\} \cup L \cup \{w_{p+1}^1 w_{p+1}^2\} \cup M_{p+1}$, $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i$ ($p + 2 \leq i \leq m - 1$), and $P_m = \{ss_2\} \cup M_m$.

Case 3.1.3.2. $2n - 2m + kp - 1 \leq |F_2| \leq 2n - 2m + (k + 1)p - 2$ for $2p \leq kp \leq m$.

Then, $|F_1| \leq 2n - m - 1 - (2n - 2m + kp - 1) = m - kp$. Since $|T_2| - |F_1| \geq (m - p) - (m - kp) = (k - 1)p$, there are $(k - 1)p$ vertices $t_i \in T_2$ such that $t_i^1 \notin F_1$. Without loss of generality, let $Z = \{t_i: p + 1 \leq i \leq kp\}$ satisfy above requirement. Let $Z_1 = \{t_i^1: p + 1 \leq i \leq kp\}$. Then, $|Z_1| = |Z| = (k - 1)p$. Let $T_2' = T_2 \setminus Z$. Let $W_1 = \{w_i^1: p + 2 \leq i \leq kp\}$ for $(k - 1)p \geq 2$ and $W_1 = \emptyset$ for $(k - 1)p = 1$. Note $|F_1| \leq m - kp \leq 2(n - 1) - 2(k - 1)p$ for $2 \leq 2p \leq kp \leq m \leq n$. By Theorem 1, there are $(k - 1)p$ disjoint paths $L_{p+1} = (s_1, t_{p+1}^1)$ and $L_i = (w_i^1, t_i^1)$ ($p + 2 \leq i \leq kp$) from $W_1 \cup \{s_1\}$ to Z_1 covering $Q[1] - F_1$.

Case 3.1.3.2.1. $2p \leq kp < m$.

Let $W_2 = \{w_i^2: kp + 1 \leq i \leq m - 1\}$ for $m - kp \geq 2$ and $W_2 = \emptyset$ for $m - kp = 1$. Note $|\{s_2\} \cup W_2| = m - kp = |T_2 \setminus Z|$ and $|F_2 \cup Z| \leq (2n - 2m + (k + 1)p - 2) + (k - 1)p = 2n - 2m + 2kp - 2 = 2(n - 1) - 2(m - kp)$. By Theorem 1, there are $(m - kp)$ disjoint paths $M_i = (w_i^2, t_i)$ ($kp + 1 \leq i \leq m - 1$) and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T_2' covering $Q[2] - F_2 - Z$.

Let $W_0 = \{w_i^0: p + 2 \leq i \leq m - 1\}$ for $m - p \geq 3$ and $W_0 = \emptyset$ for $m - p = 2$. Then, $|W_0| = m - p - 2$. Note $|T_0| + |W_0| = p + (m - p - 2) = m - 2$ and $|F_0| \leq 2n - m - 1 \leq 2(n - 1) - (m - 2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i)$ ($1 \leq i \leq p$) and $R_i = (s, w_i^0)$ ($p + 2 \leq i \leq m - 1$) from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i$ ($1 \leq i \leq p$), $P_{p+1} = \{ss_1\} \cup L_{p+1} \cup \{t_{p+1}^1 t_{p+1}^2\}$, $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i \cup \{t_i^1 t_i^2\}$ ($p + 2 \leq i \leq kp$), $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i$ ($kp + 1 \leq i \leq m - 1$), and $P_m = \{ss_2\} \cup M_m$.

Case 3.1.3.2.2. $kp = m$.

$2n - m - 1 = 2n - 2m + kp - 1 \leq |F_2| \leq 2n - 2m + (k + 1)p - 2 = 2n - m + p - 2 \leq 2n - m - 1$. Thus, $|T_0| = p = 1$, $|T_2| = m - p = m - 1$, $|F_2| = 2n - m - 1$ and $|F_0| = |F_1| = 0$. Let $T_2' = T_2 \setminus \{t_m\}$. Then, $|T_2'| = m - 2$. Note $|F_2| = 2n - m - 1 = 2(n - 1) - (m - 2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $M_i = (s_2, t_i)$ ($2 \leq i \leq m - 1$) from s_2 to T_2' covering $Q[2] - F_2$. Then, $t_m \in R_i$ for some $2 \leq i \leq m - 1$. Without loss of generality, let $t_m \in R_{m-1}$. Let u and z_i be the next vertex of t_m and s_2 in R_{m-1} and R_i for $2 \leq i \leq m - 2$, respectively. Let $u_1 = N(u) \cap V(Q[1])$ and $z_i^0 = N(z_i) \cap V(Q[0])$. By the definition of Q_n^3 , $z_i^0 \in N(s)$. Since $st_i \notin E(Q_n^3)$, $z_i^0 \notin T_0$. Let $Z_0 = \{z_i^0: 2 \leq i \leq m - 2\}$. By Lemma 2, there is a Hamilton path $L = (s_1, u_1)$ in $Q[1]$. Note

$|T_0| + |Z_0| = m - 2$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_1 = (s, t_1)$ and $R_i = (s, z_i^0)$ ($2 \leq i \leq m - 2$) from s to $T_0 \cup Z_0$ covering $Q[0] - F_0$.

Let $P_1 = R_1$, $P_i = R_i \cup \{z_i^0 z_i^1\} \cup M_i[z_i, t_i]$ ($2 \leq i \leq m - 2$), $P_{m-1} = \{ss_1\} \cup L \cup \{u_1 u\} \cup M_{m-1}[u, t_{m-1}]$ and $P_m = \{ss_2\} \cup M_{m-1}[s_2, t_m]$.

Case 3.2. $|T_1| = |T_2| = 1$.

Then, $|T_0| = m - 2$. Let $T_0 = \{t_1, \dots, t_{m-2}\}$, $T_1 = \{t_{m-1}\}$ and $T_2 = \{t_m\}$. Note $|F_i| \leq 2n - 5$ for $1 \leq i \leq 2$. By Lemma 2, there is a Hamilton path $L = (s_1, t_{m-1})$ and $M = (s_2, t_m)$ in $Q[1] - F_1$ and $Q[2] - F_2$, respectively. Note $|F_0| \leq 2n - m - 1 = 2(n - 1) - (m - 2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i)$ ($1 \leq i \leq m - 2$) covering $Q[0] - F_0$.

Let $P_i = R_i$ ($1 \leq i \leq m - 2$), $P_{m-1} = \{ss_1\} \cup L$ and $P_m = \{ss_2\} \cup M$.

Case 3.3. $|T_i| \geq 2$ for $i = 1, 2$.

Let $|T_1| = p$ and $|T_2| = q$. Since $1 \leq |T_0| = m - p - q \leq m - 4$, $4 \leq p + q \leq m - 1$. Without loss of generality, let $|T_1| = p \geq q = |T_2|$. Then, $m - 3 \geq p \geq q \geq 2$. Since $1 \leq m - p - q \leq m - 2q$, we get $2q + 1 \leq m$. Let $T_0 = \{t_1, \dots, t_{m-p-q}\}$, $T_1 = \{t_{m-p-q+1}, \dots, t_{m-q}\}$ and $T_2 = \{t_{m-q+1}, \dots, t_m\}$.

Since $|V(Q[i])| - |F_i| - |T_i| - |\{s_i\}| \geq 3^{n-1} - (2n - m - 1) - m - 1 \geq m - 3 \geq p + q - 2$, there are $(p + q - 2)$ cycles $C_i = \langle w_i^0, w_i^1, w_i^2, w_i^0 \rangle$ where $w_i^j \in V(Q[j] - F - T - s_j)$ for $m - p - q + 2 \leq i \leq m - 1$ and $1 \leq j \leq 2$.

Let $W_2 = \{w_i^2: m - q + 1 \leq i \leq m - 1\}$ for $q \geq 2$. Then, $|W_2 \cup \{s_2\}| = q = |T_2|$. Note $|F_2| \leq 2n - m - 1 \leq 2(n - 1) - 2q$ for $m \geq 2q + 1$. By Theorem 1, there are q disjoint paths $M_i = (w_i^2, t_i)$ ($m - q + 1 \leq i \leq m - 1$) and $M_m = (s_2, t_m)$ from $W_2 \cup \{s_2\}$ to T_2 covering $Q[2] - F_2$.

Case 3.3.1. $m \geq 2p + 1$.

Let $W_1 = \{w_i^1: m - p - q + 2 \leq i \leq m - q\}$ for $p \geq 2$. Then, $|W_1 \cup \{s_1\}| = p = |T_1|$. Note $|F_1| \leq 2n - m - 1 \leq 2(n - 1) - 2p$ for $m \geq 2p + 1$. By Theorem 1, there are p disjoint paths $L_{m-p-q+1} = (s_1, t_{m-p-q+1})$ and $L_i = (w_i^1, t_i)$ ($m - p - q + 2 \leq i \leq m - q$) from $\{s_1\} \cup W_1$ to T_1 covering $Q[1] - F_1$.

Let $W_0 = \{w_i^0: m - p - q + 2 \leq i \leq m - 1\}$. Then, $|W_0| = p + q - 2$. Note $|T_0| + |W_0| = (m - p - q) + (p + q - 2) = m - 2$ and $|F_0| \leq 2n - m - 1 = 2(n - 1) - (m - 2) - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i)$ ($1 \leq i \leq m - p - q$) and $R_i = (s, w_i^0)$ ($m - p - q + 2 \leq i \leq m - 1$) from s to $T_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i$ ($1 \leq i \leq m - p - q$), $P_{m-p-q+1} = \{ss_1\} \cup L_{m-p-q+1}$, $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i$ ($m - p - q + 2 \leq i \leq m - q$), $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i$ ($m - q + 1 \leq i \leq m - 1$) and $P_m = \{ss_2\} \cup L_m$.

Case 3.3.2. $2p \geq m$.

Case 3.3.2.1. $|F_0| \geq 2p - m + 1$.

Note $|F_1| \leq 2n - m - 1 - (2p - m + 1) = 2n - 2p - 2 = 2(n - 1) - 2p$. Then, it is similar to Case 3.3.1.

Case 3.3.2.2. $2p - m - (k + 1)q + 1 \leq |F_0| \leq 2p - m - kq$ for $0 \leq kp \leq 2p - m$.

Then, $|F_1| \leq 2n - m - 1 - (2p - m - (k + 1)q + 1) = 2n - 2p + (k + 1)q - 2$ and $|F_0| + |T_0| \leq (2p - m - kq) + (m - p - q) = p - (k + 1)q < p = |T_1|$. Since $|T_2| - |T_0| - |F_0| \geq p - (p - (k + 1)q) = (k + 1)q$, there are $(k + 1)q$ vertices $t_i \in T_1$ such that $t_i^0 \notin (F_0 \cup T_0)$. Let t_i ($m - p - q + 1 \leq i \leq m - p + kq$) satisfy above requirement. Let $Z = \{t_i: m - p - q + 1 \leq i \leq m - p + kq\}$ and $Z_0 = \{t_i^0: m - p - q + 1 \leq i \leq m - p + kq\}$. Then, $|Z| = |Z_0| = (k + 1)q$. Note $|T_1 \setminus Z| = p - (k + 1)q \geq p - (2p - m) - q = m - p - q \geq 1$ and $|F_1| + |Z| \leq (2n - 2p + (k + 1)q - 2) + (k + 1)q = 2n - 2p + 2(k + 1)q - 2 = 2(n - 1) - 2(p - (k + 1)q)$.

Let $W_1 = \{w_i^1: m - p + kq + 2 \leq i \leq m - q\}$ for $(k + 1)q \leq 2p - m + 1 \leq p - 2$. Then, $|W_1| = p - (k + 1)q - 1$. By Theorem 1, there are $(p - (k + 1)q)$ disjoint paths $L_{m-p+kq+1} = (s_1, t_{m-p+kq+1})$ and $L_i = (w_i^1, t_i)(m - p + kq + 2 \leq i \leq m - q)$ from $W_1 \cup \{s_1\}$ to $T_1 \setminus Z$ covering $Q[1] - F_1 - Z$.

Let $W_0 = \{w_i^0: m - p + kq + 2 \leq i \leq m - 1\}$ for $kq \leq 2p - m \leq p - 3$. Then, $|W_0| = p - kq - 2$. Note $|T_0| + |Z_0| + |W_0| = (m - p - q) + (k + 1)q + (p - kq - 2) = m - 2$ and $|F_0| \leq 2n - m - 1$. By the induction hypothesis, there are $(m - 2)$ disjoint paths $R_i = (s, t_i)(1 \leq i \leq m - p - q)$ and $R_i = (s, t_i^0)(m - p - q + 1 \leq i \leq m - p + kq)$, $R_i = (s, w_i^0)(m - p + kq + 2 \leq i \leq m - 1)$ from s to $T_0 \cup Z_0 \cup W_0$ covering $Q[0] - F_0$.

Let $P_i = R_i(1 \leq i \leq m - p - q)$, $P_i = R_i \cup \{t_i^0 t_i\}(m - p - q + 1 \leq i \leq m - p + kq)$, $P_{m-p+kq+1} = \{ss_1\} \cup L_{m-p+kq+1}$, $P_i = R_i \cup \{w_i^0 w_i^1\} \cup L_i(m - p + kq + 2 \leq i \leq m - q)$, $P_i = R_i \cup \{w_i^0 w_i^2\} \cup M_i(m - q + 1 \leq i \leq m - 1)$ and $P_m = \{ss_2\} \cup M_m$. $\square$

5. Conclusions

In this paper, we studied m disjoint paths from S to T covering $Q_n^3 - F$ with $|F| \leq 2n - 2m$ for $2 \leq |S| = |T| = m \leq n$. Furthermore, we determined there are m disjoint paths from s to T covering $Q_n^3 - F$ with $|F| \leq 2n - m - 1$ where $2 \leq |T| = m \leq n$ and s is in a fault-free cycle of length three. In future research, scholars can continue to study vertex disjoint path problem of Q_n^k or other graphs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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