Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 3965721, 8 pages
https://doi.org/10.1155/2020/3965721

Research Article

Existence and Multiplicity of Positive Solutions for Kirchhoff-Type Equations with the Critical Sobolev Exponent

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Received 25 October 2019; Revised 7 December 2019; Accepted 16 December 2019; Published 20 January 2020

Academic Editor: Rosa M. Benito

In this paper, we consider the following Kirchhoff-type problems involving critical exponent $\begin{cases} -(a+b\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u + V(x)u = \mu u^{2^*-1} + \lambda g(x,u), & x \in \Omega \\ u > 0, & x \in \Omega \\ u = 0, & x \in z\Omega \end{cases}$. The existence and multiplicity of positive solutions for Kirchhoff-type equations with a nonlinearity in the critical growth are studied under some suitable assumptions on $V(x)$ and $g(x,u)$. By using the mountain pass theorem and Brézis–Lieb lemma, the existence and multiplicity of positive solutions are obtained.

1. Introduction

We consider the following nonlinear boundary value problem for second-order impulsive differential equations:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = \mu u^{2^*-1} + \lambda g(x,u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain and $\partial\Omega$ is a smooth boundary of Ω. $\lambda, \mu > 0, a, b \geq 0, a + b > 0$. $V \in C(\Omega, \mathbb{R})$ and $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. $2^* = 2N/(N-2)$ is the critical Sobolev exponent for the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. Moreover, $V(x)$ and $g(x,u)$ satisfy some conditions which will be given later.

Over the past decades, the following Kirchhoff equation:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = g(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

has been extensively considered. With various assumptions about the nonlinearity $g(x,u)$, the existence and multiplicity of solutions for system (2) are obtained by variational methods, see [1–5] and the references therein.

To our best knowledge, system (2) is related to the stationary analogue of the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

which was proposed by Kirchhoff in [6]. In fact, (3) is an extension and generalization of the classical D'Alembert wave equation in some ways. This model is widely used in many fields, such as non-Newtonian mechanics, cosmophysics, elastic theory, and electromagnetics. It is worth noting that equation (2) has a nonlocal term $\int_0^L |\partial u/\partial x|^2 \mathrm{d}x$. Only after Lions [7] proposed an abstract functional analysis framework about the following equation:

$$u_{tt} - \left(a+b\int_\Omega |Du|^2\right)\Delta u = g(x,u), \tag{4}$$

problem (4) received much attention, and we refer the readers to [8–15] for more details and the references therein. More precisely, Bisci and Pizzimenti [13] studied the existence of infinitely many solutions for a class of Kirchhoff-type problems involving the p-Laplacian by using variational methods. In [15], Bisci considered the existence of (weak) solutions for some Kirchhoff-type problems on a geodesic ball of the hyperbolic space and the main technical approach is based on variational and topological methods. In [16], the authors firstly used the variation method to study the existence of positive solution of the Kirchhoff-type problems with the Sobolev critical exponent. After that, there are many works on the existence and multiplicity of solutions for Kirchhoff-type problems with the Sobolev critical exponent (one can see [17–21] and the references therein).

In [17], the author considered the following Kirchhoff-type elliptic equation:

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2 \mathrm{d}x\right)\Delta u = \mu g(x,u) + u^5, & u>0,\ x\in\Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{5}$$

and by using the variation method, the existence of positive solutions of system (5) is obtained. To our best knowledge, a nonlinear elliptic boundary value problem has a critical term, which is a difficulty to prove the existence of solutions for the problem. The difficulty is caused by the lack of compactness of the embedding $H_0^1(\Omega)\hookrightarrow L^{2^*}(\Omega)$, which makes the PS condition cannot be checked directly. In [16], the authors make the parameter μ large enough to make a critical value below a certain level. In [17], under the AR condition, the authors restored the compactness of the embedding by using the second concentration compactness lemma, which is an extension of the work in [8].

In [18], the authors used the variational method to consider system (5) with $\mu = 1$ and the existence and multiplicity of solutions for the system are obtained.

Moreover, problems on the unbounded domain R^N have also been widely studied by some researchers, for example, [22–26]. More precisely, in [25], Liu and He used the variant version of fountain theorem to get the existence of infinitely many high energy solutions of the system. In [26], the authors studied the concentration behavior of positive solutions. For more information about this problem, we refer the readers to [11, 20, 27, 28] and the reference therein.

Motivated by the above facts, we want to consider the positive solutions of system (1). By using the mountain pass theorem and Brézis–Lieb lemma, the existence and multiplicity of positive solutions of system (1) are obtained.

To show our main results, we introduce some conditions on nonlinearity $g(x,u)$ and $V(x)$.

$(V1)$ $V(x)$ is 1-periodic in each of $x_i\,(i=1,2,\ldots,N)$, and there exists a positive constant V_0 such that

$$V(x) \geq V_0 > 0, \quad x\in\Omega. \tag{6}$$

$(F1)$ If $s\leq 0$, then $g(x,s)\equiv 0$; if $s\geq 0$, then $g(x,s)\geq 0$.

$(F2)$

$$\lim_{s\longrightarrow 0^+}\frac{g(x,s)}{s} = 0,$$
$$\lim_{s\longrightarrow +\infty}\frac{g(x,s)}{s^5} = 0, \tag{7}$$
$$x\in\overline{\Omega}.$$

$(F3)$ $\forall (x,s)\in\Omega\times R_+$, there is

$$g(x,s)s - 4G(x,s) \geq -a\lambda_1 s^2, \tag{8}$$

where $G(x,s)=\int_0^s g(x,t)\mathrm{d}t$ and $\lambda_1>0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

$(G1)$ $g(x,u) = f(x)u^{q-1}\,(2\leq q<2^*)$.
$f(x)\in L^{(2^*-q)/2^*}(\Omega)$ and $f(x)\geq 0$, $f(x)\equiv 0$.

$(G2)$ $V(x)\in L^2(\Omega)$. In addition, $V(x)>0$ is bounded in Ω.

In the next section, we will present our main results.

Theorem 1. *Let $N=3$ and $\lambda=\mu=1$. If $(V1)$ and $(F1)-(F3)$ hold, then system (1) has a positive ground state solution.*

Remark 1. In [17], the author used a condition which is stronger than $(F3)$, that is,

$(F3)'$ There exists a constant $\theta\in(4,6)$, such that

$$g(x,s)s - \theta G(x,s)\geq 0, \quad \forall (x,s)\in\Omega\times R_+. \tag{9}$$

Theorem 2. *Let $N=4$ and $0<\mu<bS^2$. If $(G1)$ and $(G2)$ hold, then there exists a constant $\lambda_*>0$ (we will give in the proof of Theorem 2 in Section 3) such that $\forall\lambda>\lambda_*$, and system (1) has at least two positive solutions.*

Remark 2. In reference [21], the authors considered system (1) as $f(x)\equiv 1$, $1<q<2$, $V(x)=0$, and $\mu=1$. Underlying the condition $\lambda<\lambda_0$ (a constant the authors given in their paper), two positive solutions are obtained. However, our results are very different from those in [21]. In our paper, $2\leq q<2^*$, and if $\lambda>\lambda_*$ (a constant we give in the proof of Theorem 2) is sufficiently large, two positive solutions are obtained. Besides, in [21], $N=3$; in our paper, for $N=4$, the multiple solutions of higher dimensional space are obtained.

Remark 3. When $a=1, b=0$, and $V(x)=0$, system (1) degenerates to a classical semilinear elliptic problem. Theorem 2 can be the generalization of the corresponding results in [8] of Kirchhoff-type problems.

The reminder of this paper is organized as follows. In Section 2, some preliminary results are presented. The proof of main results will be given in Section 3.

2. Preliminaries

In this paper, we make some notations as follows:

(i) The space $H_0^1(\Omega)$ (denoted by E) is equipped with the norm $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2}$, and we also define the equivalent norm by $\|u\|_E = (\int_\Omega (|\nabla u|^2 + V(x)u^2)dx)^{1/2}$. In addition, $L^p(\Omega)\,(1 \le p < +\infty)$ is a Lebesgue space with the norm $|u|_p = (\int_\Omega |u|^p ds)^{1/p}$.

(ii) The sequence $\{x_n\}$ in $H_0^1(\Omega)$ is a $(PS)_c$ sequence if $I(x_n) \longrightarrow c$ and $I'(x_n) \longrightarrow 0$ as $n \longrightarrow \infty$. We say that if the functional satisfies $(PC)_c$ condition for any $(PS)_c$ sequence, it has a convergent subsequence.

(iii) $C, C_1, C_2, \ldots$, denote various positive constants.

(iv) $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \max\{-u(x), 0\}$.

(v) $o(1)$ shows when $n \longrightarrow \infty, o(1) \longrightarrow 0$.

(vi) Let S be the best Sobolev constant, that is,

$$S := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{|u|_{2^*}^2}. \tag{10}$$

Now, we give the energy functional corresponding to problem (1), that is,

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{2}\int_\Omega V(x)u^2 dx - \lambda \int_\Omega G(x,u)dx - \frac{\mu}{2^*}\int_\Omega (u^+)^{2^*} dx. \tag{11}$$

It is obvious that $I \in C^1(E,R)$ and has the following derivative:

$$\langle I'(u), v\rangle = \left(a + b\|u\|^2\right)\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega V(x)uv \, dx - \lambda\int_\Omega g(x,u)v \, dx - \mu\int_\Omega (u^+)^{2^*-1} v \, dx. \tag{12}$$

Using the continuity of $g(x,u)$ and $V(x)$, it shows that $u \in E$ is a critical point of I, if it is a solution of problem (1).

Lemma 1. *Let $N = 3$ and $\lambda = \mu = 1$. If (V1), (F1), and (F2) are satisfied, then the following hold:*

(1) There exist constants ρ, $\alpha > 0$, such that

$$I(u) \ge \alpha, \quad \forall u \in H_0^1(\Omega), \ \|u\| = \rho. \tag{13}$$

(2) There exists $u \in E$ such that

$$I(u) < 0 \quad (\|u\| > \rho). \tag{14}$$

Proof. From $(F1)$ and $(F2)$, it shows that there has a constant $C_1 > 0$ such that

$$|G(x,s)| \le \frac{a\lambda_1}{4}|s|^2 + C_1|s|^6, \quad \forall (x,s) \in \overline{\Omega} \times R. \tag{15}$$

By (10), (15), $(V1)$, and Sobolev inequality, it follows that

$$\begin{aligned} I(u) &\ge \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{2}\int_\Omega V(x)u^2 - \int_\Omega G(x,u)dx \\ &\quad - \frac{1}{6}\int_\Omega |u|^6 dx, \ge \frac{a}{2}\|u\|^2 - \frac{a}{4}\|u\|^2 - C_1|u|^6 - \frac{1}{6}|u|^6 \\ &\ge \frac{a}{4}\|u\|^2 - C_2\|u\|^6. \end{aligned} \tag{16}$$

(1) Taking $\rho > 0$ small enough, there exists a constant $\alpha > 0$ such that

$$I(u) \ge \alpha, \quad \forall u \in H_0^1(\Omega), \ \|u\| = \rho. \tag{17}$$

(2) If we take $v_0 \in H_0^1(\Omega)$ and $v_0 \equiv 0$, then one gets the following:

$$\begin{aligned} I(tv_0) &= \frac{a}{2}t^2\|v_0\|^2 + \frac{b}{4}t^4\|v_0\|^4 + \frac{1}{2}t^2\int_\Omega V(x)v_0^2 \\ &\quad - \int_\Omega G(x,tv_0)dx - \frac{1}{6}t^6\int_\Omega (v_0^+)^6 dx \\ &\le \frac{t^2}{2}\max\{a,1\}\|v_0\|_E^2 + \frac{b}{4}t^4\|v_0\|_E^4 - \frac{1}{6}t^6\int_\Omega (v_0^+)^6 dx. \end{aligned} \tag{18}$$

Since $\|\cdot\|$ and the $\|\cdot\|_E$ are equivalent, then

$$I(tv_0) \le C_3\frac{t^2}{2}\max\{a,1\}\|v_0\|^2 + C_4\frac{b}{4}t^4\|v_0\|^4 - \frac{1}{6}t^6\int_\Omega (v_0^+)^6 dx. \tag{19}$$

It is obvious that

$$I(tv_0) \longrightarrow -\infty \quad (t \longrightarrow +\infty). \tag{20}$$

Therefore, we can find a positive constant t_0, and $\|t_0v_0\| > \rho$, such that

$$I(t_0v_0) < 0. \tag{21}$$

Let $u = t_0v_0$ and the conclusion is satisfied. □

Lemma 2. *Let $N = 3$ and $\lambda = \mu = 1$. $V(x)$ satisfies (V1) and $g(x,u)$ satisfies (F1)—(F3). Suppose*

$$\Lambda = \frac{ab}{4}S^3 + \frac{b^3}{24}S^6 + \frac{aS}{6}\sqrt{b^2S^4 + 4aS} + \frac{b^2}{24}S^4\sqrt{b^2S^4 + 4aS}, \quad c \in (0,\Lambda), \tag{22}$$

then I satisfies the $(PS)_c$ condition.

Proof. By ($F1$) and ($F2$), there exists a constant $C_5 > 0$, such that $\forall (x, s) \in \overline{\Omega} \times R$, and it has

$$\left|\frac{1}{5} g(x, s)s - G(x, s)\right| \le \frac{1}{30}|s|^6 + C_5. \tag{23}$$

Suppose $\{u_n\}$ is a $(PS)_c$ sequence, $c \in (0, \Lambda)$,

$$I(u_n) \longrightarrow c, I'(u_n) \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{24}$$

The next work is to prove the boundness of $\{u_n\}$. Clearly, one has

$$\begin{aligned} 1 + c + o(1)\|u_n\| &\ge I(u_n) - \frac{1}{5}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{a}{2} - \frac{a}{5}\right)\|u_n\|^2 + \left(\frac{b}{4} - \frac{b}{5}\right)\|u_n\|^4 \\ &\quad + \left(\frac{1}{2} - \frac{1}{5}\right)\int_\Omega V(x)u_n^2 \mathrm{d}x \\ &\quad + \left(\frac{1}{5} - \frac{1}{6}\right)\int_\Omega (u_n^+)^6 \mathrm{d}x \\ &\quad + \int_\Omega \left[\frac{1}{5} g(x, u_n^+)u_n^+ - G(x, u_n^+)\right]\mathrm{d}x \\ &\ge \frac{3}{10} a\|u_n\|^2 + \frac{b}{20}\|u_n\|^4 - C_5|\Omega|. \end{aligned} \tag{25}$$

That is to say $\|u_n\|$ is bounded in E. Going necessary to a subsequence, it has

$$\begin{cases} u_n \rightharpoonup u, & u \in H_0^1(\Omega), \\ u_n \longrightarrow u, & u \in L^p(\Omega)(1 \le p < 2^* = 6), \\ u_n(x) \longrightarrow u(x), & \text{a.e. } x \in \Omega. \end{cases} \tag{26}$$

By ($V1$) and ($F2$), one has

$$\begin{gathered} \int_\Omega g(x, u_u)u_n \mathrm{d}x \longrightarrow \int_\Omega g(x, u)u\, \mathrm{d}x \quad (n \longrightarrow \infty), \\ \int_\Omega G(x, u_u)\mathrm{d}x \longrightarrow \int_\Omega G(x, u)\mathrm{d}x \quad (n \longrightarrow \infty), \\ \int_\Omega V(x)u_n^2 \mathrm{d}x \longrightarrow \int_\Omega V(x)u^2 \mathrm{d}x \quad (n \longrightarrow \infty). \end{gathered} \tag{27}$$

Let $v_n = u_n - u$, then we can claim $\|v_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Otherwise, there exist a subsequence (for convenience, we still denote it by v_n) such that

$$\lim_{n \longrightarrow \infty} \|v_n\|^2 = l, \tag{28}$$

where $l > 0$; then

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1). \tag{29}$$

By using Brézis–Lieb lemma in [29], it has

$$\int_\Omega (u_n^+)^6 = \int_\Omega (v_n^+)^6 + \int_\Omega (u^+)^6 + o(1). \tag{30}$$

Because $I'(u_n) \longrightarrow 0$ in $(E)^*$, one has

$$\begin{aligned} \langle I'(u_n), u_n\rangle &= a\|u_n\|^2 + b\|u_n\|^4 + \int_\Omega V(x)u_n^2 \mathrm{d}x \\ &\quad - \int_\Omega g(x, u_n)u_n \mathrm{d}x - \int_\Omega (u_n^+)^6 \mathrm{d}x = o(1), \end{aligned} \tag{31}$$

which shows that

$$\begin{aligned} & al + a\|u\|^2 + bl^2 + b\|u\|^4 + 2bl\|u\|^2 + \int_\Omega V(x)u^2 \mathrm{d}x \\ &\quad - \int_\Omega g(x, u)u\, \mathrm{d}x - \int_\Omega (v_n^+)^6 \mathrm{d}x - \int_\Omega (u^+)^6 \mathrm{d}x = o(1). \end{aligned} \tag{32}$$

It also has

$$\begin{aligned} \lim_{n \longrightarrow \infty} \langle I'(u_n), u\rangle &= a\|u\|^2 + bl\|u\|^2 + b\|u\|^4 \\ &\quad + \int_\Omega V(x)u^2 \mathrm{d}x - \int_\Omega g(x, u)u\, \mathrm{d}x \\ &\quad - \int_\Omega (u^+)^6 \mathrm{d}x = 0. \end{aligned} \tag{33}$$

Combining (32), (33), and (10), one gets

$$\begin{aligned} al + bl^2 + bl\|u\|^2 &= \int_\Omega (v_n^+)^6 \mathrm{d}x + o(1) = \frac{\|v_n\|^6}{S^3} + o(1) \\ &\le \frac{l^3}{S^3} + o(1), \end{aligned} \tag{34}$$

which implies that

$$al + bl^2 + bl\|u\|^2 \le \frac{l^3}{S^3}. \tag{35}$$

By (35),

$$l \ge \frac{bS^3 + \sqrt{b^2 S^6 + 4\left(a + b\|u\|^2\right)S^3}}{2} \ge \frac{bS^3 + \sqrt{b^2 S^6 + 4aS^3}}{2}. \tag{36}$$

As $I(u_n) \longrightarrow c\,(n \longrightarrow \infty)$, we have

$$\begin{aligned} c &= \frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 + \frac{1}{2}\int_\Omega V(x)u_n^2 \mathrm{d}x - \int_\Omega G(x, u_n)\mathrm{d}x \\ &\quad - \frac{1}{6}\int_\Omega (u_n^+)^6 \mathrm{d}x + o(1) \\ &= \frac{a}{2} l + \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{b}{4} l^2 + \frac{bl}{2}\|u\|^2 + \frac{1}{2}\int_\Omega V(x)u^2 \mathrm{d}x \\ &\quad - \int_\Omega G(x, u)\mathrm{d}x - \frac{1}{6}\int_\Omega (v_n^+)^6 \mathrm{d}x - \frac{1}{6}\int_\Omega (u^+)^6 \mathrm{d}x + o(1). \end{aligned} \tag{37}$$

From (34) and (37), it has

$$
\begin{aligned}
I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{2}\int_\Omega V(x)u^2 dx - \int_\Omega G(x,u)dx \\
&\quad - \frac{1}{6}\int_\Omega (u^+)^6 dx \\
&= c - \left(\frac{a}{3}l + \frac{b}{12}l^2 + \frac{bl}{3}\|u\|^2\right) \\
&\le c - \frac{a}{3}\frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2} - \frac{b}{12}\left(\frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2}\right)^2 \\
&\quad - \frac{bl}{3}\|u\|^2 \\
&= c - \Lambda - \frac{bl}{3}\|u\|^2 < -\frac{bl}{3}\|u\|^2.
\end{aligned}
\tag{38}
$$

From the above inequality, it has

$$
I(u) + \frac{bl}{3}\|u\|^2 < 0. \tag{39}
$$

On the other hand, from $(F3)$ and (33), which deduce

$$
\begin{aligned}
\frac{bl}{3}\|u\|^2 + I(u) &= \frac{bl}{3}\|u\|^2 + \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{2}\int_\Omega V(x)u^2 dx \\
&\quad - \int_\Omega G(x,u)dx - \frac{1}{6}\int_\Omega (u^+)^6 dx \\
&= \frac{bl}{3}\|u\|^2 + \frac{a}{4}\|u\|^2 - \frac{bl}{4}\|u\|^2 + \frac{1}{4}\int_\Omega V(x)u^2 dx \\
&\quad + \int_\Omega \left(\frac{1}{4}g(x,u)u - G(x,u)\right)dx \\
&\quad + \frac{1}{12}\int_\Omega (u^+)^6 dx \\
&\ge \frac{bl}{12}\|u\|^2 + \frac{a}{4}\|u\|^2 - \frac{a\lambda_1}{4}\int_\Omega u^2 dx \\
&\quad + \frac{1}{12}\int_\Omega (u^+)^6 dx \\
&\ge \frac{bl}{12}\|u\|^2 + \frac{1}{12}\int_\Omega (u^+)^6 dx \ge 0,
\end{aligned}
\tag{40}
$$

which is a contradiction with (39).

So $l = 0$, that is to say, $u_n \longrightarrow u$ in E as $n \longrightarrow \infty$. Thus, I satisfies the $(PS)_c$ condition. □

Lemma 3. *Let $N = 4$ and $a, b > 0$. If (G1) and (G2) are satisfied, then there exists a positive constant $\mu_* = bS^2 > 0$, such that for every $\mu \in (0, \mu_*)$, the functional $I(u)$ satisfies the (PS) condition in $H_0^1(\Omega)$.*

Proof. If $\{u_n\} \subset H_0^1(\Omega)$ is a $(PS)_c$ sequence of I, that is,

$$
I(u_n) \longrightarrow c, I'(u_n) \longrightarrow 0\,(n \longrightarrow \infty). \tag{41}
$$

As $N = 4$, and by (4) and Hölder inequality, one has

$$
\begin{aligned}
I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{2}\int_\Omega V(x)u^2 dx - \frac{\mu}{4}\int_\Omega (u^+)^4 dx \\
&\quad - \frac{\lambda}{q}\int_\Omega f(x)(u^+)^q dx \ge \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\mu}{4}S^{-2}\|u\|^4 \\
&\quad - \frac{\lambda}{q}|f|_{4/4-q}S^{-q/2}\|u\|^q.
\end{aligned}
\tag{42}
$$

Choose $\mu_* = bS^2$ and $2 \le q < 2^* = 4$. For every $\mu \in (0, \mu_*)$, (42) implies that the functional I is coercive and bounded in $H_0^1(\Omega)$ for all $\lambda > 0$. Therefore, the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. It means that there exists a subsequence and we still denote it by $\{u_n\}$ for simplicity, such that

$$
\begin{cases}
u_n \rightharpoonup u, & u \in H_0^1(\Omega), \\
u_n \longrightarrow u, & u \in L^p(\Omega)\,(1 \le p < 2^* = 4), \\
u_n(x) \longrightarrow u(x), & \text{a.e. } x \in \Omega.
\end{cases}
\tag{43}
$$

The following is to prove $u_n \longrightarrow u\,(n \longrightarrow \infty)$ in $H_0^1(\Omega)$. Let $w_n = u_n - u$. By (43), one has

$$
\|u_n\|^2 = \|w_n\|^2 + \|u\|^2 + o(1), \tag{44}
$$

$$
\|u_n\|^4 = \|w_n\|^4 + 2\|w_n\|^2\|u\|^2 + \|w_n\|^4 + o(1). \tag{45}
$$

And by Brézis–Lieb's lemma in [29], one obtains

$$
\int_\Omega (u_n^+)^4 dx = \int_\Omega (w_n^+)^4 dx + \int_\Omega (u^+)^4 dx + o(1), \tag{46}
$$

and we can claim that

$$
\lim_{n\longrightarrow\infty}\int_\Omega V(x)u_n^2 dx = \int_\Omega V(x)u^2 dx, \tag{47}
$$

$$
\lim_{n\longrightarrow\infty}\int_\Omega f(x)((u_n)^+)^q dx = \int_\Omega f(x)(u^+)^q dx. \tag{48}
$$

In fact, by the Sobolev imbedding theorem, there exists a constant $C > 0$, such that $|u_n|_4 \le C < \infty$. $\forall \varepsilon > 0, \exists \delta > 0$, for any $S \subset \Omega$ with $\operatorname{meas}(S) < \delta$ and by Hölder inequality, one has

$$
\int_S V(x)u_n^2 dx \le |V|_2|u_n|_4^2 \le \varepsilon C^2, \tag{49}
$$

$$
\int_S f(x)(u_n^+)^q dx \le |f|_{4/4-q}|u_n|_4^q < \varepsilon C^q. \tag{50}
$$

In view of the absolute continuity of the integrals $\int_S |V(x)|^2 dx$ and $\int_S |f(x)|^{4/(4-q)} dx$, it means that

$$
\lim_{\operatorname{meas}(S)\longrightarrow 0}\int_S |V(x)|^2 dx = 0, \tag{51}
$$

$$
\lim_{\operatorname{meas}(S)\longrightarrow 0}\int_S |f(x)|^{4/(4-q)} dx = 0. \tag{52}
$$

By (49)–(52), it follows that $\int_S V(x)u_n^2\mathrm{d}x \le \varepsilon C^2$ and $\int_S f(x)(u_n^+)^q\mathrm{d}x \le \varepsilon C^q$. Thus, by Vitali's theorem [30], (47) and (48) hold. Similarly, one obtains

$$\lim_{n\longrightarrow\infty}\int_\Omega (u_n^+)^3 u\,\mathrm{d}x = \int_\Omega (u^+)^4\mathrm{d}x, \tag{53}$$

$$\lim_{n\longrightarrow\infty}\int_\Omega V(x)u_n u\,\mathrm{d}x = \int_\Omega V(x)u^2\mathrm{d}x, \tag{54}$$

$$\lim_{n\longrightarrow\infty}\int_\Omega f(x)(u_n^+)^{q-1}u\,\mathrm{d}x = \int_\Omega f(x)(u^+)^q\mathrm{d}x. \tag{55}$$

By (41) and (53)–(55), one gets

$$\begin{aligned}\langle I'(u_n), u_n\rangle &= a\|u_n\|^2 + b\|u_n\|^4 + \int_\Omega V(x)u_n^2\mathrm{d}x \\ &\quad - \mu\int_\Omega (u_n^+)^4\mathrm{d}x - \lambda\int_\Omega f(x)(u_n^+)^q\mathrm{d}x = o(1)\end{aligned} \tag{56}$$

$$\begin{aligned}\lim_{n\longrightarrow\infty}\langle I'(u_n), u\rangle &= a\|u\|^2 + b\|w_n\|^2\|u\|^2 + b\|u\|^4 \\ &\quad + \int_\Omega V(x)u^2\mathrm{d}x - \mu\int_\Omega (u^+)^4\mathrm{d}x \\ &\quad - \lambda\int_\Omega f(x)(u^+)^q\mathrm{d}x = 0.\end{aligned} \tag{57}$$

By (44)–(48) and (57), it follows that

$$\begin{aligned}& a\|w_n\|^2 + a\|u\|^2 + b\|w_n\|^4 + b\|u\|^4 + 2b\|w_n\|^2\|u\|^2 \\ &\quad + \int_\Omega V(x)u^2\mathrm{d}x - \mu\int_\Omega (w_n^+)^4\mathrm{d}x - \mu\int_\Omega (u^+)^4\mathrm{d}x \\ &\quad - \lambda\int_\Omega f(x)(u^+)^q\mathrm{d}x = o(1).\end{aligned} \tag{58}$$

Combining (57) and (58), one can get

$$a\|w_n\|^2 + b\|w_n\|^4 + b\|w_n\|^2\|u\|^2 - \mu\int_\Omega (w_n^+)^4\mathrm{d}x = o(1). \tag{59}$$

From (10), it can be deduced that

$$\int_\Omega (w_n^+)^4\mathrm{d}x \le \int_\Omega |w_n|^4\mathrm{d}x \le S^{-2}\|w_n\|^4. \tag{60}$$

Let $\|w_n\| = l$. Consequently, from (59) and (60), one gets

$$al^2 + bl^2\|u\|^2 + bl^4 \le \mu S^{-2} l^4. \tag{61}$$

Choosing $\mu_* = bS^2 > 0$, $\forall \mu \in (0, \mu_*)$, inequality (61) implies $l = 0$. Thus, $u_n \longrightarrow u$ in $H_0^1(\Omega)$. This completes the proof of Lemma 3. □

3. Proof of Main Results

Now we will prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Combining Lemma 1 with Lemma 2, we can say that it exists $u \in E$, such that

$$\begin{aligned} I(u) &= c, \\ I'(u) &= 0.\end{aligned} \tag{62}$$

Let $W = \{u \in E\backslash\{0\}, I'(u) = 0\}$ and $m = \inf_{u\in W} I(u)$. Then, $W \ne \emptyset, m \le c$. By $(F1)$ and $(F2)$, there exists a constant $C_6 > 0$, such that

$$|g(x,s)s| \le \frac{a\lambda_1}{2}|s|^2 + C_6|s|^6, \quad \forall (x,s) \in \overline{\Omega}\times R. \tag{63}$$

By Sobolev inequality, $\forall u \in W$, one has

$$\begin{aligned} a\|u\|^2 + b\|u\|^4 + \int_\Omega V(x)u^2\mathrm{d}x &= \int_\Omega g(x,u)u\,\mathrm{d}x + \int_\Omega (u^+)^6\mathrm{d}x \\ &\le \frac{a\lambda_1}{2}\int_\Omega |u|^2\mathrm{d}x + C_6\int_\Omega |u|^6\mathrm{d}x \\ &\quad + \int_\Omega (u^+)^6\mathrm{d}x \\ &\le \frac{a}{2}\|u\|^2 + C_7\|u\|^6.\end{aligned} \tag{64}$$

It is easy to know

$$\frac{a}{2}\|u\|^2 + b\|u\|^4 \le C_7\|u\|^6. \tag{65}$$

The above inequality can deduce that there exists a constant $C > 0$, such that

$$\|u\| \ge C, \quad \forall u \in M. \tag{66}$$

We claim that if $C_8 > 0$, such that

$$\int_\Omega (u^+)^6\mathrm{d}x \ge C_8, \quad \forall u \in M. \tag{67}$$

Otherwise, we assume that $u_n \in M$, such that

$$\lim_{n\longrightarrow\infty}\int_\Omega (u_n^+)^6\mathrm{d}x = 0. \tag{68}$$

Then,

$$\lim_{n\longrightarrow\infty}\int_\Omega (u_n^+)^2\mathrm{d}x = 0. \tag{69}$$

In addition, we can calculate

$$\begin{aligned} aC^2 &\le a\|u_n\|^2 + b\|u_n\|^4 + \int_\Omega V(x)u_n^2 = \int_\Omega g(x,u_n^+)u_n^+\mathrm{d}x \\ &\quad + \int_\Omega (u_n^+)^6\mathrm{d}x \\ &\le \frac{a\lambda_1}{2}\int_\Omega (u_n^+)^2\mathrm{d}x + (C_6+1)\int_\Omega (u_n^+)^6 \longrightarrow 0,\end{aligned} \tag{70}$$

which is a contradiction. Therefore, the assertion is established. Therefore, $\forall u \in M$, we have

$$\begin{aligned}
I(u) &= I(u) - \frac{1}{4}\langle I'(u), u\rangle \\
&= \frac{a}{4}\|u\|^2 + \frac{1}{4}\int_\Omega V(x)u^2 dx + \int_\Omega \left(\frac{1}{4}g(x,u)u - G(x,u)\right)dx \\
&\quad + \frac{1}{12}\int_\Omega (u^+)^6 dx \\
&\geq \frac{a\lambda_1}{4}\int_\Omega u^2 dx - \frac{a\lambda_1}{4}\int_\Omega u^2 dx + \frac{1}{12}\int_\Omega (u^+)^6 dx \geq \frac{1}{12}C_8,
\end{aligned} \tag{71}$$

which implies $m > 0$; by the definition of m, we can get a $(PS)_m$ sequence. By using Lemma 1 and Lemma 2, there exists $u \in E$, such that

$$\begin{aligned}
I(u) &= m, \\
I'(u) &= 0.
\end{aligned} \tag{72}$$

By $\langle I'(u), u^-\rangle = 0$, where $u^- = \max\{-u, 0\}$, we can get $u^- = 0$, so $u = u^+$. Then, by strong maximum principle, it implies $u > 0$. □

Proof of Theorem 2. It is divided into two steps to prove Theorem 2. Firstly, we claim that system (1) has a positive global minimizer solution u_1. In fact, from the proof of Lemma 3, it can be known that the functional I is coercive and bounded, so $m = \inf_{u\in H_0^1(\Omega)} I(u)$ is defined. By (G2), $V(x)$ is bounded in Ω, so there exists a constant $M_0 > 0$, such that

$$|V(x)| \leq M_0. \tag{73}$$

By Hölder inequality, (10), and the above inequality, one gets

$$\begin{aligned}
I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{2}\int_\Omega V(x)u^2 dx - \frac{\mu}{4}\int_\Omega (u^+)^4 dx \\
&\quad - \frac{\lambda}{q}\int_\Omega f(x)(u^+)^q dx \leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 \\
&\quad + \frac{1}{2}M_0|\Omega|^{1/2}S^{-1}\|u\|^2 - \frac{\lambda}{q}\int_\Omega f(x)(u^+)^q dx.
\end{aligned} \tag{74}$$

Choosing $\|u_0\| = 1$, from (74), one has

$$I(u_0) \leq \frac{2a + b + 2M_0|\Omega|^{1/2}S^{-1}}{4} - \frac{\lambda}{q}\int_\Omega f(x)(u^+)^q dx < 0, \tag{75}$$

for all $\lambda > \lambda_* = q(2a + b + 2M_0|\Omega|^{1/2}S^{-1})/4\int_\Omega f(x)(u^+)^q dx$. Therefore, $m < 0$. By Lemma 2.3 and Theorem 4.4 in [31], there exists $u_1 \in H_0^1(\Omega)$ such that $I(u_1) = m < 0$. Letting $v = u_1^-$ in (12), it follows that $u_1 > 0$. Thus, u_1 is a nonzero and nonnegative solution of system (1). Moreover, by the strong maximum principle, it has $u_1 > 0$. That is to say, u_1 is a positive global minimizer solution of system (1), such that $I(u_1) = m < 0$.

Secondly, we will prove that system (1) has another positive solution. As $0 < q < 2^* = 4$, it is easy to know 0 is a local minimum point of functional I in $H_0^1(\Omega)$. Defining c as follows: $c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t))$, where

$$\Gamma = \left\{\gamma \in C\left([0,1], H_0^1(\Omega)\right) \,\middle|\, \gamma(0) = 0, \gamma(1) = u_1\right\}. \tag{76}$$

It is obvious that $c > 0$. By the mountain pass lemma in [32], there exists $u_2 \in H_0^1(\Omega)$, such that $I(u_2) = c > 0$ and $I'(u_2) = 0$. Similarly, taking $v = u_2^-$ in (12), we can get u_2 is also a nonzero and nonnegative solution of system (1). By the strong maximum principle, it has $u_2 > 0$, such that $I(u_2) = c > 0$. This proves Theorem 2. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Junjun Zhou completed the main design and the writing of the manuscript. Xiangyun Hu provided some idea and revised the manuscript. Tiaojie Xiao did some work which was related to math formulation, derivation, and calculation.

Acknowledgments

This project was funded by the National Natural Science Foundation of China (41630317) and the National Key Research and Development Program of China (2017YFC0602405).

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