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Research Article

Pathway Fractional Integral Formulas Involving $\mathcal{S}$-Function in the Kernel

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In this paper, we present several composition formulae of pathway fractional integral operators connected with $\mathcal{S}$-function. Here, we point out important links to known outcomes for some specific cases with our key results.

1. Introduction and Preliminaries

In recent years, fractional calculus has become a significant instrument for the modeling analysis and plays a significant role in different fields, for example, material science, science, mechanics, power, economy, and control theory. In addition, a number of researchers have investigated a variety of fractional calculus operators in the depth level of properties, implementation methods, and complex modifications. Other analogous topics are also very active and extensive around the world. One may refer to the research monographs in [1, 2].

$\mathcal{S}$-function. Recently, Saxena and Daiya [3] defined and studied a special function called as $\mathcal{S}$-function (also see [4]) and its relation with other special functions, which include generalized $\mathcal{K}$-function, $\mathcal{M}$-series, k-Mittag–Leffler function, Mittag–Leffler type functions, and other many special functions. These special functions have recently found essential applications in solving problems in applied sciences, biology, physics, and engineering.

The $\mathcal{S}$-function is defined for $\sigma, \eta, \varepsilon, \tau \in \mathbb{C}$, $\mathfrak{R}(\sigma) > 0$, $k \in \mathfrak{R}$, $\mathfrak{R}(\sigma) > k\mathfrak{R}(\tau)$, $l_i\,(i = 1, 2, 3, \ldots, p)$, $m_j\,(j = 1, 2, 3, \ldots, q)$, and $p < q + 1$ as

$$\mathcal{S}^{\sigma,\eta,\varepsilon,\tau,k}_{(p,q)}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x\right] = \sum_{n=0}^{\infty} \frac{(l_1)_n \ldots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \ldots (m_q)_n \Gamma_k(n\sigma + \eta)} \frac{x^n}{n!}, \tag{1}$$

Here, k-Pochhammer symbol is as follows:

$$(\varepsilon)_{n,k} = \begin{cases} \dfrac{\Gamma_k(\varepsilon + nk)}{\Gamma_k(\varepsilon)}, & (k \in \mathfrak{R}, \varepsilon \in \mathbb{C}/\{0\}), \\ \varepsilon(\varepsilon + k)\ldots(\varepsilon + (n-1)k), & (n \in \mathbb{N}, \varepsilon \in \mathbb{C}). \end{cases} \tag{2}$$

Also, the k-gamma function is

$$\Gamma_k(\varepsilon) = k^{(\varepsilon/k)-1}\Gamma\left(\frac{\varepsilon}{k}\right), \tag{3}$$

where $\varepsilon \in \mathbb{C}, k \in \mathfrak{R}$, and $n \in \mathbb{N}$, introduced by Díaz and Pariguan [5] (see also Romero and Cerutti [6]).

Several major special cases of the $\mathcal{S}$-function are described as follows:

(i) For $p = q = 0$, the generalized k-Mittag–Leffler function from Saxena et al. [7] (see [8, 9]) is

$$\mathscr{E}_{k,\sigma,\eta}^{\varepsilon,\tau}(x) = \mathcal{S}_{(0,0)}^{\sigma,\eta,\varepsilon,\tau,k}[-;-;x] = \sum_{n=0}^{\infty} \frac{(\varepsilon)_{n\tau,k}}{\Gamma_k(n\sigma+\eta)} \frac{x^n}{n!}, \Re\left(\frac{\sigma}{k}-\tau\right) > p-q. \tag{4}$$

(ii) For $k = \tau = 1$, the $\mathcal{S}$-function is the generalized $\mathscr{K}$-function, introduced by Sharma [10] (see also [11]):

$$\begin{aligned}\mathscr{K}_{(p,q)}^{\sigma,\eta,\varepsilon}\left[l_1,\ldots,l_p;m_1,\ldots,m_q;x\right] &= \mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,1,1}\left[l_1,\ldots,l_p;m_1,\ldots,m_q;x\right] \\ &= \sum_{n=0}^{\infty} \frac{(l_1)_n\cdots(l_p)_n(\varepsilon)_n}{(m_1)_n\cdots(m_q)_n\Gamma(n\sigma+\eta)} \frac{x^n}{n!}, \quad \Re(\sigma) > p-q.\end{aligned} \tag{5}$$

(iii) For $\tau = k = \varepsilon = 1$, the $\mathcal{S}$-function reduced to generalized $\mathscr{M}$-series introduced by Sharma and Jain [12](detail [13]) is

$$\begin{aligned}\mathscr{M}_{(p,q)}^{\sigma,\eta}\left[l_1,\ldots,l_p;m_1,\ldots,m_q;x\right] &= \mathcal{S}_{(p,q)}^{\sigma,\eta,1,1,1}\left[l_1,\ldots,l_p;m_1,\ldots,m_q;x\right] \\ &= \sum_{n=0}^{\infty} \frac{(l_1)_n\cdots(l_p)_n x^n}{(m_1)_n\cdots(m_q)_n\Gamma(n\sigma+\eta)}, \quad \Re(\sigma) > p-q-1.\end{aligned} \tag{6}$$

Recently, an expending pathway fractional integral (PFI) operator introduced by Nair [14], which was earlier defined by Mathai [15] and Mathai and Haubold [16, 17], is defined as follows:

$$\left(\mathscr{P}_{0+}^{\lambda,\varsigma} f\right)(x) = x^{\lambda} \int_0^{[x/(a(1-\varsigma))]} \left(1 - \frac{a(1-\varsigma)\xi}{x}\right)^{\lambda/(1-\varsigma)} f(\xi)\mathrm{d}\xi, \tag{7}$$

where Lebesgue measurable function $f \in \mathscr{L}(a,b)$ for real or complex term valued function, $\lambda \in \mathbb{C}$, $\Re(\lambda) > 0, a > 0$, and $\varsigma < 1$ (ς is a pathway parameter).

The pathway model for a real scalar ς and scalar random variables is represented by the probability density function (p.d.f.) in the following manner:

$$f(x) = \frac{c}{|x|^{1-\nu}}\left[1 - a(1-\varsigma)|x|^{\rho}\right]^{\lambda/(1-\varsigma)}, \tag{8}$$

where $x \in (-\infty,\infty); \lambda > 0; \rho > 0; [1 - a(1-\varsigma)|x|^{\rho}]^{\lambda/(1-\varsigma)} > 0$; $\nu > 0$ and ς and c denote the pathway parameter and normalizing constant, respectively.

Additionally, for $\varsigma \in \Re$, the normalizing constants are expressed in the following way:

$$c = \begin{cases} \dfrac{1}{2}\dfrac{\rho[a(1-\varsigma)]^{\nu/\rho}\Gamma(\nu/\rho + \lambda/(1-\varsigma)+1)}{\Gamma(\nu/\rho)\Gamma(\lambda/(1-\varsigma)+1)}, & (\varsigma < 1), \\[2ex] \dfrac{1}{2}\dfrac{\rho[a(1-\varsigma)]^{\nu/\rho}\Gamma(\lambda/(\varsigma-1))}{\Gamma(\nu/\rho)\Gamma(\lambda/(\varsigma-1)-\nu/\rho)}, & \left(\dfrac{1}{\varsigma-1} - \dfrac{\nu}{\rho} > 0, \varsigma > 1\right), \\[2ex] \dfrac{1}{2}\dfrac{[a\lambda]^{\nu/\rho}}{\Gamma(\nu/\rho)}, & (\varsigma \longrightarrow 1). \end{cases} \tag{9}$$

It is noted that if $\varsigma < 1$, finite range density with $[1 - a(1-\varsigma)|x|^{\rho}]^{\lambda/(1-\varsigma)} > 0$ and (8) can be considered a member of the extended generalized type-1 beta family. Also, the triangular density, the uniform density, the extended type-1 beta density and various other probability density functions are precise special cases of the pathway density function defined in (8) for $\varsigma < 1$.

For example, if $\varsigma > 1$ and by setting $(1-\varsigma) = -(\varsigma - 1)$ in (7), then we have

$$\left(\mathscr{P}_{0+}^{\lambda,\varsigma}f\right)(x)=x^{\lambda}\int_{0}^{[x/-a(\varsigma-1)]}\left(1+\frac{a(\varsigma-1)\xi}{x}\right)^{\lambda/-(\varsigma-1)}f(\xi)\mathrm{d}\xi, \tag{10}$$

$$f(x)=\frac{c}{|x|^{1-\nu}}\left[1+a(\varsigma-1)|x|^{\rho}\right]^{\lambda/-(\varsigma-1)}, \tag{11}$$

provided that $x\in(-\infty,\infty)$; $\rho>0$; $\lambda>0$; and $\varsigma>1$ characterize the extended generalized type-2 beta model for real x. The specific cases of density function (11) include the type-2 beta density function, the p density function, and the Student's t density function. For $\varsigma\longrightarrow 1$, (7) diminishes to the Laplace integral transform.

In a similar way, if $\varsigma=0$, $a=1$, and λ takes the place of $\lambda-1$, then (7) diminishes to the familiar Riemann–Liouville (R-L) fractional integral operator $\mathscr{I}_{0+}^{\lambda}f$ (e.g., [7]):

$$\left(\mathscr{P}_{0+}^{\lambda-1,0}f\right)(x)=\Gamma(\lambda)\left(\mathscr{I}_{0+}^{\lambda}f\right)(x),\quad(\mathfrak{R}(\lambda)>1). \tag{12}$$

PFI operator (7) leads to numerous interesting illustrations such as fractional calculus associated with probability density functions and their significant in statistical theory. Nowadays, many researchers study PFI formulae associated with various special functions (see [18–27]). Motivated by these researchers, we study the $\mathcal{S}$-function, which is connected with PFI operator (7), to present their integral formulae. Suitable connections of some particular cases are also pointed out.

2. Pathway Fractional Integral Operator of $\mathcal{S}$-Function

In this section, we establish the PFI formula involving the $\mathcal{S}$-function which is stated in Theorems 1 and 2.

Theorem 1. *Suppose $w,k\in\mathfrak{R}$, $\sigma,\eta,\varepsilon,\tau\in\mathbb{C}$, $\mathfrak{R}(\sigma)>0$, $\mathfrak{R}(\lambda)>0$, $\mathfrak{R}(\sigma)>k\mathfrak{R}(\tau)$, and $p<q+1$, $\mathfrak{R}(\lambda/(1-\varsigma))>-1$; $\varsigma<1$. Then, the following formula holds true:*

$$\begin{aligned}&\mathscr{P}_{0^+}^{\lambda,\varsigma}\left[\zeta^{(\eta/k)-1}\mathcal{S}_{(p,q)}^{\sigma,\eta,\varepsilon,\tau,k}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;w\zeta^{(\sigma/k)}\right]\right](x)\\&=\frac{x^{\lambda+(\eta/k)}k^{(1+(\lambda/1-\varsigma))}\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{(\eta/k)}}\times\mathcal{S}_{(p,q)}^{\sigma,\eta+(1+\lambda/(1-\varsigma))k,\varepsilon,\tau,k}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;\frac{wx^{(\sigma/k)}}{(a(1-\varsigma))^{(\sigma/k)}}\right].\end{aligned} \tag{13}$$

Proof. We indicate the RHS of equation (13) by $\mathfrak{I}_1$, and invoking equations (1) and (7), we have

$$\begin{aligned}\mathfrak{I}_1&=x^{\lambda}\int_0^{[x/a(1-\varsigma)]}\left(1-\frac{a(1-\varsigma)\zeta}{x}\right)^{\lambda/(1-\varsigma)}\zeta^{(\eta/k)-1}\\&\times\sum_{n=0}^{\infty}\frac{(l_1)_n\ldots(l_p)_n(\varepsilon)_{n\tau,k}}{(m_1)_n\ldots(m_q)_n\Gamma_k(n\sigma+\eta)}\frac{\left(w\zeta^{(\sigma/k)}\right)^n}{n!}\mathrm{d}\zeta.\end{aligned} \tag{14}$$

Now changing the order of integration and summation, we obtain

$$\begin{aligned}\mathfrak{I}_1&=x^{\lambda}\sum_{n=0}^{\infty}\frac{(l_1)_n\ldots(l_p)_n(\varepsilon)_{n\tau,k}w^n}{(m_1)_n\ldots(m_q)_n\Gamma_k(n\sigma+\eta)n!}\\&\times\int_0^{[x/a(1-\varsigma)]}\left(1-\frac{a(1-\varsigma)\zeta}{x}\right)^{\lambda/(1-\varsigma)}\zeta^{((\eta+\sigma n)/k)-1}\mathrm{d}\zeta.\end{aligned} \tag{15}$$

Using the substitution $u=a(1-\varsigma)\zeta/x$, we can change the limit of integration into the following:

$$\begin{aligned}\mathfrak{I}_1&=x^{\lambda}\sum_{n=0}^{\infty}\frac{(l_1)_n\ldots(l_p)_n(\varepsilon)_{n\tau,k}w^n}{(m_1)_n\ldots(m_q)_n\Gamma_k(n\sigma+\eta)n!}\left(\frac{x}{a(1-\varsigma)}\right)^{(\eta+\sigma n)/k}\\&\times\int_0^1(1-u)^{\lambda/(1-\varsigma)}u^{((\eta+\sigma n)/k)-1}\mathrm{d}u.\end{aligned} \tag{16}$$

Now, by calculating the inner integral and using the beta function formula, we obtain the following:

$$\begin{aligned}\mathfrak{I}_1&=x^{\lambda}\sum_{n=0}^{\infty}\frac{(l_1)_n\ldots(l_p)_n(\varepsilon)_{n\tau,k}w^n}{(m_1)_n\ldots(m_q)_n\Gamma_k(n\sigma+\eta)n!}\left(\frac{x}{a(1-\varsigma)}\right)^{(\eta+\sigma n)/k}\\&\times\frac{\Gamma(\eta/k+n\sigma/k)\Gamma(\lambda/(1-\varsigma)+1)}{\Gamma(\eta/k+(n\sigma/k)+\lambda/(1-\varsigma)+1)}.\end{aligned} \tag{17}$$

Using (3), we obtain

$$\mathfrak{I}_1 = x^{\lambda+\eta/k}\sum_{n=0}^{\infty}\frac{(l_1)_n\cdots(l_p)_n(\varepsilon)_{n\tau,k}}{(m_1)_n\cdots(m_q)_n\Gamma(n\sigma/k+\eta/k)k^{(n\sigma/k)+(\eta/k)-1}n!}$$
$$\times\frac{\Gamma(\eta/k+n\sigma/k)\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{\eta/k}\Gamma(\eta/k+(n\sigma/k)+\lambda/(1-\varsigma)+1)}$$
$$\cdot\left(w\left(\frac{x}{a(1-\varsigma)}\right)^{(\sigma)/k}\right)^n. \tag{18}$$

Once again, using (3), we obtain

$$\mathfrak{I}_1 = \frac{x^{\lambda+(\eta/k)}k^{(1+\lambda/(1-\varsigma))}\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{(\eta/k)}}$$
$$\mathcal{S}^{\sigma,\eta+(1+\lambda/(1-\varsigma))k,\varepsilon,\tau,k}_{(p,q)}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,\right. \tag{19}$$
$$\left. m_q;\frac{wx^{(\sigma/k)}}{(a(1-\varsigma))^{(\sigma/k)}}\right],$$

which gives the required proof of Theorem 1. □

Corollary 1. *If we put $p=q=0$, then (13) leads to the subsequent result of generalized k-Mittag–Leffler function:*

$$\mathcal{P}^{\lambda,\varsigma}_{0^+}\left[\zeta^{(\eta/k)-1}\mathcal{E}^{\varepsilon,\tau}_{k,\sigma,\eta}\left(w\zeta^{(\sigma/k)}\right)\right](x)$$
$$=\frac{x^{\lambda+(\eta/k)}k^{(1+\lambda/(1-\varsigma))}\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{(\eta/k)}}\mathcal{E}^{\varepsilon,\tau}_{k,\sigma,\eta+(1+\lambda/(1-\varsigma))k}(x)$$
$$\cdot\left[\frac{wx^{(\sigma/k)}}{(a(1-\varsigma))^{(\sigma/k)}}\right]. \tag{20}$$

Proof. We consider (4) and $p=q=0$ in Theorem 1, and we obtain the desired result in (13). □

Corollary 2. *If we put $k=\tau=1$, then (13) leads to the subsequent result in terms of generalized $\mathcal{K}$-function:*

$$\mathcal{P}^{\lambda,\varsigma}_{0^+}\left[\zeta^{\eta-1}\mathcal{K}^{\sigma,\eta,\varepsilon}_{(p,q)}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;w\zeta^{\sigma}\right]\right](x)$$
$$=\frac{x^{\lambda+\eta}\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{\eta}}\mathcal{K}^{\sigma,\eta+(1+\lambda/(1-\varsigma)),\varepsilon}_{(p,q)}$$
$$\cdot\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;\frac{wx^{\sigma}}{(a(1-\varsigma))^{\sigma}}\right]. \tag{21}$$

Proof. If we set $k=\tau=1$ in Theorem 1 and using (5), we obtain the required result (21). □

Corollary 3. *If we put $k=\tau=1$, then (13) holds the formula in terms of generalized $\mathcal{M}$-series:*

$$\mathcal{P}^{\lambda,\varsigma}_{0^+}\left[\zeta^{\eta-1}\mathcal{M}^{\sigma,\eta}_{(p,q)}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;w\zeta^{\sigma}\right]\right](x)$$
$$=\frac{x^{\lambda+\eta}\Gamma(\lambda/(1-\varsigma)+1)}{(a(1-\varsigma))^{\eta}}\mathcal{M}^{\sigma,\eta+(1+\lambda/(1-\varsigma))}_{(p,q)}$$
$$\cdot\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;\frac{wx^{\sigma}}{(a(1-\varsigma))^{\sigma}}\right]. \tag{22}$$

Proof. If we put $\tau=k=\varepsilon=1$ in Theorem 1 and using (6), we obtain the result (22). □

Now, we use equation (10) to define the following theorem, by the case $\varsigma>1$.

Theorem 2. *Suppose $w,k\in\mathfrak{R};\sigma,\eta,\varepsilon,\tau\in\mathbb{C},\mathfrak{R}(\sigma)>0$, $\mathfrak{R}(\lambda)>0$, $\mathfrak{R}(\sigma)>k\,\mathfrak{R}(\tau)$ and $p<q+1$, and $\mathfrak{R}(1-(\lambda/\varsigma-1))>0;\varsigma>1$. Then, the following formula holds true:*

$$\mathcal{P}^{\lambda,\varsigma}_{0^+}\left[\zeta^{(\eta/k)-1}\mathcal{S}^{\sigma,\eta,\varepsilon,\tau,k}_{(p,q)}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;w\zeta^{(\sigma/k)}\right]\right]$$
$$(x)=\frac{x^{\lambda+(\eta/k)}k^{(1-\lambda/(\varsigma-1))}\Gamma(1-(\lambda/(\varsigma-1)))}{(-a(\varsigma-1))^{(\eta/k)}}$$
$$\times\mathcal{S}^{\sigma,\eta+(1-\lambda/(\varsigma-1))k,\varepsilon,\tau,k}_{(p,q)}\left[l_1,l_2,\ldots,l_p;m_1,m_2,\ldots,m_q;\right.$$
$$\left.\frac{wx^{(\sigma/k)}}{(-a(\varsigma-1))^{(\sigma/k)}}\right]. \tag{23}$$

Proof. We denote, for convenience, the RHS of equation (23) by $\mathfrak{I}_2$, and invoking equations (1) and (10), we have

$$\mathfrak{I}_2 = x^{\lambda}\int_0^{[x/-a(\varsigma-1)]}\left(1+\frac{a(\varsigma-1)\zeta}{x}\right)^{\lambda/-(\varsigma-1)}\zeta^{(\eta/k)-1}$$
$$\times\sum_{n=0}^{\infty}\frac{(l_1)_n\cdots(l_p)_n(\varepsilon)_{n\tau,k}}{(m_1)_n\cdots(m_q)_n\Gamma_k(n\sigma+\eta)}\frac{\left(w\zeta^{(\sigma/k)}\right)^n}{n!}d\zeta. \tag{24}$$

Now, changing the order of integration and summation, we obtain

$$\mathfrak{I}_2 = x^{\lambda} \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k} w^n}{(m_2)_n \cdots (m_q)_n \Gamma_k (n\sigma + \eta) n!} \times \int_0^{[x/-a(\varsigma-1)]} \left(1 + \frac{a(\varsigma-1)\zeta}{x}\right)^{\lambda/-(\varsigma-1)} \zeta^{((\eta+\sigma n)/k)-1} d\zeta. \tag{25}$$

By setting $v = -a(\varsigma - 1)\zeta/x$, we can change the limit of integration into the following:

$$\mathfrak{I}_2 = x^{\lambda} \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k} w^n}{(m_1)_n \cdots (m_q)_n \Gamma_k (n\sigma + \eta) n!} \left(\frac{x}{-a(\varsigma-1)}\right)^{(\eta+\sigma n)/k} \times \int_0^1 (1-v)^{\lambda/-(\varsigma-1)} v^{((\eta+\sigma n)/k)-1} dv \times \int_0^1 (1-v)^{\lambda/-(\varsigma-1)} v^{((\eta+\sigma n)/k)-1} dv. \tag{26}$$

By analyzing the internal integral and using the beta function rule, we obtain

$$\mathfrak{I}_2 = x^{\lambda} \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k} w^n}{(m_1)_n \cdots (m_q)_n \Gamma_k (n\sigma + \eta) n!} \left(\frac{x}{-a(\varsigma-1)}\right)^{(\eta+\sigma n)/k} \times \frac{\Gamma(\eta/k + n\sigma/k)\Gamma(1 - \lambda/(\varsigma-1))}{\Gamma(\eta/k + (n\sigma/k) + 1 - \lambda/(\varsigma-1))}. \tag{27}$$

Using (3), we obtain

$$\mathfrak{I}_2 = x^{\lambda+\eta/k} \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\varepsilon)_{n\tau,k}}{(m_1)_n \cdots (m_q)_n \Gamma(n\sigma/k + \eta/k) k^{(n\sigma/k)+(\eta/k)-1} n!} \times \frac{\Gamma(\eta/k + n\sigma/k)\Gamma(1 - \lambda/(\varsigma-1))}{(-a(\varsigma-1))^{\eta/k}\Gamma(\eta/k + (n\sigma/k) + 1 - \lambda/(\varsigma-1))} \cdot \left(w\left(\frac{x}{-a(\varsigma-1)}\right)^{(\sigma)/k}\right)^n. \tag{28}$$

Once again, we arrive at the target outcome by applying (3):

$$\mathfrak{I}_2 = \frac{x^{\lambda+(\eta/k)} k^{(1-\lambda/\varsigma-1)}\Gamma(1-\lambda/\varsigma-1)}{(-a(\varsigma-1))^{(\eta/k)}} S_{(p,q)}^{\sigma,\eta+(1-\lambda/\varsigma-1)k,\varepsilon,\tau,k} \cdot \left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; \frac{w x^{(\sigma/k)}}{(-a(\varsigma-1))^{(\sigma/k)}}\right]. \tag{29}$$

□

Corollary 4. *If we put $p = q = 0$, then (23) provides the result as follows:*

$$\mathscr{P}_{0^+}^{\lambda,\varsigma}\left[\zeta^{(\eta/k)-1}\mathscr{E}_{k,\sigma,\eta}^{\varepsilon,\tau}\left(w\zeta^{(\sigma/k)}\right)\right](x) = \frac{x^{\lambda+(\eta/k)}k^{(1-(\lambda/\varsigma-1))}\Gamma(1-(\lambda/\varsigma-1))}{(-a(\varsigma-1))^{(\eta/k)}}\mathscr{E}_{k,\sigma,\eta+(1-(\lambda/\varsigma-1))k}^{\varepsilon,\tau}(x) \cdot \left[\frac{w x^{(\sigma/k)}}{(-a(\varsigma-1))^{(\sigma/k)}}\right]. \tag{30}$$

Proof. We consider (4) and $p = q = 0$ in Theorem 2 and we obtain the desired result (30). □

Corollary 5. *If $k = \tau = 1$, then (23) holds the following formula:*

$$\mathscr{P}_{0^+}^{\lambda,\varsigma}\left[\zeta^{\eta-1}\mathscr{K}_{(p,q)}^{\sigma,\eta,\varepsilon}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; w\zeta^{\sigma}\right]\right](x) = \frac{x^{\lambda+\eta}\Gamma(1-(\lambda/\varsigma-1))}{(-a(\varsigma-1))^{\eta}}\mathscr{K}_{(p,q)}^{\sigma,\eta+(1-(\lambda/\varsigma-1)),\varepsilon} \cdot \left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; \frac{w x^{\sigma}}{(-a(\varsigma-1))^{\sigma}}\right]. \tag{31}$$

Proof. If we set $k = \tau = 1$ in Theorem 2 and using (5), we obtain the required result (31). □

Corollary 6. *If we put $k = \tau = \varepsilon = 1$, then resulting formula (23) holds true:*

$$\mathscr{P}_{0^+}^{\lambda,\varsigma}\left[\zeta^{\eta-1}\mathscr{M}_{(p,q)}^{\sigma,\eta}\left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; w\zeta^{\sigma}\right]\right](x) = \frac{x^{\lambda+\eta}\Gamma(1-(\lambda/\varsigma-1))}{(-a(\varsigma-1))^{\eta}}\mathscr{M}_{(p,q)}^{\sigma,\eta+(1-(\lambda/\varsigma-1))} \cdot \left[l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; \frac{w x^{\sigma}}{(-a(\varsigma-1))^{\sigma}}\right]. \tag{32}$$

Proof. If we put $\tau = k = \varepsilon = 1$ in Theorem 2 and using (6), we obtain the result (32). □

3. Concluding Remarks

In the present paper, we have established two pathway fractional integral formulae associated with the more generalized special function called as S-function. The results obtained here involve special functions such as k-Mittag–Leffler function, $\mathscr{K}$-function, and $\mathscr{M}$-series, due to their general nature and usefulness in the theory of integral operators and relevant part of computational mathematics. Also, the special functions involved here can be reduced to simpler functions, which have a number of applications in various fields of science and technology and can be found as special cases that we have not specifically stated here.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the present investigation and read and approved the final manuscript.

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