Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 4268361, 8 pages
https://doi.org/10.1155/2020/4268361

Research Article

On the Extended Hypergeometric Matrix Functions and Their Applications for the Derivatives of the Extended Jacobi Matrix Polynomial

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Received 10 September 2019; Accepted 6 November 2019; Published 24 March 2020

Academic Editor: Mariusz Michta

In this paper, we obtain some generating matrix functions and integral representations for the extended Gauss hypergeometric matrix function EGHMF and their special cases are also given. Furthermore, a specific application for the extended Gauss hypergeometric matrix function which includes Jacobi matrix polynomials is constructed.

1. Introduction

Generalizations of the classical special functions to matrix setting have become important during the previous years. Special matrix functions appear in solutions for some physical problems. Applications of special matrix functions also grow and become active areas in the recent literature including statistics, Lie groups theory, and differential equations (see, e.g., [1–4] and elsewhere). New extensions of some of the well-known special matrix functions such as gamma matrix function, beta matrix function, and Gauss hypergeometric matrix function have been extensively studied in recent papers [5–10].

Hypergeometric matrix functions are an interesting problem to study from a purely analytic point of view. These functions arise in the study of matrix-valued spherical functions and in the theory of matrix-valued orthogonal polynomials.

Moreover, they appear in the practice of various fields of mathematics and engineering, so knowledge of them is necessary for applications of theories associated with these fields.

In various areas of applications, generating functions and integral transformations for some families of hypergeometric functions is potentially useful (see [11–16]), especially in situations when these hypergeometric functions are involved in solutions of mathematical, physical, and engineering problems that can be modeled by ordinary and partial differential equations.

The main object of this paper is to investigate various properties for the extended Gauss hypergeometric matrix function EGHMF. The generating functions and integral formulas are derived for EGHMF. We also present some special cases of the main results of this work. A specific application for the extended Gauss hypergeometric matrix function which includes Jacobi matrix polynomials is constructed.

Throughout this paper, I and $\mathbf{0}$ will denote the identity matrix and null matrix in $\mathbb{C}^{r\times r}$, respectively. For a matrix $A\in\mathbb{C}^{r\times r}$, its spectrum is denoted by $\sigma(A)$. We say that if $\operatorname{Re}(\xi)$ for all $\xi\in\sigma(A)$, a matrix A in $\mathbb{C}^{r\times r}$ is a positive stable matrix. In [9, 17–19], if $f(z)$ and $g(z)$ are holomorphic functions in an open set Λ of the complex plane and if A is a matrix in $\mathbb{C}^{r\times r}$ for which $\sigma(A)\subset\Lambda$, then $f(A)g(A)=g(A)f(A)$.

Notation 1. For all A in $\mathbb{C}^{r\times r}$, and

$$A+nI,\quad \text{is invertible for all integers } n, \tag{1}$$

then the Pochhammer symbol is defined by [1, 20]

$$(A)_n = A(A+I)\cdots(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A);\quad (A)_0 \equiv I. \tag{2}$$

By inserting a regularization matrix factor $e^{-B/t}, B \in \mathbb{C}^{r\times r}$. Abul-Dahab and Bakhet [6] have introduced the following generalization of the gamma matrix function.

Definition 1. Let A and B be positive stable matrices in $C^{r\times r}$; then, the generalized Gamma matrix function $\Gamma(A,B)$ is defined by

$$\Gamma(A,B) = \int_0^\infty t^{A-I} e^{-(It+(B/t))}\mathrm{d}t,\quad t^{A-I} = \exp((A-I)\ln t), \tag{3}$$

for $B=0$ reduces gamma matrix function in [21].

Also, Abdalla and Bakhet [7] considered the extension of Euler's beta matrix function in the following definition.

Definition 2. Suppose that A, B, and $\mathbb{P}$ are positive stable and commutative matrices in $\mathbb{C}^{r\times r}$ satisfying spectral condition (1); then, the extended beta matrix function $\mathscr{B}(A,B;\mathbb{P})$ is defined by

$$\mathscr{B}(A,B;\mathbb{P}) := \int_0^1 t^{A-I}(1-t)^{B-I}\exp\left(\frac{-\mathbb{P}}{t(1-t)}\right)\mathrm{d}t. \tag{4}$$

Hence,

$$\mathscr{B}(A,B;\mathbb{P}) = \Gamma(A,\mathbb{P})\Gamma(B,\mathbb{P})\Gamma^{-1}(A+B;\mathbb{P}). \tag{5}$$

For $\mathbb{P}=0$, it obviously reduces to the beta matrix function in [21] by

$$\mathscr{B}(A,B) = \int_0^1 t^{A-I}(1-t)^{B-I}\mathrm{d}t. \tag{6}$$

For $p,q \in \mathbb{Z}^+$, we will denote $\Gamma(A_1)\ldots\Gamma(A_p)\Gamma^{-1}(B_1)\ldots\Gamma^{-1}(B_q)$ by

$$\Gamma\begin{pmatrix} A_1,\ldots,A_p \\ B_1,\ldots,B_q \end{pmatrix}. \tag{7}$$

Later, Abdalla and Bakhet [8] used $\mathscr{B}(A,B;\mathbb{P})$ to extend the Gauss hypergeometric matrix function in the following form:

$$F^{(\mathbb{P})}(A,B;C;z) = \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \times \sum_{n=0}^{\infty}(A)_n \mathscr{B}(B+nI, C-B;\mathbb{P})\frac{z^n}{n!}. \tag{8}$$

This matrix power series is seen to converge when $|z|<1$. Also, for $\mathbb{P}=0$, it reduces to the usual Gauss hypergeometric matrix function ${}_1F_2(A,B;C;z)$ in [22]:

$${}_2F_1(A,B;C;z) = \sum_{n=0}^{\infty}(A)_n(B)_n[(C)_n]^{-1}\frac{z^n}{n!}, \tag{9}$$

where A, B, and C are the matrices in $\mathbb{C}^{r\times r}$ and C satisfying condition (1).

Remark 1. ${}_2F_1(A,B;C;z)$ is the special case of the well-known generalized hypergeometric matrix power series ${}_qF_p(A_i;B_j;z)$ defined by [5, 20]

$${}_qF_p\left(A_i;B_j;z\right) = \sum_{n\geq 0}\prod_{i=1}^{p}(A_i)_n\prod_{j=1}^{q}\left[\left(B_j\right)_n\right]^{-1}\frac{z^n}{n!}. \tag{10}$$

For commutative matrices A_i, $1\leq i\leq p$, and for B_j, $1\leq j\leq q$, in $\mathbb{C}^{r\times r}$ such that

$$B_j + nI \text{ are invertible for all integers } n \geq 0. \tag{11}$$

Some integral forms of the extended Gauss hypergeometric matrix function proved in [8] are given by

$$F^{(\mathbb{P})}(A,B,C;z) = \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix}\int_0^1 t^{B-I}(1-t)^{C-B-I} \times (1-zt)^{-A}\exp\left(\frac{-\mathbb{P}}{t(1-t)}\right)\mathrm{d}t,\quad |\arg(1-z)|<\pi, \tag{12}$$

$$F^{(\mathbb{P})}(A,B,C;z) = 2\Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \times \int_0^\infty \left(\cosh^2 v - z\sinh^2 v\right)^{-A}(\sinh v)^{2B-I} \times (\cosh v)^{2(A-C)+I} \cdot \exp\left(-\mathbb{P}\cosh^2 v\coth^2 v\right)\mathrm{d}v, \cdot (|\arg(1-z)|<\pi), \tag{13}$$

where $CB=BC$ and C, B, and $C-B$ are positive stable.

For (EGHMF), we have the following differential formula [8]:

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{F^{(\mathbb{P})}(A,B;C;z)\right\} = (A)_n(B)_n(C)_n^{-1}\times F^{(\mathbb{P})} \cdot (A+nI, B+nI; C+nI; z). \tag{14}$$

Definition 3 (see [20, 23]). Let A and B be positive stable matrices in $\mathbb{C}^{r\times r}$; then, the Jacobi matrix polynomial (JMP) $P_n^{(A,B)}(z)$ is defined by

$$P_n^{(A,B)}(z) = \frac{(A+I)_n}{n!}\,{}_2F_1\left(-nI, A+B+(n+1)I; A+I; \frac{1-z}{2}\right). \tag{15}$$

2. Generating Functions of the EGHMF

In several areas in applied mathematics and mathematical physics, generating functions play an important role in the investigation of various useful properties of the sequences which they generate. They are used to find certain properties and formulas for numbers and polynomials in a wide variety of research subjects, indeed, in modern combinatorics. One can refer to the extensive work of Srivastava and Manocha [24] for a systematic introduction and several interesting and useful applications of the various methods of obtaining linear, bilinear, bilateral, or mixed multilateral generating functions for a fairly wide variety of sequences of special functions (and polynomials) in one, two, and more variables, among much abundant literature; in this regard, in fact, a remarkable large number of generating functions involving a variety of special functions have been developed by many authors (see, e.g., [13, 25–27]). Here, we present some generating functions involving the following family of the extended Gauss hypergeometric matrix functions:

Theorem 1. *Let $F^{(\mathbb{P})}(A,B;C;z)$ be given in (8); then, the following generating function holds true:*

$$\sum_{r=0}^{\infty}(A)_r F^{(\mathbb{P})}(A+rI,B,C;z)\frac{w^r}{r!} = (1-w)^{-A}F^{(\mathbb{P})}\left(A,B,C;\frac{z}{1-w}\right); \quad |z|<1, |w|<1. \tag{16}$$

Proof. For convenience, let the left-hand side of (16) be denoted by T. Applying the series expression of (8) to T, we obtain

$$T = \sum_{r=0}^{\infty}(A)_r\left\{\Gamma\begin{pmatrix} C \\ B, C-B\end{pmatrix}\times\sum_{n=0}^{\infty}(A+rI)_n \cdot \mathscr{B}(B+nI,C-B;\mathbb{P})\frac{z^n}{n!}\right\}\frac{w^r}{r!}. \tag{17}$$

By changing the order of summations in (17), we obtain

$$T = \Gamma\begin{pmatrix} C \\ B, C-B\end{pmatrix}\sum_{n=0}^{\infty}(A)_n\mathscr{B}(B+nI,C-B;\mathbb{P}) \times\left\{\sum_{r=0}^{\infty}(A+nI)_r\frac{w^r}{r!}\right\}\frac{z^n}{n!}. \tag{18}$$

Furthermore, upon using the generalized binomial expansion, we find that the inner sum in (18) yields

$$\sum_{r=0}^{\infty}(A+nI)_r\frac{w^r}{r!} = (1-w)^{-(A+nI)}; \quad |w|<1. \tag{19}$$

Finally, in view of (18) and (19), we obtain the desired result of Theorem 1.

A further generalization of the extended Gauss hypergeometric matrix functions (8) is given in the following definition. □

Definition 4. In terms of the extended Gauss hypergeometric matrix function given by (8), we define a sequence $\{\Omega_n(z)\}_{n\in\mathbb{N}_0}$ as follows:

$$\begin{aligned}\Omega_n(z) &= \Omega_n^{(\lambda)}(A,B;C;z) \\ &= F^{(\mathbb{P})}(\Delta(\lambda;A+nI),B,C;z); \quad \lambda\in\mathbb{N},\end{aligned} \tag{20}$$

where, for convenience, $\Delta(\lambda;A)$ abbreviates the array of λ matrix parameters:

$$\frac{A}{\lambda},\frac{A+I}{\lambda},\frac{A+2I}{\lambda},\ldots,\frac{A+(\lambda-1)I}{\lambda}; \quad \lambda\in\mathbb{N}. \tag{21}$$

Remark 2. In the extended Gauss hypergeometric matrix function occurring in definition (20), it is understood that the matrix parameter A of definition (8) has been replaced by a set of λ parameters which are abbreviated by $\Delta(\lambda;A+nI)$. The above definition (20) is motivated by the extensive investigation on this subject.

Now, we prove the following result, which provides the generating functions for the extended Gauss hypergeometric matrix functions defined above.

Theorem 2. *For each $\lambda\in\mathbb{N}$, the following generating function holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty}(A+kI)_n\Omega_{k+n}^{(\lambda)}(z)\frac{w^n}{n!} \\ &= (1-w)^{-(A+kI)}\Omega_k^{(\lambda)}\left(\frac{z}{(1-w)^{\lambda}}\right),\end{aligned} \tag{22}$$

where $|z|<1, |w|<1$, and $k\in\mathbb{N}_0$.

Proof. Using the definitions (20) and (8) and changing the order of summation, the left-hand side Υ of the result (22) is given by

$$\begin{aligned}\Upsilon = \sum_{r=0}^{\infty}&\left(\frac{A+kI}{\lambda}\right)_r\left(\frac{A+(k-1)I}{\lambda}\right)_r\cdots\left(\frac{A+(k+\lambda-1)I}{\lambda}\right)_r \\ &\times\Gamma\begin{pmatrix} C \\ B, C-B\end{pmatrix}\mathscr{B}(B+rI,C-B;\mathbb{P}) \\ &\times\left\{\sum_{n=0}^{\infty}(A+(k+r\lambda)I)_n\frac{w^n}{n!}\right\}\frac{z^r}{r!}.\end{aligned} \tag{23}$$

Now, by appealing once again to (19), we easily arrive to the desired result (22) asserted by Theorem 2. □

Remark 3. Furthermore, we note the following special cases of generating functions of the EGHMF as follows:

(i) It may be noted that if we set $\lambda = 1$ and replace A by $A - kI$ in (22), we readily obtain assertion (16) of Theorem 1

(ii) At $\mathbb{P} = 0$, we observe that result (16) corresponds with that in [28]

(iii) For arbitrary complex numbers $\{\alpha, \beta, \gamma \text{ and } \eta\}$, putting $\{A = \alpha \in \mathbb{C}^{1\times 1}, B = \beta \in \mathbb{C}^{1\times 1}, C = \gamma \in \mathbb{C}^{1\times 1}$, and $\mathbb{P} = \eta \in \mathbb{C}^{1\times 1}\}$ in (16) and (22), we find generating functions for the generalized Gauss hypergeometric function on [13, 27]

3. Integral Representations for the EGHMF

Integral formulas with such special matrix functions such as the beta matrix functions and the hypergeometric matrix functions are used in solving numerous applied problems. Hence, their demonstrated applications and several generalizations of integral transforms with hypergeometric matrix functions have been actively investigated. Here, by means of the extended beta matrix function $\mathscr{B}(A, B; \mathbb{P})$ given in (4), we introduce some new generalized integral formulas for the EGHMF in this section.

Theorem 3. *For $\alpha \in \mathbb{C}$, the extended Gauss hypergeometric matrix function satisfies the following integral relations:*

(i) $\int_0^\infty z^{\alpha-1} F^{(\mathbb{P})}(A, B; C; -z)dz = \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \mathscr{B}(A - \alpha I, \alpha I)\mathscr{B}(B - \alpha I, C - B; \mathbb{P})$

(ii) $\int_0^\infty \exp(-z) z^{\alpha-1} F^{(\mathbb{P})}(A, B; C; z)dz = \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \sum_{n=0}^\infty \mathscr{B}(B + nI, C - B; \mathbb{P})(\alpha)_n \Gamma(\alpha)(A)_n/n!$

(iii) $\int_0^1 (1 - z^2)^{\alpha-1} F^{(\mathbb{P})}(A, B; C; 1 - z/2)dz = \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \sum_{n=0}^\infty 2^{2\alpha+(n-2)}(A)_n \times \mathscr{B}((\alpha + n)I, \alpha I)\mathscr{B}(B + nI, C - B; \mathbb{P})(1/2)^n/n!$,

where A, B, C, and $C - B$ are the positive stable matrices in $\mathbb{C}^{r\times r}$

Proof

(i) Replacing $F^{(\mathbb{P})}(A, B; C; -z)$ by its integral representation (12) and changing the order of integration, we get

$$\int_0^\infty z^{\alpha-1} F^{(\mathbb{P})}(A, B; C; -z)dz = \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \times \int_0^1 t^{B-I}(1-t)^{C-B-I} \exp\left(\frac{-\mathbb{P}}{t(1-t)}\right) \cdot \int_0^\infty z^{\alpha-1}(1 + tz)^{-A} dt\, dz. \tag{24}$$

Now, if we integrate with respect to z using the properties of beta matrix function and substitute $u = tz$, we will have

$$\begin{aligned} \int_0^\infty z^{\alpha-1} F^{(\mathbb{P})}(A, B; C; -z)dz &= \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \cdot \int_0^1 t^{B-I}(1-t)^{C-B-I} \times \exp\left(\frac{-\mathbb{P}}{t(1-t)}\right)dt \\ &\cdot \int_0^\infty \left(\frac{u}{t}\right)^{\alpha-1}(1+u)^{-A}\frac{du}{t} \\ &= \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \mathscr{B}(A - \alpha I, \alpha I)\mathscr{B}(B - M, C - B; \mathbb{P}), \end{aligned} \tag{25}$$

which is the required result in (i).

(ii) Direct calculations using (12) yield

$$F^{(\mathbb{P})}(A, B; C; z) = \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \times \int_0^1 t^{B-I}(1-t)^{C-B-I} \cdot (1 - zt)^{-A} \exp\left(\frac{-\mathbb{P}}{t(1-t)}\right)dt. \tag{26}$$

Since $(1 - zt)^{-A} = \sum_{n=0}^\infty (A)_n (zt)^n/n!$, then we have

$$\begin{aligned} &\int_0^\infty \exp(-z) z^{\alpha-1} F^{(\mathbb{P})}(A, B; C; z)dz \\ &= \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \int_0^\infty \int_0^1 t^{B-I}(1-t)^{C-B-I} \exp(-z) z^{\alpha-1} \times \exp\left(\frac{-\mathbb{P}}{t(1-t)}\right) \sum_{n=0}^\infty \frac{(A)_n}{n!}(zt)^n dt\, dz \\ &= \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \int_{n=0}^\infty (A)_n \frac{1}{n!} \int_0^\infty \int_0^1 t^{B+(n-1)I}(1-t)^{C-B-I} \times \exp(-z) z^{\alpha+(n-1)} \exp\left(\frac{-\mathbb{P}}{t(1-t)}\right)dt\, dz \\ &= \Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \sum_{n=0}^\infty (A)_n \frac{1}{n!} \int_0^\infty \exp(-z) z^{\alpha+(n-1)} dz \times \int_0^1 t^{B+(n-1)I}(1-t)^{C-B-I} \exp\left(\frac{-\mathbb{P}}{t(1-t)}\right)dt \\ &= \Gamma(\alpha)\Gamma\begin{pmatrix} C \\ B, C-B \end{pmatrix} \sum_{n=0}^\infty (A)_n (\alpha)_n \mathscr{B}(B + nI, C - B; \mathbb{P}) \frac{1}{n!}. \end{aligned} \tag{27}$$

Thus, we get the desired assertion (ii) of Theorem 3.

(iii) By using (12), it follows that

$$F^{(\mathbb{P})}\left(A,B;C;\frac{1-z}{2}\right)=\Gamma\begin{pmatrix} C \\ B, C-B\end{pmatrix}\times\int_0^1 t^{B-I}(1-t)^{C-B-I}$$
$$\cdot\sum_{n=0}^{\infty}\frac{(A)_n}{n!}\left(\frac{1-z}{2}\right)^n t^n\exp\left(\frac{-\mathbb{P}}{t(1-t)}\right)\mathrm{d}t. \tag{28}$$

Thus,

$$\int_0^1\left(1-z^2\right)^{\alpha-1}F^{(\mathbb{P})}\left(A,B;C;\frac{1-z}{2}\right)\mathrm{d}z$$
$$=\Gamma\begin{pmatrix} C \\ B, C-B\end{pmatrix}\int_0^1\int_0^1\left(1-z^2\right)^{\alpha-1}t^{B-I}(1-t)^{C-B-I} \tag{29}$$
$$\times\exp\left(\frac{-\mathbb{P}}{t(1-t)}\right)\sum_{n=0}^{\infty}\frac{(A)_n}{n!}\left(\frac{1-z}{2}\right)^n t^n\mathrm{d}t\mathrm{d}z.$$

Now, we put $z=\cos\theta$, $1-z=2\sin^2(\theta/2)$, and $1+z=2\cos^2(\theta/2)$ in (29); we obtain

$$\int_0^1\left(1-z^2\right)^{\alpha-1}F^{(\mathbb{P})}\left(A,B;C;\frac{1-z}{2}\right)\mathrm{d}z$$
$$=\Gamma\begin{pmatrix} C \\ B, C-B\end{pmatrix}\sum_{n=0}^{\infty}\frac{(A)_n}{2^n n!}\times\int_0^{\pi/2}\int_0^1\left(2\sin^2\left(\frac{\theta}{2}\right)\right)^n$$
$$\cdot\left(4\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\right)^{\alpha-1}\times\sin\theta t^{B+(n-1)I}(1-t)^{C-B-I}$$
$$\cdot\exp\left(\frac{-\mathbb{P}}{t(1-t)}\right)\mathrm{d}\theta\,\mathrm{d}t. \tag{30}$$

Hence, we easily arrive to the desired result (iii) asserted by Theorem 3:

$$\int_0^1\left(1-z^2\right)^{\alpha-1}F^{(\mathbb{P})}\left(A,B;C;\frac{1-z}{2}\right)\mathrm{d}z$$
$$=\Gamma\begin{pmatrix} C \\ B, C-B\end{pmatrix}\sum_{n=0}^{\infty}\frac{(A)_n}{2^n n!}\times\int_0^{\pi/2}\int_0^1 2^{2\alpha+(n-1)}$$
$$\cdot\sin^{2\alpha+(2n-1)}\left(\frac{\theta}{2}\right)\times\cos^{2\alpha-1}\left(\frac{\theta}{2}\right)t^{B+(n-1)I}(1-t)^{C-B-I}$$
$$\cdot\exp\left(\frac{-\mathbb{P}}{t(1-t)}\right)\mathrm{d}\left(\frac{\theta}{2}\right)\mathrm{d}t$$
$$=\Gamma\begin{pmatrix} C \\ B, C-B\end{pmatrix}\sum_{n=0}^{\infty}2^{2\alpha+(n-2)}(A)_n\mathscr{B}((\alpha+n)I,\alpha I)$$
$$\times\mathscr{B}(B+nI,C-B;\mathbb{P})\frac{(1/2)^n}{n!}, \tag{31}$$

which proves the assertion (iii) of Theorem 3. □

Remark 4. It is worthy of note that the special cases can be obtained from formulas (i), (ii), and (ii) of Theorem 3 as follows:

(1) Taking $\mathbb{P}=0$, we find some integral representations for the Gauss hypergeometric matrix function (GHMF) (cf. [10, 20, 22])

(2) Furthermore, choosing $\{a,b,c,\text{ and }\eta\in\mathbb{C}\}$ and setting $\{A=a\in\mathbb{C}^{1\times1},B=b\in\mathbb{C}^{1\times1},C=c\in\mathbb{C}^{1\times1}$ and $\mathbb{P}=\eta\in\mathbb{C}^{1\times1}\}$ in Theorem 3, we obtain some integral representations for Gauss hypergeometric function (cf. [14])

4. An Application of the Computation of *m* Derivatives of the Extended Jacobi Matrix Polynomial

The Jacobi matrix polynomial and their special cases play important roles in approximation theory and its applications [20]. In this section, the extended Jacobi matrix polynomial ispresented and prove the following theorems for the mth derivatives of extended Jacobi matrix polynomials. Using the definition of the extended Gauss hypergeometric matrix functions EGHMF to define the extended matrix Jacobi polynomial and their special cases.

Definition 5. Let A, B, and $\mathbb{P}$ be positive stable matrices in $\mathbb{C}^{r\times r}$ whose eigenvalues, z, all satisfy $\mathrm{Re}(z)>-1$. For any positive integer n, the nth extended Jacobi matrix polynomial is

$$P_n^{(A,B;\mathbb{P})}(z)=\frac{(A+I)_n}{n!}{}_2F_1^{(\mathbb{P})}$$
$$\cdot\left(-nI,A+B+(n+1)I;A+I;\frac{1-z}{2}\right). \tag{32}$$

Theorem 4. *Let A, B, and $\mathbb{P}$ be positive stable matrices in $\mathbb{C}^{r\times r}$. For the derivatives of extended Jacobi matrix polynomial, we find*

$$D^m\left\{P_n^{(A,B;\mathbb{P})}(z)\right\}=2^{-m}(A+B+(n+1)I)_m P_{n-m}^{(A+mI,B+mI;\mathbb{P})}(z), \tag{33}$$

where $|z|<1$ and $D=d/dz$.

Proof. Using (14) and (32) with the parameters $A=-nI$, $B=A+B+(n+1)I$, and $C=A+I$, we get

$$D^m\left\{{}_2F_1^{(\mathbb{P})}\left(-nI,A+nI;A+\frac{I}{2};\frac{1-z}{2}\right)\right\}$$
$$=\left(\frac{-1}{2}\right)^m(-n)_m(A+B+(n+1)I)_m\left[(A+I)_m\right]^{-1}$$
$${}_2F_1^{(\mathbb{P})}\left((-n+m)I,A+(n+m)I;A+;A+(m+1)I;\frac{1-z}{2}\right), \tag{34}$$

from (34) and multiplying by $(A+I)_n/n!$, we obtain

$$D^{m}\left\{\frac{(A+I)_{n}}{n!}{}_{2}F_{1}^{(\mathbb{P})}\left(-nI,A+nI;A+\frac{I}{2};\frac{1-z}{2}\right)\right\}$$
$$=\frac{(A+I)_{n}}{n!}\left(\frac{-1}{2}\right)^{m}(-n)_{m}(A+B+(n+1)I)_{m}$$
$$\left[(A+I)_{m}\right]^{-1}{}_{2}F_{1}^{(\mathbb{P})}\left((-n+m)I,A+(n+m)I;A+;A+(m+1)I;\frac{1-z}{2}\right). \tag{35}$$

Now making use of the extended of Jacobi matrix polynomial (32), we find

$$D^{m}\left\{P_{n}^{(A,B;\mathbb{P})}(z)\right\}=\frac{(A+I)_{n}}{n!}\left(\frac{-1}{2}\right)^{m}(-n)_{m}(A+B+(n+1)I)_{m}$$
$$\left[(A+I)_{m}\right]^{-1}{}_{2}F_{1}^{(\mathbb{P})}\left((-n+m)I,A+(n+m)I;A+;A+(m+1)I;\frac{1-z}{2}\right). \tag{36}$$

By using (36), we get

$$D^{m}\left\{P_{n}^{(A,B;\mathbb{P})}(z)\right\}=\left(\frac{-1}{2}\right)^{m}\frac{\Gamma(-n+m)}{\Gamma(-n)n!}(A+B+(n+1)I)_{m}$$
$$\times\Gamma^{-1}(A+(m+1)I)\Gamma(A+(n+1)I)$$
$$\times{}_{2}F_{1}^{(\mathbb{P})}\left((-n+m)I,A+(n+m)I;\cdot A+;A+(m+1)I;\frac{1-z}{2}\right). \tag{37}$$

Further simplification yields

$$D^{m}\left\{P_{n}^{(A,B;\mathbb{P})}(z)\right\}=2^{-m}(A+B+(n+1)I)_{m}$$
$$\cdot\frac{\Gamma^{-1}(A+(m+1)I)\Gamma(A+(n+1)I)}{(n-m)!}$$
$$\times{}_{2}F_{1}^{(\mathbb{P})}\left((-n+m)I,A+(n+m)I;\cdot A+;A+(m+1)I;\frac{1-z}{2}\right)$$
$$=2^{-m}(A+B+(n+1)I)_{m}P_{n-m}^{(A+mI,B+mI;\mathbb{P})}(z). \tag{38}$$

This completes the proof of Theorem 4. □

Theorem 5. *Let A, B, C, D, and $\mathbb{P}$ are positive stable matrices in $\mathbb{C}^{r\times r}$ and suppose that*

$$P_{n}^{(C,D;\mathbb{P})}(z)=\sum_{m=0}^{n}\delta_{nm}(A,B,C,D)P_{m}^{(A,B;\mathbb{P})}(z). \tag{39}$$

Then,

$$\delta_{nm}(A,B,C,D)=\frac{(C+D+(n+1)I)_{k}(C+(m+1)I)_{n-m}}{(m-n)!}$$
$$\times\Gamma^{-1}(A+B+(2m+1)I)$$
$$\cdot\Gamma(A+B+(m+1)I)$$
$$\times{}_{3}F_{2}((-n+m)I,C+D+(m+n+1)I,$$
$$\cdot A+(m+1)I;$$
$$\cdot C+(m+1)I,A+B+2(m+1)I;1). \tag{40}$$

Proof. Substitution of (40) into the RHS of (39) and making use of the extended of Jacobi matrix polynomial (32), we find

$$\sum_{m=0}^{n}\delta_{nm}(A,B,C,D)P_{m}^{(A,B;\mathbb{P})}(z)$$
$$=\frac{(C+D+(n+1)I)_{k}(C+(m+1)I)_{n-m}}{(m-n)!}$$
$$\times\Gamma^{-1}(A+B+(2m+1)I)\Gamma(A+B+(m+1)I)$$
$$\times{}_{3}F_{2}((-n+m)I,C+D+(m+n+1)I,A+(m+1)I;$$
$$C+(m+1)I,A+B+2(m+1)I;1)$$
$$\frac{\Gamma^{-1}(A+I)\Gamma(A+(m+1)I)}{m!}{}_{2}F_{1}^{(\mathbb{P})}$$
$$\cdot\left(-mI,A+B+(m+1)I;A+I;\frac{1-z}{2}\right), \tag{41}$$

since

$${}_{3}F_{2}(-nI,A,B;C,D;1)=\left[(C)_{n}\right]^{-1}\left[(C-A-B)_{n}\right]^{-1}(C-A)_{n}(C-B)_{n}, \tag{42}$$

and we have

$$\sum_{m=0}^{n} \delta_{nm}(A,B,C,D) P_m^{(A,B;\mathbb{P})}(z)$$
$$= \sum_{m=0}^{n} \frac{\Gamma^{-1}(A+B+(2m+1)I)(C+D+(n+1)I)_k \Gamma(A+B+(m+1)I)}{(m-n)!}$$
$$\times [(-(A+D+(n+m+1)I))]^{-1}(-nI+D)_{n-m}(C-A)_{n-m} \tag{43}$$
$$\Gamma^{-1}(A+B+(m+1)I)\Gamma^{-1}(-(B+mI))\frac{\Gamma(A+(m+1)I)}{m!}$$
$$\sum_{r=0}^{\infty}\frac{\Gamma(-m+r)}{\Gamma(-m)}\mathscr{B}(A+B+(m+r+1)I,-B-mI;\mathbb{P})\frac{((1-z)/2)^r}{r!}.$$

Expanding (43) and collecting similar terms, we obtain

$$\sum_{m=0}^{n} \delta_{nm}(A,B,C,D) P_m^{(A,B;\mathbb{P})}(z)$$
$$= \Gamma^{-1}(C+D+(n+1)I)\Gamma^{-1}(-(D+nI))\frac{\Gamma(C+(n+1)I)}{n!},$$
$$\sum_{r=0}^{\infty}(-nI)_r \mathscr{B}(C+D+(n+r+1)I,-D-nI;\mathbb{P})\frac{((1-z)/2)^r}{r!}$$
$$= \frac{C+I)}{n!}{}_2F_1^{(\mathbb{P})}\left(-nI, C+D+(n+1)I; C+I; \frac{1-z}{2}\right)$$
$$= P_n^{(C,D;\mathbb{P})}(z). \tag{44}$$

This completes the proof of Theorem 5.

Finally, for the definition of extended of Jacobi matrix polynomial, we consider some of the extended special matrix polynomial as follows. □

Definition 6. Let A and $\mathbb{P}$ be positive stable matrices in $\mathbb{C}^{r\times r}$ whose eigenvalues, z, all satisfy $Re(z) > -1$. For any positive integer n, the nth extended ultraspherical matrix polynomials are

$$C_n^{(A;\mathbb{P})}(z) = \frac{(2A)_n}{n!}{}_2F_1^{(\mathbb{P})}\left(-nI, A+nI; A+\frac{I}{2}; \frac{1-z}{2}\right). \tag{45}$$

4.1. Special Cases. Upon assigning particular values to the parameters and variables, we interestingly get a range of special cases for (32) and (45) as discussed below:

(i) Setting $A = \mathbf{0}$ in (45), we find

$$T_n^{(0;\mathbb{P})}(z) = {}_2F_1^{(\mathbb{P})}\left(-nI, nI; \frac{I}{2}; \frac{1-z}{2}\right), \tag{46}$$

where $T_n^{(\mathbf{0};\mathbb{P})}(z)$ is called the extended Chebyshev matrix polynomial of the first kind.

(ii) Putting $A = I$ in (45), we have

$$U_n^{(\mathbf{0};\mathbb{P})}(z) = \frac{1}{n+1}\sqrt{1-nz^2}\,{}_2F_1^{(\mathbb{P})}\left((-n+1)I, (n+1)I; \frac{3I}{2}; \frac{1-z}{2}\right), \tag{47}$$

where $U_n^{(\mathbf{0};\mathbb{P})}(z)$ is called the extended Chebyshev matrix polynomial of the second kind.

(iii) Furthermore, taking $A = I/2$ in (45), we get

$$P_n^{(\mathbf{0};\mathbb{P})}(z) = {}_2F_1^{(\mathbb{P})}\left(-nI, (n+1)I; I; \frac{1-z}{2}\right), \tag{48}$$

where $P_n^{(\mathbf{0};\mathbb{P})}(z)$ is called the extended Legendre matrix polynomial.

(iv) For $\mathbb{P} = \mathbf{0}$ in (32), we get of Jacobi matrix polynomial in (15).

(v) Taking $\mathbb{P} = \mathbf{0}$ and putting $\{A = \alpha \in \mathbb{C}^{1\times1}$, and $B = \beta \in \mathbb{C}^{1\times1}, \}$ in (32), we get Jacobi polynomial as follows:

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}{}_2F_1 \cdot \left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1}{2}(1-z)\right). \tag{49}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly to writing this article. All the authors read and approved the final manuscript.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through research group program under grant (R.G.P.1/148/40). This work was also supported by the National Natural Science Foundation of China (11601525).

References

[1] M. Abul-Dahab, M. Abul-Ez, Z. Kishka, and D. Constales, "Reverse generalized Bessel matrix differential equation, polynomial solutions, and their properties," *Mathematical Methods in the Applied Sciences*, vol. 38, no. 6, pp. 1005–1013, 2015.

[2] A. M. Mathai and H. Haubold, *Special Functions for Applied Scientists*, Springer Science, New York, NY, USA, 2008.

[3] A. M. Mathai, *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*, Oxford University Press, Oxford, UK, 1993.

[4] W. Miller, *Lie Theory and Specials Functions*, Academic Press, New York, NY, USA, 1968.

[5] M. Abdalla, "Further results on the generalised hypergeometric matrix functions," *International Journal of Computing Science and Mathematics*, vol. 10, no. 1, pp. 1–10, 2019.

[6] M. Abul-Dahab and A. Bakhet, "A certain generalized gamma matrix functions and their properties," *Journal of Analysis & Number Theory*, vol. 3, pp. 63–68, 2015.

[7] M. Abdalla and A. Bakhet, "Extension of beta matrix function," *Asian Journal of Mathematics and Computer Research*, vol. 9, no. 3, pp. 253–264, 2016.

[8] M. Abdalla and A. Bakhet, "Extended Gauss hypergeometric matrix functions," *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 42, no. 3, pp. 1465–1470, 2018.

[9] M. Abdalla, "On the incomplete hypergeometric matrix functions," *The Ramanujan Journal*, vol. 43, no. 3, pp. 663–678, 2017.

[10] A. Bakhet, Y. Jiao, and F. He, "On the Wright hypergeometric matrix functions and their fractional calculus," *Integral Transforms and Special Functions*, vol. 30, no. 2, pp. 138–156, 2019.

[11] P. Agarwal, S. Dragomir, M. Jleli, and B. Samet, *Advances in Mathematical Inequalities and Applications (Trends in Mathematics)*, Birkhuser, Basel, Switzerland, 2019.

[12] P. Agarwal, J. Choi, and S. Jain, "Extended hypergeometric functions of two and three variables," *Communications of the Korean Mathematical Society*, vol. 30, no. 4, pp. 403–414, 2015.

[13] P. Agarwal, M. Chand, and S. D. Purohit, "A note on generating functions involving the generalized Gauss hypergeometric functions," *National Academy Science Letters*, vol. 37, no. 5, pp. 457–459, 2014.

[14] J. Choi and P. Agarwal, "Certain integral transform and fractional integral formulas for the generalized Gauss hypergeometric functions," *Abstract and Applied Analysis*, vol. 2014, Article ID 735946, 7 pages, 2014.

[15] M.-J. Luo, G. V. Milovanovic, and P. Agarwal, "Some results on the extended beta and extended hypergeometric functions," *Applied Mathematics and Computation*, vol. 248, pp. 631–651, 2014.

[16] M. Ruzhansky, Y. JeCho, P. Agarwal, and I. Area, *Advances in Real and Complex Analysis with Applications (Trends in Mathematics)*, Birkhuser, Basel, Switzerland, 2017.

[17] M. Abdalla, H. Abd-Elmageed, M. Abul-Ez, and Z. Kishka, "Operational formulae of the multivariable hypergeometric matrix functions and related matrix polynomials," *General Letters in Mathematics*, vol. 3, pp. 81–90, 2017.

[18] A. Bakhet and F. He, "On 2-variables Konhauser matrix polynomials and their fractional integrals," *Mathematics*, vol. 8, no. 2, p. 232, 2020.

[19] M. Hidan and M. Abdalla, "A note on the Appell hypergeometric matrix function," *Mathematical Problems in Engineering*, 2020, In press.

[20] M. Abdalla, "Special matrix functions: characteristics, achievements and future directions," *Linear and Multilinear Algebra*, vol. 68, no. 1, pp. 1–28, 2018.

[21] L. Jódar and J. C. Cortés, "Some properties of Gamma and Beta matrix functions," *Applied Mathematics Letters*, vol. 11, no. 1, pp. 89–93, 1998.

[22] L. Jódar and J. C. Cortés, "On the hypergeometric matrix function," *Journal of Computational and Applied Mathematics*, vol. 99, no. 1-2, pp. 205–217, 1998.

[23] F. He, A. Bakhet, M. Hidan, and M. Abdalla, "Two variables shivley's matrix polynomials," *Symmetry*, vol. 11, no. 2, p. 151, 2019.

[24] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, John Wiley and Sons, New York, NY, USA, 1984.

[25] F. Ayant and D. Kumar, "Generating relations and multivariable Aleph-function," *Analysis*, vol. 38, Article ID 137143, 2018.

[26] R. Srivastava, "Some classes of generating functions associated with a certain family of extended and generalized hypergeometric functions," *Applied Mathematics and Computation*, vol. 243, pp. 132–137, 2014.

[27] H. M. Srivastava, P. Agarwal, and S. Jain, "Generating functions for the generalized Gauss hypergeometric functions," *Applied Mathematics and Computation*, vol. 247, pp. 348–352, 2014.

[28] R. S. Batahan, *Study and developing some subjects in complex analysis*, Ph.D. thesis, Assiut University, Asyut, Egypt, 2004.