Research Article

A Modified Three-Term Type CD Conjugate Gradient Algorithm for Unconstrained Optimization Problems

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Conjugate gradient methods are well-known methods which are widely applied in many practical fields. CD conjugate gradient method is one of the classical types. In this paper, a modified three-term type CD conjugate gradient algorithm is proposed. Some good features are presented as follows: (i) A modified three-term type CD conjugate gradient formula is presented. (ii) The given algorithm possesses sufficient descent property and trust region property. (iii) The algorithm has global convergence with the modified weak Wolfe–Powell (MWWP) line search technique and projection technique for general function. The new algorithm has made great progress in numerical experiments. It shows that the modified three-term type CD conjugate gradient method is more competitive than the classical CD conjugate gradient method.

1. Introduction

Considering the problem

$$\min\{f(x) \mid x \in \mathcal{R}^n\}, \quad (1)$$

where $f: \mathcal{R}^n \rightarrow \mathcal{R}$ is a continuously differentiable function. This kind of model is often used to solve some problems in applied mathematics, economics, engineering, and so on. Generally, the following iteration formula is used to generate the next iteration point:

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0, 1, 2, \dots, \quad (2)$$

where x_{k+1} and x_k denote the next iteration point and current iteration point, respectively. α_k is a step-length and d_k is the search direction. The search direction d_k generated by Conjugated Gradient (CG) method is defined by the following formula:

$$d_{k+1} = \begin{cases} -g_{k+1}, & \text{if } k = 0, \\ -g_{k+1} + \beta_k d_k, & \text{if } k \geq 1, \end{cases} \quad (3)$$

where $g_k = g(x_k)$ is the gradient of $f(x)$ at x_k and β_k is a parameter. Different β_k will generate different CG methods [1–10]. There are six classical forms of β_k :

$$\begin{aligned} \beta_k^{PRP} &= \frac{g_{k+1}^T y_k}{\|g_k\|^2}, [8, 9], \\ \beta_k^{HS} &= \frac{g_{k+1}^T y_k}{y_k^T d_k}, [6], \\ \beta_k^{LS} &= \frac{g_{k+1}^T y_k}{-d_k^T g_k}, [7], \\ \beta_k^{FR} &= \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, [5], \\ \beta_k^{CD} &= \frac{\|g_{k+1}\|^2}{-d_k^T g_k}, [4], \\ \beta_k^{DY} &= \frac{\|g_{k+1}\|^2}{y_k^T d_k}, [2], \end{aligned} \quad (4)$$

where $y_k = g_{k+1} - g_k$, and $\|\cdot\|$ is the Euclidean norm. Those formulas can be divided into two categories: One includes

PRP method, HS method, and LS method, which have good numerical performance; the other includes FR method, CD method, and DY method, which have good theory convergence. About these methods, many scholars had applied them to solve nonlinear monotone equations and normal optimization problems, and some good results were achieved [11–16].

Zhang et al. [17] presented a modified PRP CG formula as follows:

$$d_{k+1} = -g_{k+1} + \frac{g_{k+1}^T y_k d_k - d_k^T g_{k+1} y_k}{g_k^T g_k}. \quad (5)$$

They proved that the modified PRP method is globally convergent with Armijo-type line search.

Yuan et al. [18] proposed another modified PRP CG formula with

$$d_{k+1} = -\gamma_k g_{k+1} + \beta_k^{\text{PRP}} d_k - \frac{d_k^T g_{k+1} y_k}{\|g_k\|^2}, \quad (6)$$

where $\gamma_k = (\theta_k / \min\{\eta_1 s_k^T y_k, \eta_2 \theta_k\})$, $\theta_k = (s_k^T y_k / \|s_k\|^2)$, $s_k = x_{k+1} - x_k$, $\eta_1 \in (0, 1)$, and $\eta_2 \in (0, 1)$. Yuan et al. [18] obtained the global convergence of the modified PRP method with a modified weak Wolfe–Powell (MWWP) line search technique, which was proposed by Yuan et al. [19]:

$$f(x_k + \alpha_k d_k) \leq f_k + \lambda \alpha_k g_k^T d_k + \alpha_k \min\left\{-\lambda_1 g_k^T d_k, \lambda \frac{\alpha_k}{2} \|d_k\|^2\right\}, \quad (7)$$

$$g(x_k + \alpha_k d_k)^T d_k \geq \mu g_k^T d_k + \min\left\{-\lambda_1 g_k^T d_k, \lambda \alpha_k \|d_k\|^2\right\}, \quad (8)$$

where $\lambda \in (0, (1/2))$, $\lambda_1 \in (0, \lambda)$, and $\mu \in (\lambda, 1)$. Yuan et al. [16] obtained the global convergence of PRP method by using the above modified weak Wolfe–Powell line search technique and a projection technique:

$$x_{k+1} = x_k + \frac{g(\psi_k)^T (\psi_k - x_k) + \chi \|\psi_k - x_k\|^2}{\|g(\psi_k) - g(x_k)\|^2} [g(\psi_k) - g(x_k)], \quad (9)$$

where $\psi_k = x_k + \alpha_k d_k$ is the next point and parameter $\chi > 0$. With this projection technique, any unsatisfactory iteration points generated by the normal PRP algorithm will be projected onto a surface to overcome the failure to converge.

Motivated by the above researches, a modified three-term CD conjugate gradient algorithm is presented with (7), (8), and (9) in this paper. Some good properties are obtained as follows:

- (i) A modified three-term type CD conjugate gradient formula is presented
- (ii) The given algorithm possesses sufficient descent property and trust region property
- (iii) The algorithm has global convergence with the MWWP line search technique and projection technique for general function

This paper is organized as follows: the next section will introduce the modified CD formula and relative algorithm; section 3 gives the proof of the global convergence of the new algorithm; numerical experiments are given in section 4; some conclusions are presented in section 5. Throughout this paper, we use $\|\cdot\|$ to denote the Euclidean norm, $g(x_k)$ and $g(x_{k+1})$ are replaced by g_k and g_{k+1} , respectively.

2. Motivation and Algorithm

The convergence of CD conjugate gradient method has been proved [4]; however, the numerical results of this method are worse than the PRP method and others. Therefore, it is necessary to propose a new search direction to improve the numerical performance of the CD method. Meanwhile, the sufficient descent property is significant for obtaining the convergence of the conjugate gradient method:

$$g_k^T d_k \leq -\eta \|g_k\|^2. \quad (10)$$

Then, we also hope that we can propose a new method that possesses this property. Inspired by the above discussion, a modified three-term type CD conjugate gradient formula is designed by the following:

$$d_{k+1} = \begin{cases} -g_{k+1}, & \text{if } k = 0, \\ -\gamma_k g_{k+1} + \frac{g_{k+1}^T g_{k+1} d_k}{\max\{-d_k^T g_k, \eta_1 \|g_{k+1}\| \|d_k\|\}} - \frac{g_{k+1}^T d_k g_{k+1}}{\max\{-d_k^T g_k, \eta_1 \|g_{k+1}\| \|d_k\|\}}, & \text{if } k \geq 1, \end{cases} \quad (11)$$

where $\gamma_k = (\theta_k / \min\{\eta_2 s_k^T y_k, \eta_3 \theta_k\})$, $\theta_k = \|s_k\|^2$, $s_k = x_{k+1} - x_k$, $y_k = g_{k+1} - g_k$, $\eta_1 > 0$, and $\eta_2, \eta_3 \in (0, 1)$. The steps of the given algorithm are listed as follows.

3. Convergence Analysis

In this section, we are going to analyse the convergence of the proposed algorithm. The following assumptions are needed.

Assumption 1

- (i) The level set $\Omega = \{x \in \mathbb{R}^n: f(x) \leq f(x_0)\}$ is bounded.
- (ii) $f(x)$ is twice continuously differentiable and bounded below, and the gradient function $g(x)$ is

Lipschitz continuous, which means that there exists a constant $L > 0$ such that

$$\|g(x) - g(y)\| \leq L\|x - y\|, \quad \text{for all } x, y \in \Omega. \quad (12)$$

Lemma 1. Let the search direction d_k be generated by formula (11), then, the following relations hold:

$$g_{k+1}^T d_{k+1} \leq -\eta \|g_{k+1}\|^2. \quad (13)$$

$$\|d_{k+1}\| \leq \xi \|g_{k+1}\|, \quad (14)$$

where constants $\eta, \xi > 0$.

Proof. According to the definition of d_k ,

$$\begin{aligned} g_{k+1}^T d_{k+1} &= g_{k+1}^T \left[-\gamma_k g_{k+1} + \frac{g_{k+1}^T g_{k+1} d_k}{\max\{-d_k^T g_k, \eta_1 \|g_{k+1}\| \|d_k\|\}} - \frac{g_{k+1}^T d_k g_{k+1}}{\max\{-d_k^T g_k, \eta_1 \|g_{k+1}\| \|d_k\|\}} \right] \\ &= -\gamma_k \|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2 g_{k+1}^T d_k}{\max\{-d_k^T g_k, \eta_1 \|g_{k+1}\| \|d_k\|\}} - \frac{g_{k+1}^T d_k \|g_{k+1}\|^2}{\max\{-d_k^T g_k, \eta_1 \|g_{k+1}\| \|d_k\|\}} \\ &= -\gamma_k \|g_{k+1}\|^2. \end{aligned} \quad (15)$$

For the parameter γ_k , we have the following:

$$\gamma_k = \frac{\theta_k}{\min\{\eta_2 s_k^T y_k, \eta_3 \theta_k\}} \geq \frac{\theta_k}{\eta_3 \theta_k} = \frac{1}{\eta_3}, \quad (16)$$

then $g_{k+1}^T d_{k+1} \leq -(1/\eta_3) \|g_{k+1}\|^2$ holds. Letting $\eta = (1/\eta_3)$, the relation (13) is obtained.

Using the definition of the parameter γ_k , we analyse the value of γ_k in two cases:

Case 1. $\min\{\eta_2 s_k^T y_k, \eta_3 \theta_k\} = \eta_2 s_k^T y_k$. Similar to Lemma 1 of [20], $s_k^T y_k \geq \rho \|s_k\|^2$, where $\rho > 0$ is a scalar; then

$$\gamma_k = \frac{\theta_k}{\min\{\eta_2 s_k^T y_k, \eta_3 \theta_k\}} = \frac{\theta_k}{\eta_2 s_k^T y_k} \leq \frac{\theta_k}{\eta_2 \rho \|s_k\|^2} = \frac{\|s_k\|^2}{\eta_2 \rho \|s_k\|^2} = \frac{1}{\eta_2 \rho}. \quad (17)$$

Case 2. $\min\{\eta_2 s_k^T y_k, \eta_3 \theta_k\} = \eta_3 \theta_k$. Then

$$\gamma_k = \frac{\theta_k}{\min\{\eta_2 s_k^T y_k, \eta_3 \theta_k\}} = \frac{\theta_k}{\eta_3 \theta_k} = \frac{1}{\eta_3}. \quad (18)$$

Letting $\xi_0 = \max\{(1/\eta_2 \rho), (1/\eta_3)\}$, we have $0 < \gamma_k \leq \xi_0$. Thus,

$$\begin{aligned} \|d_{k+1}\| &= \left\| -\gamma_k g_{k+1} + \frac{g_{k+1}^T g_{k+1} d_k}{\max\{-d_k^T g_k, \eta_1 \|g_{k+1}\| \|d_k\|\}} - \frac{g_{k+1}^T d_k g_{k+1}}{\max\{-d_k^T g_k, \eta_1 \|g_{k+1}\| \|d_k\|\}} \right\| \\ &\leq \gamma_k \|g_{k+1}\| + \frac{\|g_{k+1}\| \|g_{k+1}\| \|d_k\|}{\max\{-d_k^T g_k, \eta_1 \|g_{k+1}\| \|d_k\|\}} + \frac{\|g_{k+1}\| \|d_k\| \|g_{k+1}\|}{\max\{-d_k^T g_k, \eta_1 \|g_{k+1}\| \|d_k\|\}} \\ &\leq \gamma_k \|g_{k+1}\| + \frac{\|g_{k+1}\| \|g_{k+1}\| \|d_k\|}{\eta_1 \|g_{k+1}\| \|d_k\|} + \frac{\|g_{k+1}\| \|d_k\| \|g_{k+1}\|}{\eta_1 \|g_{k+1}\| \|d_k\|} \\ &\leq \gamma_k \|g_{k+1}\| + \frac{2}{\eta_1} \|g_{k+1}\| \\ &\leq \left(\xi_0 + \frac{2}{\eta_1} \right) \|g_{k+1}\|. \end{aligned} \quad (19)$$

Step 1: Given the initial point $x_0 \in \mathfrak{R}^n$, constants $\epsilon \in (0, 1)$, $0 < \lambda_1 < (\lambda/2) < \lambda < (1/2)$, $\lambda < \mu < 1$, $\eta_1 > 0$, $\eta_2, \eta_3 \in (0, 1)$, let $d_0 = -g_0$, $k := 0$.

Step 2: If $\|g_k\| \leq \epsilon$, then, stop; otherwise, proceed to the next step.

Step 3: Compute step size α_k by the line search (7) and (8).

Step 4: Let $\psi_k = x_k + \alpha_k d_k$.

Step 5: If $-\lambda_1 g_k^T d_k > \lambda \alpha_k \|d_k\|^2$ holds, let $x_{k+1} = \psi_k$, $y_k = g_{k+1} - g_k$, and go to Step 7; otherwise, go to Step 6.

Step 6: Let x_{k+1} be defined by (9), $y_k^* = g(\psi_k) - g(x_k)$, and $y_k = y_k^*$.

Step 7: If $\|g_{k+1}\| \leq \epsilon$, stop; otherwise, proceed to next step.

Step 8: Calculate the search direction d_k by (11).

Step 9: Let $k := k + 1$, and go to step 2.

ALGORITHM 1: The modified three-term type CD conjugate gradient algorithm.

Let $\xi = \xi_0 + (2/\eta_1)\xi = \xi_0 + (2/\eta_1)$; then (14) holds. The proof is complete.

Remark 1. The relation (14) shows that the optimization algorithm possesses the trust region feature.

The following theorem is obtained for the global convergence of Algorithm 1.

Theorem 1. Assume that α_k, d_k and x_k are generated by Algorithm 1 and Lemma 1 holds. Then,

$$\lim_{k \rightarrow +\infty} \|g_k\| = 0. \quad (20)$$

Proof. From the line search (8), we have the following:

$$g(x_k + \alpha_k d_k)^T d_k \geq \mu g_k^T d_k + \min \left\{ -\lambda_1 g_k^T d_k, \lambda \alpha_k \|d_k\|^2 \right\} \geq \mu g_k^T d_k. \quad (21)$$

According to (ii) of Assumption 1,

$$\alpha_k L \|d_k\|^2 \geq (g_{k+1} - g_k)^T d_k \geq -(1 - \mu) g_k^T d_k. \quad (22)$$

Then, we have the following:

$$\alpha_k \geq \frac{-(1 - \mu)}{L \|d_k\|^2} g_k^T d_k. \quad (23)$$

Using Lemma 1,

$$\alpha_k \geq \frac{-(1 - \mu)}{L \|d_k\|^2} g_k^T d_k \geq \frac{(1 - \mu) \eta \|g_k\|^2}{L \|d_k\|^2} \geq \frac{(1 - \mu) \eta}{L \xi^2}. \quad (24)$$

From Assumption 1, line search (7) and sufficient descent property (13),

$$\begin{aligned} f(x_k + \alpha_k d_k) &\leq f_k + \lambda \alpha_k g_k^T d_k + \alpha_k \min \left\{ -\lambda_1 g_k^T d_k, \lambda \frac{\alpha_k}{2} \|d_k\|^2 \right\} \\ &\leq f_k + (\lambda - \lambda_1) \alpha_k g_k^T d_k \\ &\leq f_k - \eta (\lambda - \lambda_1) \alpha_k \|g_k\|^2 \\ &\leq f_k - \frac{\eta^2 (\lambda - \lambda_1) (1 - \mu)}{L \xi^2} \|g_k\|^2. \end{aligned} \quad (25)$$

Summing these inequalities from $k = 0$ to $+\infty$,

$$\sum_{k=0}^{\infty} \|g_k\|^2 < f_0 - f_{+\infty}. \quad (26)$$

From Assumption 1, it is easy to know that $f(x)$ is bounded. Then, $\|g_k\| \rightarrow 0$. The proof is complete.

4. Numerical Results

This section will report the numerical experiments with some classical optimization problems, the nonlinear Muskingum model, and the application in image restoration problems. All the tests are coded in MATLAB R2014a, run on a PC with a 2.50 GHz CPU, and 4.00 GB of memory running the Windows 10 operating system.

4.1. Normal Unconstrained Optimization Problems. In this subsection, the numerical experiments would be done with some test problems from [20], and all test problems are listed in Table 1. We compare Algorithm 1 with the classical CD conjugated method (called Algorithm 2) and the classical PRP conjugated method (called Algorithm 3). The detailed experimental data are list in Table 2. Figures 1–3 show the performance of these three algorithms related to CPU, NI, and NFG. Some columns of Tables 1 and 2 and Figures 1–3 have the following meanings:

- (i) **No**: the serial number of the problem.
- (ii) **Dim**: the dimension of the variable x .
- (iii) **NI**: the iteration numbers.
- (iv) **NFG**: the sum of function value and gradient value.
- (v) **CPU**: the calculation time in seconds.
- (vi) **Dimension**: the dimensions are 3000, 9000, and 15000.
- (vii) **Initialization**: the parameters of the algorithms are chosen to be $\lambda = 0.1$, $\lambda_1 = 0.01$, $\mu = 0.9$, $\eta_1 = 0.01$, $\eta_2 = 0.1$, $\eta_3 = 0.1$, $e_1 = e_2 = 10^{-5}$, and $\epsilon = 10^{-6}$, and the initial search direction $d_0 = -g_0$.
- (viii) **Stop rules**: the following Himmeblau stop rule [21] is used: if $|f(x_k)| > e_1$, let $\text{stop1} = (|f(x_k) - f(x_{k+1})|/f(x_k))$; otherwise, let $\text{stop1} = |f(x_k) - f(x_{k+1})|$. For every problem, if the conditions

TABLE 1: Test problems.

No	Test problems
1	Extended freudenstein and roth function
2	Extended trigonometric function
3	Extended rosenbrock function
4	Extended white and holst function
5	Extended beale function
6	Extended penalty function
7	Perturbed quadratic function
8	Raydan 1 function
9	Raydan 2 function
10	Diagonal 1 function
11	Diagonal 2 function
12	Diagonal 3 function
13	Hager function
14	Generalized tridiagonal 1 function
15	Extended tridiagonal 1 function
16	Extended three exponential terms function
17	Generalized tridiagonal 2 function
18	Diagonal 4 function
19	Diagonal 5 function
20	Extended himmelblau function
21	Generalized PSC1 function
22	Extended PSC1 function
23	Extended powell function
24	Extended block diagonal BD1 function
25	Extended maratos function
26	Extended cliff function
27	Quadratic diagonal perturbed function
28	Extended wood function
29	Extended hiebert function
30	Quadratic function QF1 function
31	Extended quadratic penalty QP1 function
32	Extended quadratic penalty QP2 function
33	A quadratic function QF2 function
34	Extended EP1 function
35	Extended Tridiagonal-2 function
36	BDQRTIC function (CUTE)
37	TRIDIA function (CUTE)
38	ARWHEAD function (CUTE)
39	ARWHEAD function (CUTE)
40	NONDQUAR function (CUTE)
41	DQDRTIC function (CUTE)
42	EG2 function (CUTE)
43	DIXMAANA function (CUTE)
44	DIXMAANB function (CUTE)
45	DIXMAANC function (CUTE)
46	DIXMAANE function (CUTE)
47	Partial perturbed quadratic function
48	Broyden tridiagonal function
49	Almost perturbed quadratic function
50	Tridiagonal perturbed quadratic function
51	EDENSCH function (CUTE)
52	VARDIM function (CUTE)
53	STAIRCASE S1 function
54	LIARWHD function (CUTE)
55	DIAGONAL 6 function
56	DIXON3DQ function (CUTE)
57	DIXMAANF function (CUTE)
58	DIXMAANG function (CUTE)
59	DIXMAANH function (CUTE)
60	DIXMAANI function (CUTE)

TABLE 1: Continued.

No	Test problems
61	DIXMAANJ function (CUTE)
62	DIXMAANK function (CUTE)
63	DIXMAANL function (CUTE)
64	DIXMAAND function (CUTE)
65	ENGVAL1 function (CUTE)
66	FLETCHCR function (CUTE)
67	COSINE function (CUTE)
68	Extended DENSCHNB function (CUTE)
69	DENSCHNF function (CUTE)
70	SINQUAD function (CUTE)
71	BIGGSB1 function (CUTE)
72	Partial perturbed quadratic PPQ2 function
73	Scaled quadratic SQ1 function

$\|g(x_k)\| < \varepsilon$ or $\text{stop1} < e_2$ are satisfied, then the program is stopped. This program is also stopped when the number of iterations is greater than one thousand.

From the detailed experimental data of Table 2, it is obvious to see that most of the problems can be solved quickly. For most of the problems, it takes less CPU time to solve those problems with the proposed algorithm. Meanwhile, progress has also been made in NI and NFG. Generally, the algorithm of the proposed method is promising versus the other algorithms. About the numerical results, the algorithm of Dolan and More [22] will be used to more directly show the performance profiles of these algorithms. In Figure 1, the curve of Algorithm 1 is always above the other algorithms. In Figure 2, the curves have the same trend. Algorithm 1 almost solves about 97% of the test problems in $\tau = 30$, while Algorithm 2 just solves 89.9% and Algorithm 3 just solves 87% in $\tau = 30$. Figure 3 shows a similar trend as Figure 2. All the above pictures show that the modified CD conjugate gradient algorithm is more robust and effective compared with the normal CD method and PRP method. In summary, Algorithm 1 is more competitive versus others.

4.2. The Muskingum Model. It is generally known that parameter estimation is a significant task in engineering applications. The nonlinear Muskingum model will be discussed as a common example of such an application in this subsection.

The Muskingum Model [23] is defined by the following:

$$\min f(x_1, x_2, x_3) = \sum_{i=1}^{n-1} \left(\left(1 - \frac{\Delta t}{6} \right) x_1 (x_2 I_{i+1} + (1 - x_2) Q_{i+1})^{x_3} - \left(1 - \frac{\Delta t}{6} \right) x_1 (x_2 I_i + (1 - x_2) Q_i)^{x_3} - \frac{\Delta t}{2} (I_i - Q_i) + \frac{\Delta t}{2} \left(1 - \frac{\Delta t}{3} \right) (I_{i+1} - Q_{i+1}) \right)^2. \quad (27)$$

TABLE 2: The numerical results of Algorithms 1–3.

No.	Dim	Algorithm 1			Algorithm 2			Algorithm 3		
		NI	NFG	CPU time	NI	NFG	CPU time	NI	NFG	CPU time
1	3000	7	41	0.484375	20	73	0.078125	16	65	0.109375
1	9000	13	89	0.328125	20	70	0.265625	16	62	0.28125
1	15000	7	41	0.203125	42	144	0.8125	20	85	0.5625
2	3000	169	1138	1.46875	83	260	0.328125	89	361	0.484375
2	9000	108	396	1.34375	91	286	0.875	89	298	0.890625
2	15000	125	590	2.765625	93	294	1.328125	95	306	1.453125
3	3000	9	49	0.03125	109	415	0.203125	32	106	0.0625
3	9000	5	26	0.078125	91	348	0.265625	6	29	0.015625
3	15000	5	26	0.015625	96	371	0.4375	9	50	0.078125
4	3000	43	200	0.203125	167	572	0.515625	29	104	0.109375
4	9000	21	91	0.203125	51	202	0.515625	29	109	0.21875
4	15000	21	91	0.359375	45	176	0.625	43	164	0.625
5	3000	2	14	0.015625	37	130	0.109375	88	367	0.6875
5	9000	41	128	0.296875	36	125	0.28125	29	109	0.71875
5	15000	3	26	0.046875	38	131	0.421875	53	171	0.734375
6	3000	92	327	0.171875	113	358	0.15625	124	391	0.3125
6	9000	104	340	0.390625	133	420	0.40625	144	453	0.5625
6	15000	116	370	0.65625	151	474	0.640625	145	466	0.765625
7	3000	423	4746	2.40625	968	3023	1.203125	1000	3041	1.265625
7	9000	102	373	0.359375	1000	3075	2.453125	1000	3062	3.09375
7	15000	272	987	1.578125	214	718	0.921875	1000	3046	4.296875
8	3000	18	65	0.046875	23	74	0.03125	25	79	0.046875
8	9000	17	62	0.078125	23	74	0.09375	25	79	0.109375
8	15000	17	62	0.15625	23	74	0.125	25	79	0.171875
9	3000	19	65	0.03125	9	28	0.015625	3	25	0.03125
9	9000	3	26	0.046875	9	28	0.046875	23	100	0.09375
9	15000	3	26	0.046875	9	28	0.03125	24	110	0.234375
10	3000	2	14	0.015625	2	14	0.015625	2	14	0
10	9000	2	14	0	2	14	0	2	14	0
10	15000	2	14	0.03125	2	14	0.015625	2	14	0.03125
11	3000	3	19	0.03125	75	263	0.140625	4	21	0
11	9000	3	19	0.03125	175	593	0.828125	6	36	0.0625
11	15000	3	19	0.140625	22	88	0.140625	7	37	0.078125
12	3000	17	57	0.03125	15	46	0.0625	17	55	0.046875
12	9000	16	54	0.125	16	49	0.09375	17	56	0.1875
12	15000	16	54	0.125	16	49	0.109375	17	56	0.15625
13	3000	2	14	0	23	138	0.078125	2	14	0.03125
13	9000	2	14	0.03125	3	18	0.046875	2	14	0
13	15000	2	14	0.015625	2	14	0.015625	2	14	0.0625
14	3000	18	85	0.5625	8	27	0.15625	37	169	1.0625
14	9000	7	39	0.796875	4	14	0.21875	45	274	4.890625
14	15000	7	41	1.296875	4	14	0.359375	50	343	9.75
15	3000	17	59	0.484375	52	177	1.046875	33	175	0.796875
15	9000	25	83	1.328125	50	169	2.65625	33	175	2.359375
15	15000	29	95	2.5625	41	138	3.578125	33	175	3.796875
16	3000	3	16	0	13	40	0.03125	40	151	0.109375
16	9000	8	23	0.046875	17	52	0.0625	24	73	0.09375
16	15000	17	59	0.15625	15	46	0.09375	27	82	0.171875
17	3000	2	14	0.015625	40	122	0.1875	53	164	0.234375
17	9000	36	148	0.578125	35	107	0.375	47	149	0.546875
17	15000	5	42	0.15625	27	83	0.546875	49	155	1.078125
18	3000	36	163	0.078125	3	12	0	384	4591	1.875
18	9000	40	229	0.15625	3	12	0.046875	391	4675	3.65625
18	15000	51	325	0.359375	3	12	0	394	4711	5.421875
19	3000	4	30	0.015625	3	11	0.015625	2	14	0
19	9000	4	39	0.078125	3	11	0	2	14	0.015625
19	15000	4	39	0.09375	3	11	0.046875	2	14	0.015625
20	3000	38	134	0.09375	22	70	0.03125	3	18	0
20	9000	40	142	0.15625	30	94	0.0625	3	18	0.015625

TABLE 2: Continued.

No.	Dim	Algorithm 1			Algorithm 2			Algorithm 3		
		NI	NFG	CPU time	NI	NFG	CPU time	NI	NFG	CPU time
20	15000	40	137	0.1875	38	118	0.125	3	18	0.046875
21	3000	21	64	0.046875	38	121	0.09375	27	103	0.125
21	9000	21	64	0.265625	38	121	0.21875	27	103	0.1875
21	15000	21	64	0.1875	38	121	0.296875	26	92	0.234375
22	3000	2	14	0.015625	6	35	0.0625	6	28	0.09375
22	9000	2	14	0.046875	6	35	0.15625	6	28	0.171875
22	15000	2	14	0.09375	6	35	0.328125	6	28	0.515625
23	3000	23	109	0.078125	180	610	0.546875	20	71	0.09375
23	9000	23	109	0.28125	190	657	1.5	1000	3003	7.625
23	15000	23	109	0.359375	206	707	2.734375	1000	3003	11.65625
24	3000	9	44	0.015625	50	191	0.109375	6	68	0.015625
24	9000	9	44	0.125	25	107	0.125	6	69	0.125
24	15000	9	44	0.125	31	129	0.25	6	69	0.15625
25	3000	6	27	0	18	53	0.046875	33	98	0.046875
25	9000	9	35	0.078125	28	83	0.046875	51	165	0.171875
25	15000	6	26	0.0625	34	101	0.09375	15	69	0.078125
26	3000	86	273	0.28125	96	305	0.171875	95	304	0.15625
26	9000	103	351	0.59375	107	338	0.453125	94	301	0.40625
26	15000	96	402	1.03125	98	313	0.671875	95	304	0.640625
27	3000	244	2809	1.625	22	94	0.03125	29	119	0.046875
27	9000	195	2290	2.40625	64	256	0.203125	1000	3004	3.25
27	15000	946	3784	5.984375	50	211	0.296875	21	80	0.078125
28	3000	9	46	0.03125	44	151	0.09375	36	126	0.078125
28	9000	9	46	0.0625	50	178	0.15625	36	123	0.09375
28	15000	9	46	0.078125	37	124	0.171875	34	125	0.203125
29	3000	3	18	0	4	25	0	4	24	0.046875
29	9000	3	18	0	4	24	0	4	24	0.046875
29	15000	3	18	0.03125	4	32	0.03125	4	24	0.03125
30	3000	135	1388	0.609375	1000	3063	0.921875	1000	3013	1.125
30	9000	591	4199	4.0625	1000	3066	1.9375	1000	3013	2.328125
30	15000	107	394	0.421875	1000	3083	2.609375	1000	3013	3.359375
31	3000	32	108	0.078125	40	125	0.0625	44	139	0.0625
31	9000	31	98	0.09375	43	136	0.109375	45	142	0.125
31	15000	33	104	0.1875	44	139	0.21875	48	151	0.25
32	3000	31	94	0.078125	33	102	0.046875	50	153	0.078125
32	9000	27	84	0.140625	64	195	0.328125	41	128	0.234375
32	15000	29	90	0.265625	45	140	0.3125	46	143	0.34375
33	3000	4	13	0	4	12	0	3	9	0
33	9000	2	6	0.046875	2	6	0	2	6	0
33	15000	2	6	0	2	6	0	2	6	0.015625
34	3000	4	12	0.03125	4	11	0	3	16	0
34	9000	4	12	0.015625	5	14	0.015625	7	20	0.03125
34	15000	4	12	0.015625	6	17	0.03125	9	26	0.09375
35	3000	3	17	0.015625	8	23	0.03125	14	41	0.03125
35	9000	3	17	0.03125	12	35	0.03125	21	62	0.09375
35	15000	5	14	0.03125	15	44	0.078125	25	74	0.140625
36	3000	88	344	1.75	20	81	0.359375	7	38	0.203125
36	9000	35	149	2.125	17	69	0.875	7	38	0.453125
36	15000	94	394	9	33	123	2.796875	7	38	0.84375
37	3000	380	1413	0.734375	1000	3111	1.328125	23	85	0.046875
37	9000	321	1183	1.375	344	1155	1.109375	1000	3013	3.671875
37	15000	1000	3686	6.40625	1000	3078	4.375	1000	3005	4.90625
38	3000	94	374	0.1875	9	30	0.015625	5	19	0.03125
38	9000	215	858	1	22	74	0.046875	4	14	0.03125
38	15000	294	1173	2.125	17	57	0.078125	4	14	0.046875
39	3000	6	18	0	18	53	0.015625	38	113	0.046875
39	9000	6	18	0.03125	19	56	0.03125	41	122	0.109375
39	15000	6	18	0.0625	20	59	0.09375	43	128	0.1875
40	3000	59	234	0.40625	1000	3116	6.046875	3	19	0.046875

TABLE 2: Continued.

No.	Dim	Algorithm 1			Algorithm 2			Algorithm 3		
		NI	NFG	CPU time	NI	NFG	CPU time	NI	NFG	CPU time
40	9000	3	18	0.0625	1000	3109	16.03125	3	17	0.046875
40	15000	3	17	0.078125	1000	3102	25.765625	3	17	0.078125
41	3000	16	96	0.0625	28	98	0.03125	16	60	0.046875
41	9000	23	139	0.125	23	78	0.0625	16	60	0.0625
41	15000	34	242	0.34375	19	66	0.078125	16	60	0.078125
42	3000	6	47	0.015625	4	28	0	6	42	0.03125
42	9000	6	47	0.078125	4	28	0.046875	6	42	0.03125
42	15000	7	60	0.125	4	28	0.0625	6	42	0.0625
43	3000	21	188	0.296875	19	60	0.21875	58	186	0.421875
43	9000	2	14	0.0625	20	63	0.359375	68	216	1.234375
43	15000	24	81	0.765625	20	63	0.59375	72	228	2.3125
44	3000	2	14	0.015625	27	84	0.203125	66	216	0.4375
44	9000	2	14	0.046875	28	87	0.5	75	241	1.328125
44	15000	2	14	0.078125	28	87	0.78125	77	247	2.3125
45	3000	13	40	0.09375	43	132	0.28125	75	228	0.515625
45	9000	13	40	0.328125	45	138	0.8125	79	240	1.484375
45	15000	13	40	0.375	46	141	1.34375	81	246	2.28125
46	3000	86	663	0.96875	60	194	0.40625	24	149	0.28125
46	9000	2	14	0.046875	153	510	2.90625	178	1091	5.453125
46	15000	2	14	0.078125	154	519	4.703125	200	1355	10.203125
47	3000	76	230	13.859375	76	229	13.515625	20	71	3.71875
47	9000	55	188	92.234375	34	111	53.96875	11	45	17.125
47	15000	82	304	397.6875	86	276	382.3125	92	306	416.421875
48	3000	12	52	0.09375	30	97	0.140625	23	82	0.109375
48	9000	119	665	2.71875	35	115	0.4375	37	133	0.5
48	15000	177	1485	8.859375	33	109	0.6875	126	383	2.546875
49	3000	516	5422	2.796875	1000	3093	0.890625	1000	3013	1.09375
49	9000	153	560	0.5	1000	3171	1.9375	1000	3013	2.390625
49	15000	487	1800	2.109375	1000	3072	2.890625	1000	3013	3.453125
50	3000	194	1609	8.40625	1000	3025	14.765625	1000	3013	15.203125
50	9000	320	3100	43	169	560	8.484375	1000	3013	42.484375
50	15000	241	888	20.484375	1000	3107	71.375	1000	3013	70.25
51	3000	49	218	0.28125	16	49	0.078125	28	85	0.125
51	9000	2	14	0.03125	15	46	0.171875	28	85	0.28125
51	15000	2	14	0.078125	15	46	0.234375	28	85	0.46875
52	3000	149	474	0.421875	173	548	0.46875	182	575	0.453125
52	9000	170	541	1.140625	192	607	1.15625	206	649	1.28125
52	15000	178	565	2.046875	201	636	2.03125	215	678	1.984375
53	3000	3	26	0.03125	441	1487	0.453125	3	26	0.046875
53	9000	3	27	0.015625	430	1444	1.03125	56	193	0.140625
53	15000	2	14	0.03125	1000	3078	3.46875	2	14	0.015625
54	3000	381	1403	0.765625	21	93	0.03125	3	18	0.046875
54	9000	48	185	0.203125	98	354	0.3125	3	18	0.03125
54	15000	1000	3978	7	65	238	0.390625	3	18	0.015625
55	3000	19	65	0.109375	14	43	0.046875	3	25	0.0625
55	9000	3	26	0.09375	15	46	0.171875	23	100	0.390625
55	15000	3	26	0.140625	15	46	0.34375	23	104	0.75
56	3000	3	16	0	1000	3125	0.984375	1000	2999	1.0625
56	9000	3	16	0.046875	1000	3151	2.078125	1000	2999	2.1875
56	15000	3	16	0.015625	1000	3125	3.359375	1000	2999	3.71875
57	3000	2	14	0.015625	71	226	0.5	137	1398	2.203125
57	9000	2	14	0.046875	67	211	1.234375	102	602	3.046875
57	15000	2	14	0.078125	69	221	1.984375	210	1546	10.84375
58	3000	13	40	0.09375	109	358	0.796875	584	1985	3.921875
58	9000	13	40	0.21875	119	398	2.234375	22	88	0.453125
58	15000	13	40	0.390625	149	490	4.65625	22	88	0.6875
59	3000	3	18	0.015625	48	170	0.34375	189	2038	3.140625
59	9000	3	18	0.09375	46	163	0.875	239	2683	10.328125
59	15000	3	18	0.09375	54	187	1.71875	256	2912	17.8125

TABLE 2: Continued.

No.	Dim	Algorithm 1			Algorithm 2			Algorithm 3		
		NI	NFG	CPU time	NI	NFG	CPU time	NI	NFG	CPU time
60	3000	44	377	1	110	357	0.734375	80	466	0.859375
60	9000	2	14	0.046875	184	619	3.515625	97	743	3.453125
60	15000	2	14	0.078125	184	624	5.859375	104	1000	6.4375
61	3000	2	14	0.015625	73	236	0.515625	175	1891	2.875
61	9000	2	14	0.046875	78	255	1.453125	188	1193	5.796875
61	15000	2	14	0.109375	69	229	2.03125	114	649	4.96875
62	3000	3	18	0.015625	86	280	0.625	27	103	0.1875
62	9000	3	18	0.078125	71	240	1.28125	126	702	3.609375
62	15000	3	18	0.078125	47	160	1.390625	75	415	3.25
63	3000	15	46	0.15625	139	455	1	157	627	1.1875
63	9000	15	46	0.265625	252	818	4.734375	72	309	1.515625
63	15000	15	46	0.453125	221	742	6.953125	87	363	3.046875
64	3000	18	55	0.140625	73	222	0.4375	24	86	0.1875
64	9000	18	55	0.34375	77	234	1.328125	24	86	0.484375
64	15000	18	55	0.578125	79	240	2.328125	24	86	0.859375
65	3000	3	19	0.109375	31	94	0.515625	33	102	0.578125
65	9000	3	19	0.515625	30	91	1.4375	31	96	1.5
65	15000	3	19	0.6875	30	91	2.484375	31	96	2.484375
66	3000	6	30	0.203125	4	21	0.078125	4	19	0.078125
66	9000	6	30	0.5	4	21	0.234375	4	19	0.265625
66	15000	120	479	11.015625	4	21	0.453125	4	19	0.375
67	3000	10	64	0.09375	7	69	0.03125	11	72	0.046875
67	9000	7	42	0.046875	38	258	0.796875	11	69	0.15625
67	15000	22	136	0.890625	24	170	0.71875	11	67	0.234375
68	3000	24	80	0.078125	22	67	0.03125	44	491	0.328125
68	9000	2	14	0	23	70	0.046875	2	14	0.03125
68	15000	2	14	0.0625	23	70	0.078125	2	14	0.03125
69	3000	5	33	0	53	163	0.109375	28	88	0.0625
69	9000	5	33	0.046875	51	157	0.21875	20	71	0.140625
69	15000	5	33	0.046875	51	157	0.359375	20	71	0.171875
70	3000	66	277	1.125	43	167	0.671875	13	70	0.203125
70	9000	10	45	0.5	21	90	1.03125	15	77	0.71875
70	15000	68	245	4.96875	9	34	0.59375	25	98	2.03125
71	3000	3	16	0	42	133	0.046875	3	16	0.046875
71	9000	3	16	0.015625	42	133	0.0625	3	16	0
71	15000	3	16	0.015625	42	133	0.125	3	16	0.046875
72	3000	206	765	41.4375	163	583	31.453125	15	60	2.78125
72	9000	1000	3694	1734.984375	1000	3036	1571.609375	15	60	24.125
72	15000	1000	3611	4739.8125	163	568	751.625	15	60	66.703125
73	3000	562	6118	2.671875	320	1055	0.28125	1000	3017	0.984375
73	9000	1000	9069	8.703125	1000	3048	1.90625	1000	3017	2.375
73	15000	798	5688	6.53125	1000	3041	2.71875	1000	3017	3.078125

Some of the variables have the following meanings:

- (i) n : the total time number.
- (ii) x_1 : the storage time constant.
- (iii) I_i : the observed inflow discharge.
- (iv) x_2 : the weighting factor.
- (v) Q_i : the observed outflow discharges.
- (vi) x_3 : an additional parameter.
- (vii) Δt : the time step at time t_i ($i = 1, 2, \dots, n$).

In the experiment, the observed data of the flood run-off process from Chenggouwan and Linqing of Nanyunhe River in the Haihe Basin, Tianjin, China, are used. We choose the initial point $x = [0, 1, 1]^T$; detailed data about I_i and Q_i for

the years 1960, 1961, and 1964 were obtained (see [24] in detail). $\Delta t = 12(h)$ was selected. The results of these three algorithms are listed in Table 3. The performance of the presented algorithm is shown in Figures 4–6.

Some conclusions are obtained from this experiment: (1) from Figures 4–6, we conclude that we can calculate approximations of flood outflows by using Algorithm 1, and Algorithm 1 is effective for the nonlinear Muskingum model; (2) the final points (x_1 , x_2 and x_3) of these three algorithms are interesting, and they are competitive with the final points of similar algorithms; and (3) the final points of Algorithm 1 are different from those of the BFGS method and HIWO method, which implies that the nonlinear Muskingum model has different optimum approximation solutions.

FIGURE 1: Performance profiles of these methods (CPU).

FIGURE 2: Performance profiles of these methods (NI).

FIGURE 3: Performance profiles of these methods (NGF).

TABLE 3: Results of the three algorithms.

Algorithms	x_1	x_2	x_3
BFGS [25]	10.8156	0.9826	1.0219
HIWO [23]	13.2813	0.8001	0.9933
Algorithm 1	11.1848	1.0000	0.9995

FIGURE 4: Performance of date in 1960.

FIGURE 5: Performance of date in 1961.

FIGURE 6: Performance of date in 1961.

FIGURE 7: Restoration of the Banoon image, Barbara image, and Lena image by Algorithms 1-3. From left to right: a noisy image with 30% salt-and-pepper noise, the restorations obtained with Algorithms 1-3.

FIGURE 8: Restoration of the Banoon image, Barbara image, and Lena image by Algorithms 1-3. From left to right: a noisy image with 50% salt-and-pepper noise, the restorations obtained with the Algorithms 1-3.

FIGURE 9: Restoration of the Banoon image, Barbara image, and Lena image by Algorithms 1-3. From left to right: a noisy image with 70% salt-and-pepper noise, the restorations obtained with the Algorithms 1-3.

TABLE 4: CPU time of those algorithms in seconds.

30% noise	<i>Lena</i>	<i>Barbara</i>	<i>Banoon</i>	Total
Algorithm 1	4.438	4.390	4.656	13.484
Algorithm 2	4.563	5.234	5.406	15.203
Algorithm 3	5.156	6.313	6.781	18.250
50% noise	<i>Lena</i>	<i>Barbara</i>	<i>Banoon</i>	Total
Algorithm 1	8.609	7.891	8.141	24.641
Algorithm 2	9.219	10.688	10.547	30.454
Algorithm 3	11.297	14.391	14.859	40.547
70% noise	<i>Lena</i>	<i>Barbara</i>	<i>Banoon</i>	Total
Algorithm 1	15.750	10.578	11.984	38.312
Algorithm 2	13.891	14.438	14.703	43.032
Algorithm 3	14.859	19.922	20.375	55.156

4.3. Image Restoration Problems. In this subsection, the above algorithms will be applied to image restoration problems. The original images corrupted by impulse noise are treated as objects here. These problems are regarded as one of the most difficult problems in optimization fields. Related parameter settings are similar to the above subsections, and the program will be stopped when the condition $(\|f_{k+1}\| - \|f_k\|/\|f_k\|) < 10^{-3}$ or $(\|x_{k+1}\| - \|x_k\|/\|x_k\|) < 10^{-3}$ holds. The following three images are selected as processing objects: Banoon (512×512), Barbara (512×512), and Lena (512×512). The detail performances are shown in Figures 7-9. The CPU time taken to process images is listed in Table 4.

It is easy to see that all the algorithms are successful for image restoration problems. The results in Table 4 reveal that

the CPU time of Algorithm 1 is less than that of other algorithms, whether for 30% noise problems, 50% noise problems, or for 70% noise problems.

5. Conclusion

In this paper, a modified three-term type CD conjugate gradient algorithm is presented. Some good features are also presented: (i) sufficient descent property holds, (ii) trust region feature also holds, (iii) the algorithm has global convergence with the MWWP line search technique and projection technique for general function, and (iv) numerical results reveal that the new algorithm is more competitive than the normal CD algorithm and PRP algorithm.

In recent years, there have been considerable researches about other types of CG method, while the study of CD method is not enough, and it should not be ignored. We have many works to do in the future: whether this method is suitable for other line search technique (such as Armijo line search, nonmonotone line search), or whether there exist other better modification methods to improve the numerical results of the CD method. All these are worth studying in the next work.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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