Research Article

The Centrosymmetric Matrices of Constrained Inverse Eigenproblem and Optimal Approximation Problem

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In this paper, a kind of constrained inverse eigenproblem and optimal approximation problem for centrosymmetric matrices are considered. Necessary and sufficient conditions of the solvability for the constrained inverse eigenproblem of centrosymmetric matrices in real number field are derived. A general representation of the solution is presented for a solvable case. The explicit expression of the optimal approximation problem is provided. Finally, a numerical example is given to illustrate the effectiveness of the method.

1. Introduction

Inverse eigenproblems arise in a remarkable variety of applications, including control theory [1, 2], vibration theory [3, 4], structural design [5], molecular spectroscopy [6], in developing numerical methods, and the ordinary and partial differential equation solving [7, 8]. Centrosymmetric matrices are applied in information theory, linear system theory, and numerical analysis theory [9]. The unconstrained centrosymmetric matrices' problems have been discussed [9–14], a class of unconstrained matrices' inverse eigenproblems has been obtained [15–18], and the constrained inverse eigenproblems have been discussed [19–22], but only when the eigenvalues are real or imaginary numbers. For general real matrices, the eigenvalues are not necessarily real or imaginary numbers, so when the eigenvalue is complex, it is difficult to find the constraint solution. In this paper, we will use the real Schur decomposition theorem and the similar decomposition theorem and introduce a new norm to get the corresponding expression of the best approximation solution.

Throughout the paper, we denote the set of real $n \times m$ matrices, real $n \times n$ orthogonal matrices, $n \times n$ centrosymmetric matrices, and real numbers, respectively, by $R^{n \times m}$, $OR^{n \times n}$, $CSR^{n \times n}$, and R . A^T , A^+ , $\text{rank}(A)$, and $\|A\|_F$ denote the transpose, the Moore–Penrose generalized inverse, the rank, and the Frobenius norm of a matrix A , respectively. I is the identity matrix. $\lambda(A)$ denotes the set of eigenvalues of the

matrix A . $\lambda(A)/\lambda(B)$ denotes the set of the difference of $\lambda(A)$ and $\lambda(B)$. D_r is a closed disc with radius r and center origin, $[A]_{D_r}$ denotes the square matrix A with all of its eigenvalues located in the closed disc D_r , and $R_{D_r}^{n \times n}$ denotes the set of $n \times n$ matrices with their eigenvalues located in the disc D_r . The notation $A_{11} \oplus A_{22} \oplus \dots \oplus A_{kk}$ denotes the direct sum of the matrices $A_{11}, A_{22}, \dots, A_{kk}$, where $A_{jj} \in R^{n_j \times n_j}$. Let e_i be the i th standard unit vector, and the matrix $S_n = (e_n, e_{n-1}, \dots, e_1)$. $[x]$ represents the largest integer less than or equal to x . The following definition is given in [23].

Definition 1. $A = (a_{ij}) \in R^{n \times n}$ is called centrosymmetric matrix if $a_{ij} = a_{n+1-i, n+1-j}$, $i, j = 1, 2, \dots, n$.

Clearly, each eigenvalue of matrix $A \in CSR^{n \times n}$ is either a real number or a complex number, if A has complex eigenvalues, and they must occur in complex conjugate pairs, note that $\lambda_j = \tilde{a}_j + i\tilde{b}_j$, $\bar{\lambda}_j = \tilde{a}_j - i\tilde{b}_j$ ($\tilde{b}_j \neq 0$, $j = 1, 2, \dots, k$) are eigenvalues of A , $x_j = \xi_j + \eta_j i$ and $\bar{x}_j = \xi_j - \eta_j i$ ($j = 1, 2, \dots, k$) are eigenvectors associated with λ_j and $\bar{\lambda}_j$, respectively; also, we have

$$A[\xi_j \eta_j] = [\xi_j \eta_j] \begin{bmatrix} \tilde{a}_j & \tilde{b}_j \\ -\tilde{b}_j & \tilde{a}_j \end{bmatrix}, \quad j = 1, 2, \dots, k, \quad (1)$$

where let

$$\begin{aligned} X_j &= [\xi_j, \eta_j], \\ \Lambda_j &= \begin{bmatrix} \tilde{a}_j & \tilde{b}_j \\ -\tilde{b}_j & \tilde{a}_j \end{bmatrix}. \end{aligned} \quad (2)$$

If A has real eigenvalues, then $\lambda_j \in R (j = k+1, \dots, l)$ are eigenvalues of A and $x_j \in R^n (j = k+1, \dots, l)$ are eigenvectors associated with λ_j ; also, we have

$$Ax_j = \lambda_j x_j, \quad j = k+1, \dots, l, \quad (3)$$

where let

$$\begin{aligned} X_j &= x_j, \\ \Lambda_j &= \lambda_j, \end{aligned} \quad (4)$$

Then, $2k + (l - k) = n$.

Remark 1. Here, r -multiple eigenvalues are counted r -times and their eigenvectors may be linearly dependent.

Definition 2. Let $A \in R^{n \times n}$, given $Y \in R^{n \times n}$, and $\text{rank}(Y) = n$. Then, we define a new norm called the Y norm of matrix A as

$$\|A\|_Y = \|Y^{-1}AY\|_F. \quad (5)$$

In contrast to the definition of matrix norm (see [24], Definition 5.1.1), it is easy to show that $\|\cdot\|_Y$ is a kind of matrix norm.

Now, we can present the optimal approximation of constrained inverse eigenproblem of centrosymmetric matrices as follows:

Constrained Inverse Eigenproblem. Given a real number $r > 0$. Let $X = (X_1, X_2, \dots, X_l) \in R^{n \times m}$, $\Lambda = \text{diag}(\Lambda_1, \Lambda_2, \dots, \Lambda_l) \in R^{m \times m}$, $l \leq m \leq n$, where X_j and Λ_j satisfy equation (2) or equation (4). $D_r = \{Z \mid |Z| \leq r, Z \in C\}$ is a given closed disc. Find matrix A such that the set

$$S = \{A \in \text{CSR}^{n \times n} \mid AX = X\Lambda\}. \quad (6)$$

is nonempty, and find the subset $S_{D_r} \subset S$ such that the remaining eigenvalues of any matrix in S are located in the disc D_r .

Optimal Approximation Problem. Given $A^* \in R^{n \times n}$, $\|\cdot\|_Y$ is the norm which has been defined as equation (5), Y is an invertible matrix concerning with A^* and $A \in S_{D_r}$; find a matrix $\hat{A} \in S_{D_r}$, such that

$$\|A^* - \hat{A}\|_Y = \min_{A \in S_{D_r}} \|A^* - A\|_Y, \quad (7)$$

where S_{D_r} is the solution set of the Constrained Inverse Eigenproblem.

This paper is organized as follows. In Section 2, we provide the solvability conditions of the Constrained Inverse Eigenproblem and its general solution in that case. In Section 3, we get the expression of the solution for Optimal Approximation Problem. In Section 4, we give an algorithm of Constrained Inverse Eigenproblem and Optimal

Approximation Problem and give a numerical example of Optimal Approximation Problem.

2. The Solvability Conditions and General Solution of Constrained Inverse Eigenproblem

Firstly, let $k = \lfloor n/2 \rfloor$ and characterize the set of all centrosymmetric matrices as follows.

When $n = 2k$, let

$$D = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ S_k & -S_k \end{bmatrix}. \quad (8)$$

When $n = 2k + 1$, let

$$D = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{bmatrix}. \quad (9)$$

Clearly, D is an orthogonal matrix for all of the n .

Secondly, given $X \in R^{n \times m}$, denote

$$D^T X = \begin{bmatrix} X_{D_1} \\ X_{D_2} \end{bmatrix}, \quad X_{D_1} \in R^{(n-k) \times m}, \quad X_{D_2} \in R^{k \times m}. \quad (10)$$

Decomposing the matrices X , X_{D_1} , and X_{D_2} by the SVD, we have

$$X = \tilde{U} \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \tilde{V}^T = \tilde{U}_1 \Delta \tilde{V}_1^T, \quad (11)$$

where $\tilde{U} = [\tilde{U}_1, \tilde{U}_2] \in \text{OR}^{n \times n}$, $\tilde{V} = [\tilde{V}_1, \tilde{V}_2] \in \text{OR}^{m \times m}$, $r_0 = \text{rank}(X)$, and

$$\Delta = \text{diag}(\delta_1, \delta_2, \dots, \delta_{r_0}), \quad \delta_i > 0, \quad 1 \leq i \leq r_0, \quad (12)$$

$$X_{D_1} = U \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} V^T = U_1 \Gamma V_1^T, \quad (13)$$

where $U = [U_1, U_2] \in \text{OR}^{(n-k) \times (n-k)}$, $V = [V_1, V_2] \in \text{OR}^{m \times m}$, $r_1 = \text{rank}(X_{D_1})$, and $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_{r_1})$, ($\gamma_i > 0, 1 \leq i \leq r_1$).

$$X_{D_2} = P \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} Q^T = P_1 \Sigma Q_1^T, \quad (14)$$

where $P = [P_1, P_2] \in \text{OR}^{k \times k}$, $Q = [Q_1, Q_2] \in \text{OR}^{m \times m}$, $r_2 = \text{rank}(X_{D_2})$, and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_{r_2})$, ($\sigma_i > 0, 1 \leq i \leq r_2$).

Lemma 1 (see [9], Lemma 2). $A \in \text{CSR}^{n \times n}$ if and only if A can be expressed as

$$A = D \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} D^T, \quad B \in R^{(n-k) \times (n-k)}, \quad C \in R^{k \times k}, \quad (15)$$

where, if $n = 2k$, then D is the form of equation (8), and if $n = 2k + 1$, then D is the form of (9).

Lemma 2 (see [15], Theorems 7.2 and 7.3). Given $X \in R^{n \times m}$, $\Lambda \in R^{m \times m}$, and X decomposed as equation (11), let

$$\tilde{U}^T A \tilde{U} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad (16)$$

where $A_{ij} = \tilde{U}_i^T A \tilde{U}_j$ ($i, j = 1, 2$). Equation $AX = X\Lambda$ is consistent if and only if

$$X\Lambda\tilde{V}_2 = 0. \quad (17)$$

If the condition is satisfied, the general solution can be expressed as

$$A = X\Lambda X^+ + G\tilde{U}_2^T, \quad (18)$$

and $G = \tilde{U} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} \in R^{n \times (n-r_0)}$ is arbitrary.

Lemma 3. Given $X \in R^{n \times m}$ and $\Lambda \in R^{m \times m}$. Equation $AX = X\Lambda$ is consistent with $A \in CSR^{n \times n}$ if and only if

$$\begin{aligned} X_{D_1}\Lambda V_2 &= 0, \\ X_{D_2}\Lambda Q_2 &= 0, \end{aligned} \quad (19)$$

where X_{D_1}, X_{D_2}, V_2 , and Q_2 is the same as equations (10)–(14). Let

$$\begin{aligned} U^T B U &= \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \\ P^T C P &= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \end{aligned} \quad (20)$$

where $B_{ij} = U_i^T B U_j$ and $C_{ij} = P_i^T C P_j$ ($i, j = 1, 2$), and the set of the general solution can be expressed as

$$S = \left\{ A | A = D \begin{bmatrix} X_{D_1}\Lambda X_{D_1}^+ + G_1 U_2^T & 0 \\ 0 & X_{D_2}\Lambda X_{D_2}^+ + G_2 P_2^T \end{bmatrix} D^T \right\}, \quad (21)$$

where $G_1 = U \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} \in R^{(n-k) \times (n-k-r_1)}$ and $G_2 = P \begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix} \in R^{k \times (k-r_2)}$ are arbitrary.

Proof. If S is nonempty, by Lemma 1, we have

$$A = D \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} D^T, \quad B \in R^{(n-k) \times (n-k)}, C \in R^{k \times k}. \quad (22)$$

Using equation (10), equation $AX = X\Lambda$ is equivalent to

$$\begin{aligned} BX_{D_1} &= X_{D_1}\Lambda, \\ CX_{D_2} &= X_{D_2}\Lambda. \end{aligned} \quad (23)$$

It follows from Lemma 2, and we can obtain $AX = X\Lambda$ is solvable if and only if equation (19) holds, and the solution set can be expressed as equation (21) easily. $\square$

Remark 2. Equation (21) implies that $\lambda(A)/\lambda(\Lambda) \in S_{D_r}$ if and only if arbitrary matrices $\lambda(B_{22}) \in S_{D_r}$ and $\lambda(C_{22}) \in S_{D_r}$. From Lemmas 1–3, we have the following theorem.

Theorem 1. Constrained Inverse Eigenproblem is solvable if and only if

$$\begin{aligned} X_{D_1}\Lambda V_2 &= 0, \\ X_{D_2}\Lambda Q_2 &= 0. \end{aligned} \quad (24)$$

The general solution can be expressed as

$$A = D \begin{bmatrix} X_{D_1}\Lambda X_{D_1}^+ + U \begin{bmatrix} B_{12} \\ [B_{22}]_{D_r} \end{bmatrix} & 0 \\ 0 & X_{D_2}\Lambda X_{D_2}^+ + P \begin{bmatrix} C_{12} \\ [C_{22}]_{D_r} \end{bmatrix} \end{bmatrix} D^T, \quad (25)$$

where $B_{12} \in R^{r_1 \times (n-k-r_1)}$, $[B_{22}]_{D_r} \in R_{D_r}^{(n-k-r_1) \times (n-k-r_1)}$, and $[C_{22}]_{D_r} \in R_{D_r}^{(k-r_2) \times (k-r_2)}$ are arbitrary matrices.

3. The Solution of Optimal Approximation Problem

Lemma 5. Given a matrix

$$\begin{aligned} \tilde{B}_{jj} &= \begin{bmatrix} c_j & d_{1j} \\ -d_{2j} & c_j \end{bmatrix}, \\ d_{1j}d_{2j} &> 0, \\ |\lambda(\tilde{B}_{jj})|^2 &= c_j^2 + d_{1j}d_{2j}. \end{aligned} \quad (26)$$

By similarity invariant, we can derive an invertible matrix $Y_{\tilde{B}_{jj}}$ and $\|Y_{\tilde{B}_{jj}}\|_F = 1$ such that

$$\tilde{B}_{jj} = Y_{\tilde{B}_{jj}} B_{jj} Y_{\tilde{B}_{jj}}^{-1}, \quad (27)$$

where $B_{jj} = \begin{bmatrix} c_j & d_j \\ -d_j & c_j \end{bmatrix}$ and $|\lambda(B_{jj})|^2 = c_j^2 + d_j^2 = |\lambda(\tilde{B}_{jj})|^2$.

Lemma 6. Given $B_{jj} = \begin{bmatrix} c_j & d_j \\ -d_j & c_j \end{bmatrix}$ and $|\lambda(B_{jj})| = \sqrt{c_j^2 + d_j^2} = s_j > r > 0$, then, there exists a unique matrix:

$$A_{jj} = \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}, \quad \sqrt{a_j^2 + b_j^2} = r, \quad (28)$$

such that

$$\|A_{jj} - B_{jj}\|_F = \min, \quad (29)$$

$$A_{jj} = \frac{r}{|\lambda(B_{jj})|} B_{jj}. \quad (30)$$

Proof. From

$$\|A_{jj} - B_{jj}\|_F^2 = \min, \quad (31)$$

we may obtain

$$(a_j - c_j)^2 + (b_j - d_j)^2 = \min. \quad (32)$$

Let $(d_j/c_j) = \tan \theta_j$, $(b_j/a_j) = \tan \theta$, $0 \leq \theta, \theta_j \leq \pi$, $s_j > r$, then equation (32) is equivalent to

$$\begin{aligned} \min f(\theta, r) &= \min(r \cos \theta - s_j \cos \theta_j)^2 + (r \sin \theta - s_j \sin \theta_j)^2 \\ &= \min(r^2 + s_j^2 - 2rs_j \cos(\theta - \theta_j)). \end{aligned} \quad (33)$$

It is easy to see that $f(\theta, r) = \min$ if and only if $\theta = \theta_j$ and $r = s_j$.

Note $[B_{jj}]_{D_r} = A_{jj}$, from equation (33), we may show that equation (29) implies that

$$[B_{jj}]_{D_r} = A_{jj} = \frac{r}{s_j} B_{jj} = \frac{r}{|\lambda(B_{jj})|} B_{jj}. \quad (34)$$

There, we note $[B_{jj}]_{D_r} = A_{jj}$ just for the sake of writing.

To show Theorem 2, we first introduce some notations as follows:

$$D^T A^* D = \begin{bmatrix} A_{11}^* & A_{12}^* \\ A_{21}^* & A_{22}^* \end{bmatrix}, \quad (35)$$

where $A_{11}^* \in R^{(n-k) \times (n-k)}$, $A_{12}^* \in R^{(n-k) \times k}$, $A_{21}^* \in R^{k \times (n-k)}$, and $A_{22}^* \in R^{k \times k}$. $\square$

Lemma 7 (see [16], Theorem 7.6). *If $H \in R^{n \times n}$, then there exists an orthogonal matrix $\tilde{Q} \in OR^{n \times n}$, such that $\tilde{Q}^T H \tilde{Q}$ is upper quasi-triangular, that is, $\tilde{Q}^T H \tilde{Q}$ is block upper-triangular where each diagonal block is either a 1×1 matrix or a 2×2 matrix having complex conjugate eigenvalues.*

Lemma 8. *Given matrix $A^* \in R^{n \times n}$, $D^T A^* D$ is decomposed as equation (35), the orthogonal matrices U and P are given in equations (13) and (14), respectively; then, there exist orthogonal matrices Q_1^* and Q_2^* and inverse matrices Y_1 and Y_2 , such that*

$$U_2^T A_{11}^* U_2 = Q_1^{*T} Y_1 \begin{pmatrix} \bar{\Lambda}_{11} & * & \cdots & \cdots & \cdots & * \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \bar{\Lambda}_{ss} & \ddots & & \vdots \\ \vdots & & \ddots & \lambda_{s+1} & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & * \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda_{n-k-r_1-2s} \end{pmatrix} Y_1^{-1} Q_1^*, \quad (36)$$

where $\bar{\Lambda}_{jj} = \begin{pmatrix} l_j & f_j \\ -f_j & l_j \end{pmatrix}$, $j = 1, 2, \dots, s$.

$$P_2^T A_{22}^* P_2 = Q_2^{*T} Y_2 \begin{pmatrix} \bar{\Gamma}_{11} & * & \cdots & \cdots & \cdots & * \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \bar{\Gamma}_{tt} & \ddots & & \vdots \\ \vdots & & \ddots & \mu_{t+1} & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & * \\ 0 & \cdots & \cdots & \cdots & 0 & \mu_{k-r_2-2t} \end{pmatrix} Y_2^{-1} Q_2^*, \quad (37)$$

where $\bar{\Gamma}_{jj} = \begin{pmatrix} g_j & h_j \\ -h_j & g_j \end{pmatrix}$, $j = 1, 2, \dots, t$.

Proof. For Lemma 7, there exist orthogonal matrices $Q_1^* \in OR^{(n-k-r_1) \times (n-k-r_1)}$ and $Q_2^* \in OR^{(k-r_2) \times (k-r_2)}$ such that

$$\begin{aligned} U_2^T A_{11}^* U_2 &= Q_1^{*T} \begin{pmatrix} \Lambda_{11} & * & \cdots & \cdots & \cdots & * \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \Lambda_{ss} & \ddots & & \vdots \\ \vdots & & \ddots & \lambda_{s+1} & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & * \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda_{n-k-r_1-2s} \end{pmatrix} Q_1^*, \\ P_2^T A_{22}^* P_2 &= Q_2^{*T} \begin{pmatrix} \Gamma_{11} & * & \cdots & \cdots & \cdots & * \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \Gamma_{tt} & \ddots & & \vdots \\ \vdots & & \ddots & \mu_{t+1} & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & * \\ 0 & \cdots & \cdots & \cdots & 0 & \mu_{k-r_2-2t} \end{pmatrix} Q_2^*, \end{aligned} \quad (38)$$

where

$$\begin{aligned} \Lambda_{jj} &= \begin{pmatrix} l_j & f_{1j} \\ -f_{2j} & l_j \end{pmatrix}, \quad \lambda_j \in R, \quad j = s+1, \dots, n-k-r_1-2s, \\ \Gamma_{jj} &= \begin{pmatrix} g_j & h_{1j} \\ -h_{2j} & g_j \end{pmatrix}, \quad \mu_j \in R, \quad j = t+1, \dots, k-r_2-2t. \end{aligned} \quad (39)$$

$\square$

For Lemma 5, there exist inverse matrices Y_{1j} and Y_{2j} , satisfying

$$\begin{aligned} Y_{1j}^{-1} \Lambda_{jj} Y_{1j} &= \begin{pmatrix} l_j & f_j \\ -f_j & l_j \end{pmatrix} = \bar{\Lambda}_{jj}, \\ \|Y_{1j}\|_F &= 1, \quad j = 1, 2, \dots, s, \\ Y_{2j}^{-1} \Gamma_{jj} Y_{2j} &= \begin{pmatrix} g_j & h_j \\ -h_j & g_j \end{pmatrix} = \bar{\Gamma}_{jj}, \\ \|Y_{2j}\|_F &= 1, \quad j = 1, 2, \dots, t. \end{aligned} \quad (40)$$

Let $Y_1 = Y_{11} \oplus \cdots \oplus Y_{1s} \oplus I_{n-k-r_1-2s}$ and $Y_2 = Y_{21} \oplus \cdots \oplus Y_{2t} \oplus I_{k-r_2-2t}$, I_{n-k-r_1-2s} and I_{k-r_2-2t} are identity matrices. Then, it is easy to see that equations (36) and (37) exist.

Theorem 2. *Given $A^* \in R^{n \times n}$, suppose given matrices X and Λ , real number $r > 0$ make Constrained Inverse Eigenproblem solvable, and the solution expression is the same as equation (25). Let*

$$Y = D \begin{pmatrix} U & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} I_{r_1} & 0 & 0 & 0 \\ 0 & Q_1^* Y_1 & 0 & 0 \\ 0 & 0 & I_{r_2} & 0 \\ 0 & 0 & 0 & Q_2^* Y_2 \end{pmatrix}. \quad (41)$$

Then, Optimal Approximation Problem has a unique solution which can be expressed as

$$\hat{A} = D \begin{pmatrix} \hat{B} & 0 \\ 0 & \hat{C} \end{pmatrix} D^T, \quad (42)$$

where

$$\begin{aligned} \hat{B} &= U \begin{pmatrix} \Sigma_1 V_1^T \Lambda V_1 \Sigma_1^{-1} & U_1^T A_{11}^* U_2 \\ 0 & [B_{22}]_{D_r} \end{pmatrix} U^T, \\ \hat{C} &= P \begin{pmatrix} \Sigma_2 Q_1^T \Lambda Q_1 \Sigma_2^{-1} & P_1^T A_{22}^* P_2 \\ 0 & [C_{22}]_{D_r} \end{pmatrix} P^T. \end{aligned} \quad (43)$$

$$\begin{aligned} & \left\| \Sigma_1 V_1^T \Lambda V_1 \Sigma_1^{-1} - U_1^T A_{11}^* U_1 \right\|^2 + \left\| B_{12} Q_1^{*T} Y_1 - U_1^T A_{11}^* U_2 Q_1^{*T} Y_1 \right\|^2 + \left\| Y_1^{-1} Q_1^* [B_{22}]_{D_r} Q_1^{*T} Y_1 - Y_1^{-1} Q_1^* U_2^T A_{11}^* U_2 Q_1^{*T} Y_1 \right\|^2 \\ & + \left\| \Sigma_2 Q_1^T \Lambda Q_1 \Sigma_2^{-1} - P_1^T A_{22}^* P_1 \right\|^2 + \left\| C_{12} Q_2^{*T} Y_2 - P_1^T A_{22}^* P_2 Q_2^{*T} Y_2 \right\|^2 + \left\| Y_2^{-1} Q_2^* [C_{22}]_{D_r} Q_2^{*T} Y_2 - Y_2^{-1} Q_2^* P_2^T A_{22}^* P_2 Q_2^{*T} Y_2 \right\|^2 \\ & = \min. \end{aligned} \quad (45)$$

Obviously, we may derive that equation (44) has the solution $\hat{A}$ if and only if

$$\left\| Y_1^{-1} Q_1^* U_2^T A_{11}^* U_2 Q_1^{*T} Y_1 - Y_1^{-1} Q_1^* [B_{22}]_{D_r} Q_1^{*T} Y_1 \right\|_F^2 = \min, \quad (46)$$

$$\left\| Y_2^{-1} Q_2^* P_2^T A_{22}^* P_2 Q_2^{*T} Y_2 - Y_2^{-1} Q_2^* [C_{22}]_{D_r} Q_2^{*T} Y_2 \right\|_F^2 = \min, \quad (47)$$

$$\left\| U_1^T A_{11}^* U_2 Q_1^{*T} Y_1 - B_{12} Q_1^{*T} Y_1 \right\|_F = \min, \quad (48)$$

Proof. Let $A \in S_{D_r}$, by Definition 2, we have

$$\|A - A^*\|_Y^2 = \|Y^{-1}AY - Y^{-1}A^*Y\|_F^2 = \min_{A \in S_{D_r}}. \quad (44)$$

The expression of A is the same as equation (25) and $D^T A^* D$ is decomposed as equation (35); then, equation (44) is equivalent to

$$\left\| P_1^T A_{22}^* P_2 Q_2^{*T} Y_2 - C_{12} Q_2^{*T} Y_2 \right\|_F = \min. \quad (49)$$

From Lemmas 5 and 6, equations (36) and (37), we may see that equations (46) and (47) imply that

$$[B_{22}]_{D_r} = Q_1^{*T} Y_1 \begin{pmatrix} [\bar{\Lambda}_{11}]_{D_r} & * & \cdots & \cdots & \cdots & * \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & [\bar{\Lambda}_{ss}]_{D_r} & \ddots & & \vdots \\ \vdots & & \ddots & [\bar{\Lambda}_{s+1}]_{D_r} & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & * \\ 0 & \cdots & \cdots & \cdots & 0 & [\bar{\Lambda}_{n-k-r_1-2s}]_{D_r} \end{pmatrix} Y_1^{-1} Q_1^*, \quad (50)$$

Step 1. Input X, Λ, A^*, r ;
 Step 2. Decompose $D^T X$ as equation (10);
 Step 3. Calculate the singular value decomposition X_{D_1}, X_{D_2} as equations (13) and (14), verify that equation (24) is true and A can be expressed as equation (25);
 Step 4. Decompose $D^T A^* D$ as equation (35);
 Step 5. Decompose $U_2^T A_{11}^* U_2, P_2^T A_{22}^* P_2$ as equations (36), (37), respectively;
 Step 6. Calculate $[B_{22}]_{D_r}, [C_{22}]_{D_r}$ as equations (50) and (52), respectively;
 Step 7. Calculate $\hat{A}$ by Theorem 2.

ALGORITHM 1: The solution of (7).

where

$$[\bar{\Lambda}_{jj}]_{D_r} = \begin{cases} \frac{r}{\lambda(\bar{\Lambda}_{jj})} \bar{\Lambda}_{jj}, & |\lambda(\bar{\Lambda}_{jj})| > r, \\ \bar{\Lambda}_{jj}, & |\lambda(\bar{\Lambda}_{jj})| \leq r, \end{cases} \quad (51)$$

$$[\bar{\lambda}_{jj}]_{D_r} = \begin{cases} \text{sign}(\bar{\lambda}_i) r, & |\lambda_i| > r, \\ \lambda_i, & |\lambda_i| \leq r, \end{cases}$$

$$j = 1, 2, \dots, s, i = s + 1, \dots, n - k - r_1 - 2s.$$

$$[C_{22}]_{D_r} = Q_2^{*T} Y_2 \begin{pmatrix} [\bar{\Gamma}_{11}]_{D_r} & * & \cdots & \cdots & \cdots & * \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & [\bar{\Gamma}_{tt}]_{D_r} & \ddots & & \vdots \\ \vdots & & \ddots & [\bar{\mu}_{t+1}]_{D_r} & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & * \\ 0 & \cdots & \cdots & \cdots & 0 & [\bar{\mu}_{k-r_2-2t}]_{D_r} \end{pmatrix} Y_2^{-1} Q_2^*, \quad (52)$$

where

$$[\bar{\Gamma}_{jj}]_{D_r} = \begin{cases} \frac{r}{\lambda(\bar{\Gamma}_{jj})} \bar{\Gamma}_{jj}, & |\lambda(\bar{\Gamma}_{jj})| > r, \\ \bar{\Gamma}_{jj}, & |\lambda(\bar{\Gamma}_{jj})| \leq r, \end{cases} \quad (53)$$

$$[\bar{\mu}_{jj}]_{D_r} = \begin{cases} \text{sign}(\bar{\mu}_i) r, & |\mu_i| > r, \\ \bar{\mu}_i, & |\mu_i| \leq r, \end{cases}$$

$$j = 1, \dots, t, i = t + 1, \dots, k - r_2 - 2t.$$

By equations (48) and (49), we have

$$\begin{aligned} B_{12} &= U_1^T A_{11}^* U_2, \\ C_{12} &= P_1^T A_{22}^* P_2. \end{aligned} \quad (54)$$

Note

$$\hat{B} = U \begin{pmatrix} \Sigma_1 V_1^T \Lambda V_1 \Sigma_1^{-1} & U_1^T A_{11}^* U_2 \\ 0 & [B_{22}]_{D_r} \end{pmatrix} U^T, \quad (55)$$

$$\hat{C} = P \begin{pmatrix} \Sigma_2 Q_1^T \Lambda Q_1 \Sigma_2^{-1} & P_1^T A_{22}^* P_2 \\ 0 & [C_{22}]_{D_r} \end{pmatrix} P^T. \quad (56)$$

Equations (55), (56), and (25) imply equation (42). $\square$

4. Numerical Example

Based on Theorems 1 and 2, we propose the following algorithm for solving Constrained Inverse Eigenproblem and Optimal Approximation Problem (Algorithm 1).

In this section, we will give a numerical example to illustrate our results. All the tests are performed by MATLAB6.5.

The matrices X, Λ , and A^* and the radius r are given by following:

$$\begin{aligned}
 X &= \begin{pmatrix} -0.0638 & -0.1385 & -0.0249 \\ -0.0593 & -0.3027 & -0.0061 \\ -0.3301 & 0.0521 & 0.0347 \\ 0.6192 & 0.6202 & 0.0000 \\ 0.6192 & -0.6202 & 0.0000 \\ -0.3301 & -0.0521 & -0.0347 \\ -0.0593 & 0.3027 & 0.0061 \\ -0.0638 & 0.1385 & 0.0249 \end{pmatrix}, \\
 \Lambda &= \begin{pmatrix} 0.7611 & 0.0000 & 0.0000 \\ 0.0000 & 0.5529 & 0.0702 \\ 0.0000 & -0.0702 & 0.5529 \end{pmatrix}, \\
 A^* &= \begin{pmatrix} 0.9501 & 0.8214 & 0.9355 & 0.1389 & 0.4451 & 0.8381 & 0.3046 & 0.3784 \\ 0.2311 & 0.4447 & 0.9169 & 0.2028 & 0.9318 & 0.0196 & 0.1897 & 0.8600 \\ 0.6068 & 0.6154 & 0.4103 & 0.1987 & 0.4660 & 0.6813 & 0.1934 & 0.8537 \\ 0.4860 & 0.7919 & 0.8936 & 0.6038 & 0.4186 & 0.3795 & 0.6822 & 0.5936 \\ 0.8913 & 0.9218 & 0.0579 & 0.2722 & 0.8462 & 0.8318 & 0.3028 & 0.4966 \\ 0.7621 & 0.7382 & 0.3529 & 0.1988 & 0.5252 & 0.5028 & 0.5417 & 0.8998 \\ 0.4565 & 0.1763 & 0.8132 & 0.0153 & 0.2026 & 0.7095 & 0.1509 & 0.8216 \\ 0.0185 & 0.4057 & 0.0099 & 0.7468 & 0.6721 & 0.4289 & 0.6979 & 0.6449 \end{pmatrix}, \\
 r &= 0.4.
 \end{aligned} \tag{57}$$

By algorithm, the constrained optimal approximation solution $\hat{A}$ by Optimal Approximation Problem is

$$\hat{A} = \begin{pmatrix} 0.4539 & 0.2884 & -0.1728 & 0.0469 & -0.1080 & 0.0782 & 0.0071 & -0.0710 \\ 0.2569 & -0.0060 & -0.1473 & -0.0179 & -0.0476 & 0.2079 & -0.3727 & -0.1464 \\ 0.1450 & 0.1242 & -0.0729 & -0.2928 & -0.2391 & -0.3526 & 0.0205 & 0.6977 \\ 0.0832 & 0.2522 & 0.3522 & 0.5442 & 0.2411 & -0.5450 & 0.3983 & 0.5450 \\ 0.5450 & 0.3983 & -0.5450 & 0.2411 & 0.5442 & 0.3522 & 0.2522 & 0.0832 \\ 0.6977 & 0.0205 & -0.3526 & -0.2391 & -0.2928 & -0.0729 & 0.1242 & 0.1450 \\ -0.1464 & -0.3727 & 0.2079 & -0.0476 & -0.0179 & -0.1473 & -0.0060 & 0.2569 \\ -0.0710 & 0.0071 & 0.0782 & -0.1080 & 0.0469 & -0.1728 & 0.2884 & 0.4539 \end{pmatrix}. \tag{58}$$

In the practical engineering problems, it is usually required that the matrix is a centrosymmetric matrix and the partial eigenvalues of the matrix must be located in a given closed disc or interval. We verify that $\hat{A}$ in Optimal Approximation Problem is a centrosymmetric matrix and the eigenvalues of $\lambda(\hat{A})/\lambda(\Lambda)$ are lost in the disc D_r .

Data Availability

All data generated or analyzed during this study are included in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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