Research Article

Soliton Solutions and Collisions for the Multicomponent Gross–Pitaevskii Equation in Spinor Bose–Einstein Condensates

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In this paper, we investigate a five-component Gross–Pitaevskii equation, which is demonstrated to describe the dynamics of an $F = 2$ spinor Bose–Einstein condensate in one dimension. By employing the Hirota method with an auxiliary function, we obtain the explicit bright one- and two-soliton solutions for the equation via symbolic computation. With the choice of polarization parameter and spin density, the one-soliton solutions are divided into four types: one-peak solitons in the ferromagnetic and cyclic states and one- and two-peak solitons in the polar states. For the former two, solitons share the similar shape of one peak in all components. Solitons in the polar states have the one- or two-peak profiles, and the separated distance between two peaks is inversely proportional to the value of polarization parameter. Based on the asymptotic analysis, we analyze the collisions between two solitons in the same and different states.

1. Introduction

Bose–Einstein condensates (BECs) of the alkali-metal-atom gases have attracted certain attention in both experimental [1, 2] and theoretical studies [3–5]. BECs can have the internal degrees of freedom associated with the hyperfine spin and such condensates are usually called the spinor BECs [6]. The spinor BECs are classified according to the relative values of certain characteristic scattering lengths [7]. For the spinor BEC with the hyperfine spin $F = 1$, the condensate can have either ferromagnetic state and polar state, such as ^{87}Rb and ^{23}Na , respectively, and the corresponding ground-state structure and dynamical properties of those two cases are distinct [8]. In the description of [7, 8], a spinor condensate with the spin $F = 2$ bosons in an optical trap can be one of the three types: ferromagnetic, polar, and cyclic. Dynamics of the spinor BEC have been addressed in [9–14].

The dynamics of nonlinear phenomena can be analyzed by means of the corresponding nonlinear evolution equations. There has been considerable work carried out on the control problem of nonlinear systems, such as

those in chaotic and stochastic systems [15–19]. Solitons, resulted from the balance between the effects of nonlinearity and dispersion [20], have been studied in fields such as nonlinear optics, plasma physics, and condensed matter physics [3, 21–23]. In addition, breathers and rogue waves have been concerned in recent research [24, 25]. In the spinor BECs, matter-wave solitons are thought to be useful for their application in the atom laser, atom interferometry, and coherent atom transport, which can contribute to the realization of quantum information processors or computation [26]. The dynamics of magnetic soliton, dark soliton, and dark-bright vector soliton in spinor Bose–Einstein condensates have been investigated [27–30]. A frontier is the model extension from the single-component condensate to the multicomponent, which opens up new fields for the study of quantum matter waves and fluids [11]. The spinor BECs with spin F is described by a $(2F + 1)$ -component mean-field wave function [31].

In this paper, we will consider a five-component Gross–Pitaevskii (GP) equation for the dynamics of an $F = 2$ spinor BEC in one dimension [32]:

$$i\partial_t \Phi_{\pm 2} + \partial_x^2 \Phi_{\pm 2} + 2n\Phi_{\pm 2} - \Theta \Phi_{\mp 2}^* = 0, \quad (1a)$$

$$i\partial_t \Phi_{\pm 1} + \partial_x^2 \Phi_{\pm 1} + 2n\Phi_{\pm 1} + \Theta \Phi_{\mp 1}^* = 0, \quad (1b)$$

$$i\partial_t \Phi_0 + \partial_x^2 \Phi_0 + 2n\Phi_0 - \Theta \Phi_0^* = 0, \quad (1c)$$

where $\Phi_{\pm j}$'s ($j = 0, 1, 2$) are the wave functions of the five spin components, t and x , respectively, denote the time and spatial co-ordinate, and the asterisk represents the complex conjugation, while the number density n and singlet-pair amplitude Θ are defined by [10, 33]

$$n = \sum_{k=-2}^2 \Phi_k \Phi_k^*, \quad (2)$$

$$\Theta = 2\Phi_2 \Phi_{-2} - 2\Phi_1 \Phi_{-1} + \Phi_0^2.$$

Equation (1) is the integrable version of the multi-component GP equation mentioned in [31] and simplified under the condition $2m = 1$, $\hbar = 1$, $c_2 = 0$, $c_4 = 1$, and $c_0 = -2 < 0$, which corresponds to the attractive interaction. Due to the integrability, equation (1) has the infinite conservation laws, which restricts the dynamics of the system in an important way. The normalization is imposed as $\int dx n = N_T$, where N_T is the total number of atoms. For equation (1), bright one- and two-soliton solutions have been given via the Zakharov-Shabat dressing [32]. However, the solitons are not classified according to the forms of the solutions and then the properties of soliton collisions are not clearly analyzed, that is the aim of this paper.

In this paper, we will concentrate on the soliton types and collisions in the same and different states in an $F = 2$ spinor BEC. For equation (1), we will present the Hirota form with an auxiliary function and explicit one- and two-soliton solutions. With symbolic computation [34, 35], this paper will be organized as follows. In Section 2, the Hirota bilinearization procedure for equation (1) will be presented. In Section 3, based on the bilinear form, we will obtain and analyze the one-soliton solutions, which reveal the one- or two-peak solitons in three different states. In Section 4, making the asymptotic analysis on the obtained two-soliton solutions, we will discuss the soliton collisions in the same and different states. Section 5 will be our conclusions.

2. Hirota Bilinearization Method for Equation (1)

In order to understand the dynamics of equation (1), it is essential to obtain the soliton solutions. The Hirota bilinear method is a tool to construct the soliton solutions for certain nonlinear evolution equations [36]. For equation (1), we will utilize the Hirota method with an auxiliary function to obtain the one- and two-soliton solutions. The procedure can be extended to obtain N -soliton solutions [37].

To begin with, equation (1) can be expressed in the bilinear form:

$$(iD_t + D_x^2)(G_{\pm 2} \cdot F) = SG_{\mp 2}^*, \quad (3a)$$

$$(iD_t + D_x^2)(G_{\pm 1} \cdot F) = -SG_{\mp 1}^*, \quad (3b)$$

$$(iD_t + D_x^2)(G_0 \cdot F) = SG_0^*, \quad (3c)$$

$$D_x^2(F \cdot F) = 2 \sum_{k=-2}^2 G_k G_k^*, \quad (3d)$$

$$SF = 2G_2 G_{-2} - 2G_1 G_{-1} + G_0^2, \quad (3e)$$

with an auxiliary function S and the following transformations:

$$\Phi_{\pm j} = \frac{G_{\pm j}}{F}, \quad \text{for } j = 0, 1, 2, \quad (4)$$

where G_j 's are the complex functions of t and x and F is a real one. D_x and D_t are the bilinear differential operators [36] defined by

$$D_x^l D_t^m (f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^l \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m f(x, t) g(x', t')|_{x'=x, t'=t}. \quad (5)$$

where l and m are the positive integers and x' and t' are the formal variables. Multisoliton solutions for equation (1) can be generated by solving equation (3) with the following power series expansions as

$$G_{\pm j} = \varepsilon g_{\pm j}^{(1)} + \varepsilon^3 g_{\pm j}^{(3)} + \varepsilon^5 g_{\pm j}^{(5)} + \dots,$$

$$F = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \varepsilon^6 f_6 + \dots, \quad (6)$$

$$S = \varepsilon^2 s_2 + \varepsilon^4 s_4 + \varepsilon^6 s_6 + \dots,$$

where ε is the formal parameter, $g_{\pm j}^{(l)}$'s ($l = 1, 3, 5, \dots$) are the complex functions of t and x , and f_m 's ($m = 2, 4, 6, \dots$) are the real ones which will be determined later.

3. Bright One-Soliton Solutions

In order to obtain one-soliton solutions for equation (1), the power series expansions for $G_{\pm j}$, F , and S are terminated as

$$G_{\pm j} = \varepsilon g_{\pm j}^{(1)} + \varepsilon^3 g_{\pm j}^{(3)},$$

$$S = \varepsilon^2 S_2, \quad (7)$$

$$F = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4.$$

It is noted that equation (7) are usually taken to obtain the two-soliton solutions. Substituting equation (7) into equation (3) and collecting the terms with the same power of ε , we obtain the one-soliton solutions for equation (1):

$$\Phi_{\pm j} = \frac{b_1^{(\pm j)} e^{\theta_1} + e^{2\theta_1 + \theta_1^* + d_1^{(\pm j)}}}{1 + e^{\theta_1 + \theta_1^* + l_1} + e^{2\theta_1 + 2\theta_1^* + c_{11}}}, \quad (8)$$

and the auxiliary function S given as

$$S = \Lambda_1 e^{2\theta_1}, \quad (9)$$

with

$$\begin{aligned}
 \theta_1 &= k_1 x + i k_1^2 t, \\
 e^{d_1^{(\pm j)}} &= \frac{\delta(j) b_1^{(\mp j)*} \Lambda_1}{2(k_1 + k_1^*)^2}, \\
 \delta(j) &= \begin{cases} -1 & j = 1 \\ 1 & j = 0, 2, \end{cases} \\
 e^{l_1} &= \frac{\sum_{k=0}^2 |b_1^{(\pm k)}|^2}{(k_1 + k_1^*)^2}, \\
 e^{c_{11}} &= \frac{|\Lambda_1|^2}{4(k_1 + k_1^*)^4}, \\
 \Lambda_1 &= 2b_1^{(+1)}b_1^{(-1)} + (b_1^{(0)})^2 - 2b_1^{(+2)}b_1^{(-2)},
 \end{aligned} \tag{10}$$

where $b_1^{(\pm j)}$'s and k_1 are all complex constants. Here, we give the spin densities $f = (f^x, f^y, f^z)$ with [32]

$$\begin{aligned}
 f^x &= 2 \left[\Phi_{-2} \Phi_{-1}^* + \frac{\sqrt{6}}{2} \Phi_0^* (\Phi_{+1} + \Phi_{-1}) + \Phi_2 \Phi_1^* \right]_R, \\
 f^y &= 2i \left[\Phi_{-2} \Phi_{-1}^* + \frac{\sqrt{6}}{2} \Phi_0^* \Phi_{-1} + \frac{\sqrt{6}}{2} \Phi_0 \Phi_1^* + \Phi_1 \Phi_2^* \right]_I, \\
 f^z &= 2|\Phi_{-2}|^2 + |\Phi_{-1}|^2 - |\Phi_1|^2 - 2|\Phi_2|^2,
 \end{aligned} \tag{11}$$

where the subscripts R and I , respectively, denote the real and imaginary parts and the total spin can be given with $\int dx f = F_T$. For equation (1), there exist different types of solitons according to the values of Λ_1 . When $\Lambda_1 = 0$, equation (8) admit the one-peak solitons in the ferromagnetic and cyclic states corresponding to the spin densities $|f| > 0$ and $f = 0$, respectively. For the case of $\Lambda_1 \neq 0$, we can find the one- and two-peak solitons in the polar state, while only the two-peak soliton in the polar state for the $F = 1$ spinor BECs [38].

3.1. One-Peak Solitons in the Ferromagnetic and Cyclic States. When $\Lambda_1 = 0$, equation (9) can be expressed as

$$\Phi_{\pm j} = A_{\pm j} e^{i\theta_{1I}} \operatorname{sech} \left(\theta_{1R} + \frac{l_1}{2} \right), \tag{12}$$

with

$$\begin{aligned}
 A_{\pm j} &= \frac{1}{2} b_1^{(\pm j)} e^{-(l_1/2)}, \\
 k_1 &= k_{1R} + i k_{1I}, \\
 \theta_{1R} &= k_{1R} (x - 2k_{1I} t), \\
 \theta_{1I} &= k_{1I} x + (k_{1R}^2 - k_{1I}^2) t.
 \end{aligned} \tag{13}$$

Here, $A_{\pm j}$, $2k_{1I}$, and $l_1/2k_{1R}$, respectively, characterize the amplitude, velocity, and initial phase of the soliton. For $|f| > 0$, it corresponds to the one-peak soliton in the ferromagnetic state [31]. When choosing suitable parameters to make $f = 0$, we find that the soliton is in the cyclic state. Those two types of solitons possess the similar intensity profile of one peak in all components, as shown in Figures 1 and 2. For simplicity, we choose the parameters to make $|\Phi_{+j}| = |\Phi_{-j}|$ in Figure 1. It is noted that the auxiliary function S vanishes in this case, as given in equation (12).

3.2. One- and Two-Peak Solitons in the Polar State. For the case $\Lambda_1 \neq 0$, equation (8) can be expressed as

$$\Phi_{\pm j} = A_{\pm j} e^{i\theta_{1I}} \frac{\cosh(\theta_{1R} + (c_{11}/4) + P_{\pm j})}{4 \cosh^2(\theta_{1R} + (c_{11}/4)) + L}, \tag{14}$$

where $A_{\pm j} = 2e^{(d_1^{(\pm j)} + \ln b_1^{(\pm j)} - c_{11})/2}$, $L = e^{(l_1 - (c_{11}/2))} - 2$, and $P_{\pm j} = e^{(2d_1^{(\pm j)} - 2 \ln b_1^{(\pm j)} - c_{11})/4}$. The form of equation (14) will be of use in the following asymptotic analysis of the two-soliton solutions. When $f = 0$ under certain constraint, equation (14) depicts the one-peak soliton in the polar state, as the short dashed lines shown in Figure 3. When $f \neq 0$ but the spin $F = \int f dx = 0$, we can observe the two-peak soliton in the polar state, as the solid lines shown in Figure 3. In addition, as the value of Λ_1 decreases, the separated distance between two peaks increases, and it looks like that there exist two ferromagnetic solitons traveling parallel in the same velocity and amplitudes as Λ_1 approaches to zero, as the long dashed lines shown in Figure 3.

4. Two-Soliton Solutions and Soliton Collisions

In this paper, we will concentrate on the two-soliton solutions and analyze the collisions between two solitons. Employing the following expansions,

$$\begin{aligned}
 G_{\pm j} &= \varepsilon g_{\pm j}^{(1)} + \varepsilon^3 g_{\pm j}^{(3)} + \varepsilon^5 g_{\pm j}^{(5)} + \varepsilon^7 g_{\pm j}^{(7)}, \\
 S &= \varepsilon^2 s_2 + \varepsilon^4 s_4 + \varepsilon^6 s_6, \\
 F &= 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \varepsilon^6 f_6 + \varepsilon^8 f_8.
 \end{aligned} \tag{15}$$

We obtain the two-soliton solutions for equation (1) in the form of

$$\Phi_{\pm j} = \frac{G_{\pm j}}{F}, \quad j = 0, 1, 2, \tag{16}$$

where the functions $G_{\pm j}$ and F are expressed as

$$\begin{aligned}
G_{\pm j} = & b_1^{(\pm j)} e^{\theta_1} + b_2^{(\pm j)} e^{\theta_2} + e^{2\theta_1 + \theta_1^* + d_1^{(\pm j)}} + e^{2\theta_1 + \theta_2^* + d_{12}^{(\pm j)}} + e^{2\theta_2 + \theta_1^* + d_{21}^{(\pm j)}} \\
& + e^{2\theta_2 + \theta_2^* + d_2^{(\pm j)}} + e^{\theta_1 + \theta_1^* + \theta_2 + D_{12}^{(\pm j)}} + e^{\theta_2 + \theta_2^* + \theta_1 + D_{21}^{(\pm j)}} + e^{2\theta_1 + 2\theta_1^* + \theta_2 + h_{12}^{(\pm j)}} + e^{2\theta_2 + 2\theta_2^* + \theta_1 + h_{21}^{(\pm j)}} + e^{2\theta_2 + 2\theta_1^* + \theta_1 + H_{12}^{(\pm j)}} \\
& + e^{2\theta_1 + 2\theta_2^* + \theta_2 + H_{21}^{(\pm j)}} + e^{2\theta_1 + \theta_1^* + \theta_2 + \theta_2^* + w_{12}^{(\pm j)}} + e^{2\theta_2 + \theta_2^* + \theta_1 + \theta_1^* + w_{21}^{(\pm j)}} + e^{2\theta_1 + 2\theta_1^* + 2\theta_2 + \theta_2^* + W_{12}^{(\pm j)}} + e^{2\theta_1 + 2\theta_2 + 2\theta_2^* + \theta_1^* + W_{21}^{(\pm j)}},
\end{aligned} \tag{17}$$

$$\begin{aligned}
F = & 1 + e^{\theta_1 + \theta_1^* + l_1} + e^{\theta_1 + \theta_2^* + l_0} + e^{\theta_2 + \theta_1^* + l_0^*} + e^{\theta_2 + \theta_2^* + l_2} + e^{2\theta_1 + 2\theta_1^* + c_{11}} + e^{2\theta_1 + 2\theta_2^* + c_{12}} + e^{2\theta_2 + 2\theta_1^* + c_{12}^*} + e^{2\theta_2 + 2\theta_2^* + c_{22}} \\
& + e^{2\theta_1 + \theta_1^* + \theta_2^* + l_{12}} + e^{2\theta_1^* + \theta_1 + \theta_2 + l_{12}^*} + e^{2\theta_2 + \theta_1^* + \theta_2^* + l_{21}} + e^{2\theta_2^* + \theta_1 + \theta_2 + l_{21}^*} + e^{\theta_1 + \theta_1^* + \theta_2 + \theta_2^* + l_3} + e^{2\theta_1 + 2\theta_1^* + \theta_2 + \theta_2^* + L_{12}} \\
& + e^{2\theta_2 + 2\theta_2^* + \theta_1 + \theta_1^* + L_{21}} + e^{2\theta_1 + 2\theta_2^* + \theta_2 + \theta_1^* + l_4} + e^{2\theta_2 + 2\theta_1^* + \theta_1 + \theta_2^* + l_4^*} + e^{2\theta_1 + 2\theta_1^* + 2\theta_2 + 2\theta_2^* + l_5},
\end{aligned} \tag{18}$$

and the auxiliary function S takes the form

$$\begin{aligned}
S = & \Lambda_1 e^{2\theta_1} + \Lambda_2 e^{2\theta_2} + e^{\theta_1 + \theta_2 + r_1} + e^{\theta_1 + \theta_1^* + 2\theta_2 + r_{21}} + e^{\theta_2 + \theta_2^* + 2\theta_1 + r_{12}} \\
& + e^{\theta_1 + \theta_2^* + 2\theta_2 + r_{21}} + e^{\theta_2 + \theta_1^* + 2\theta_1 + r_{12}} + e^{2\theta_1 + 2\theta_1^* + 2\theta_2 + r_3} \\
& + e^{2\theta_1 + 2\theta_2 + 2\theta_2^* + r_4} + e^{2\theta_1 + \theta_1^* + 2\theta_2 + \theta_2^* + r_5},
\end{aligned} \tag{19}$$

with

$$\begin{aligned}
\theta_m &= k_m x + i k_m^2 t, \\
\Lambda_m &= 2b_m^{(+1)} b_m^{(-1)} + (b_m^{(0)})^2 - 2b_m^{(+2)} b_m^{(-2)}, \\
m &= 1, 2, \\
e^{d_m^{(\pm j)}} &= \frac{\delta(j) b_m^{(\mp j)*} \Lambda_m}{2(k_m + k_m^*)^2}, \\
e^{d_{lm}^{(\pm j)}} &= \frac{\delta(j) b_l^{(\mp j)*} \Lambda_m}{(k_m + k_l^*)^2}, \\
l = 1, 2, m \neq l, \delta(j) &= \begin{cases} -1, & j = 1, \\ 1, & j = 0, 2, \end{cases} \\
e^{D_{lm}^{(\pm j)}} &= \frac{1}{(k_m + k_m^*)^2 (k_l + k_m^*)^2} \left\{ 2(k_m - k_l) \left[b_m^{(\pm j)} (k_m + k_m^*) \sum_{k=0}^2 b_m^{(\pm k)*} b_l^{(\pm k)} - b_l^{(\pm j)} (k_l + k_m^*) \sum_{k=0}^2 \delta(k) |b_m^{(\pm k)}|^2 \right] \right. \\
& \quad \left. + \delta(j) b_m^{(\mp j)*} (k_m + k_m^*) (k_l + k_m^*) \sum_{k=0}^2 \delta(k) b_m^{(\pm k)} b_l^{(\pm k)} \right\}, \\
e^{h_{lm}^{(\pm j)}} &= \frac{b_l^{(\pm j)} |\Lambda_m|^2 (k_m^* - k_l^*)^2}{4(k_m + k_m^*)^4 (k_l + k_m^*)^2}, \\
e^{H_{lm}^{(\pm j)}} &= \frac{b_m^{(\mp j)*} \Lambda_m \Lambda_l^* (k_m^* - k_l^*)^2}{4(k_m + k_m^*)^2 (k_m + k_l^*)^4},
\end{aligned}$$

$$\begin{aligned}
e_{lm}^{w(\pm j)} &= \frac{\Lambda_m (k_m - k_l)^2}{2(k_m + k_m^*)^2 |k_m + k_l^*|^4 (k_l + k_l^*)^2} \left\{ b_l^{(\pm j)} (k_l + k_l^*) (k_l + k_m^*) \sum_{k=0}^2 \delta(k) b_m^{(\mp k)*} b_l^{(\pm k)*} + \delta(j) (k_m^* - k_l^*) \right. \\
&\quad \left. \left[b_m^{(\mp j)*} (k_l + k_m^*) \sum_{k=0}^2 |b_l^{(\pm k)}|^2 - b_l^{(\mp j)*} (k_l + k_l^*) \sum_{k=0}^2 b_m^{(\pm k)*} b_l^{(\pm k)} \right] \right\}, \\
e_{lm}^{W(\pm j)} &= \frac{\delta(j) b_l^{(\mp j)*} |\Lambda_m|^2 \Lambda_l (k_m - k_l)^4 (k_m^* - k_l^*)^2}{8(k_m + k_m^*)^4 (k_l + k_m^*)^4 (k_m + k_l^*)^2 (k_l + k_l^*)^2}, \\
e^{l_0} &= \frac{\sum_{k=0}^2 b_1^{(\pm k)} b_2^{(\pm k)*}}{(k_1 + k_2^*)^2}, \\
e^{l_m} &= \frac{\sum_{k=0}^2 |b_m^{(\pm k)}|^2}{(k_m + k_m^*)^2}, \\
e^{c_{lm}} &= \frac{\Lambda_m \Lambda_l^*}{4(k_m + k_l^*)^4}, \\
e^{l_3} &= \frac{2}{(k_1 + k_1^* + k_2 + k_2^*)^2} \left[(|k_1|^2 + |k_2|^2 - 2k_1^* k_2) (|e^{c_0}|^2 - e^{c_1} e^{c_2}) - (k_1^2 + k_2^2 - 2k_1 k_2) (|e^{c_0}|^2 + e^{c_1} e^{c_2}) - 2 \sum_{m,l=1, m \neq l}^2 \sum_{k=0}^2 b_m^{(\pm k)*} e^{D_{lm}^{(sk)}} \right]_R, \\
e^{l_{lm}} &= \frac{\Lambda_m \sum_{k=0}^2 \delta(k) b_1^{(\mp k)*} b_2^{(\pm k)*}}{2(k_m + k_m^*)^2 (k_m + k_l^*)^2}, \\
e^{L_{lm}} &= \frac{|\Lambda_m|^2 \sum_{k=0}^2 |b_l^{(\pm k)}|^2 |k_1 - k_2|^4}{4(k_m + k_m^*)^4 |k_m + k_l|^4 (k_l + k_l^*)^2}, \\
e^{l_4} &= \frac{\Lambda_1 \Lambda_2^* \sum_{k=0}^2 b_2^{(\pm k)} b_1^{(\pm k)*} |k_1 - k_2|^4}{4(k_1 + k_1^*)^2 |k_1 + k_2|^4 (k_1 + k_2^*)^2 (k_2 + k_2^*)^2}, \\
e^{m_5} &= \frac{|\Lambda_1|^2 |\Lambda_2|^2 |k_1 - k_2|^8}{16(k_1 + k_1^*)^4 |k_1 + k_2^*|^8 (k_2 + k_2^*)^4}, \\
e^{r_1} &= 2 \sum_{k=0}^2 \delta(k) b_1^{(\mp k)} b_2^{(\pm k)}, \\
e^{r_{lm}} &= \frac{\Lambda_m \sum_{k=0}^2 |b_l^{(\pm k)}|^2 (k_m - k_l)^2}{(k_m + k_l^*)^2 (k_l + k_l^*)^2}, \\
e^{R_{lm}} &= \frac{\Lambda_m \sum_{k=0}^2 b_l^{(\pm k)} b_m^{(\pm k)*} (k_1 - k_2)^2}{(k_l + k_m^*)^2 (k_m + k_m^*)^2}, \\
e^{r_3} &= \frac{\Lambda_2 |\Lambda_1|^2 (k_1 - k_2)^4}{4(k_1 + k_1^*)^4 (k_2 + k_1^*)^4}, \\
e^{r_4} &= \frac{\Lambda_1 |\Lambda_2|^2 (k_1 - k_2)^4}{4(k_2 + k_2^*)^4 (k_1 + k_2^*)^4}, \\
e^{r_5} &= \frac{\Lambda_1 \Lambda_2 \sum_{k=0}^2 \delta(k) b_1^{(\mp k)*} b_2^{(\pm k)*} (k_1 - k_2)^4}{2(k_1 + k_1^*)^2 |k_1 + k_2^*|^4 (k_2 + k_2^*)^2},
\end{aligned} \tag{20}$$

where k_1 , k_2 , $b_1^{(\pm j)}$'s, and $b_2^{(\pm j)}$'s are all complex constants. To understand the collision property of solitons in the $F = 2$ spinor BEC, we will make the asymptotic analysis on two-

soliton solution (16) as $t \longrightarrow \mp\infty$, which are associated with the initial and final state, respectively. Without loss of generality, we set

FIGURE 1: Intensity plots of the one soliton in the ferromagnetic state for the components $\Phi_{\pm j}$ via solution (8). Parameters are $k_1 = 1.2 + i$, $b_1^{(\pm 2)} = 0.5$, $b_1^{(0)} = 0.5$, and $b_1^{(\pm 1)} = \sqrt{0.78}$.

FIGURE 2: Intensity plots of the one soliton in the cyclic state for the components Φ_2 and Ψ_{-1} via solution (8). Parameters are $k_1 = 1.2 + i$, $b_1^{(2)} = 0.5 + 0.7i$, $b_1^{(-2)} = b_1^{(1)} = b_1^{(0)} = 0$, and $b_1^{(-1)} = \sqrt{2}(0.5 - 0.7i)$.

FIGURE 3: Intensity plots of the one soliton in the polar state for the components $\Phi_{\pm j}$ via Solutions (8). Parameters are $k_1 = 1.2 + i$ with (1) $b_1^{(2)} = b_1^{(-2)*} = 0.5 + 0.8i$, $b_1^{(0)} = 1.01$, and $b_1^{(1)} = -b_1^{(-1)*} = 0.75 + 0.6i$, short dashed lines; (2) $b_1^{(\pm 2)} = 0.6$, $b_1^{(0)} = 1$, and $b_1^{(\pm 1)} = \sqrt{0.75}$, solid lines; (3) $b_1^{(\pm 2)} = 0.6$, $b_1^{(0)} = 1$, and $b_1^{(\pm 1)} = \sqrt{0.85999}$, long dashed lines.

$$\begin{aligned}
 k_{1R} &> 0, \\
 k_{1I} &> 0, \\
 k_{2R} &> 0, \\
 k_{2I} &< 0.
 \end{aligned} \tag{21}$$

Following the classification on the one-soliton solutions, we naturally consider the collisions between the same and different types of solitons, which are determined by Λ_m and f . Since the one-peak soliton in the cyclic state is obtained under certain strong restriction, its collisions with other types of soliton are omitted here.

4.1. Collisions between Two Solitons in the Ferromagnetic State. At first, we will consider the collision between two solitons in the ferromagnetic state. Under the condition $\Lambda_m = 0$,

the asymptotic forms for equation (16) are given as follows:

(A) Before the collision ($t \rightarrow -\infty$).

(1) Soliton S^{1-} ($\theta_{1R} \sim 0, \theta_{2R} \rightarrow -\infty$):

$$\Phi_{\pm j} \rightarrow A_{\pm j}^{1-} e^{i\theta_{1I}} \operatorname{sech}\left(\theta_{1R} + \frac{l_1}{2}\right). \tag{22}$$

(2) Soliton S^{2-} ($\theta_{2R} \sim 0, \theta_{1R} \rightarrow \infty$):

$$\Phi_{\pm j} \rightarrow A_{\pm j}^{2-} e^{i\theta_{2I}} \operatorname{sech}\left(\theta_{2R} + \frac{l_3 - l_1}{2}\right). \tag{23}$$

(B) After the collision ($t \rightarrow +\infty$).

(1) Soliton S^{1+} ($\theta_{1R} \sim 0, \theta_{2R} \rightarrow \infty$):

$$\Phi_{\pm j} \rightarrow A_{\pm j}^{1+} e^{i\theta_{1I}} \text{sech}\left(\theta_{1R} + \frac{l_3 - l_2}{2}\right). \quad (24)$$

(2) Soliton S^{2+} ($\theta_{2R} \sim 0, \theta_{1R} \rightarrow -\infty$):

$$\Phi_{\pm j} \rightarrow A_{\pm j}^{2+} e^{i\theta_{2I}} \text{sech}\left(\theta_{2R} + \frac{l_2}{2}\right), \quad (25)$$

with

$$\begin{aligned} A_{\pm j}^{1-} &= \frac{b_1^{(\pm j)}}{2} e^{-(l_1/2)}, \\ A_{\pm j}^{2-} &= \frac{1}{2} e^{((D_{12}^{(\pm j)} - l_2 - l_3)/2)}, \\ A_{\pm j}^{1+} &= \frac{1}{2} e^{((D_{21}^{(\pm j)} - l_2 - l_3)/2)}, \\ A_{\pm j}^{2+} &= \frac{b_2^{(\pm j)}}{2} e^{-(l_2/2)}. \end{aligned} \quad (26)$$

The superscripts $m\pm$ ($m=1,2$) represent the solitons indicated as S_1 and S_2 at $t \rightarrow \pm\infty$. Setting $b_1^{(\pm j)}/b_2^{(\pm j)} = \text{constant}$, we find that the amplitudes $A_{\pm j}^{m\pm}$ before and after the collision can be related through $A_{\pm j}^{1-} = (A_{\pm j}^{1+} (k_1 - k_2) (k_1^* + k_2^*)) / ((k_1^* - k_2^*) (k_1 + k_2^*))$ and $A_{\pm j}^{2-} = (A_{\pm j}^{2+} (k_1^* - k_2^*) (k_1 + k_2^*)) / ((k_1 - k_2) (k_1 + k_2^*))$, namely, $|A_{\pm j}^{m+}| = |A_{\pm j}^{m-}|$, which means that the collision is elastic in this case. Besides, the solitons S_1 and S_2 , respectively, experience a phase shift of $\eta_1 = (l_3 - l_1 - l_2)/4k_{1R} = \ln[(k_1 - k_2) (k_1^* - k_2^*) / (k_1^* + k_2) (k_1 + k_2^*)] / k_{1R}$ and $\eta_2 = -\eta_1 k_{1R} / k_{2R}$ during the collision. For other choices of $b_m^{(\pm j)}$ satisfying $\Lambda_m = 0$, we will take an example and give Figure 4. The intensity of soliton S_2 in component Φ_0 decreases while those of S_2 in components $\Phi_{\pm 1}$ increase after the collision, as shown in Figures 4(b) and 4(c). This process might be of value for the realization of soliton switching devices [39]. The similar changes of soliton intensities can be observed for soliton S_1 . For this case, the collision is similar to that with an intensity redistribution among all components of each soliton in the multicomponent NLS equation [40].

4.2. Collisions between Soliton in the Ferromagnetic State and Solitons in the Polar State. We will analyze the collisions between soliton in the ferromagnetic state and one- and two-peak solitons in the polar state. Here, we choose S_1 to be the soliton in the ferromagnetic state ($\Lambda_1=0$), and S_2 to be the soliton in the polar state corresponding to the condition $\Lambda_2 \neq 0$. The asymptotic forms of solitons S_1 and S_2 are given as follows:

(A) Before the collision.

(1) Soliton S^{1-} :

$$\Phi_{\pm j} \rightarrow A_{\pm j}^{1-} e^{i\theta_{1I}} \text{sech}\left(\theta_{1R} + \frac{l_1}{2}\right), \quad (27)$$

where $A_{\pm j}^{1-} = (1/2)b_1^{(\pm j)} e^{-(l_1/2)}$.

(2) Soliton S^{2-} :

$$\Phi_{\pm j} \rightarrow A_{\pm j}^{2-} e^{i\theta_{2I}} \frac{\cosh(\theta_{2R} + P_{\pm j}^{2-})}{4 \cosh^2(\theta_{2R}) + L^{2-}}, \quad (28)$$

where $A_{\pm j}^{2-} = 2e^{(W_{21}^{(\pm j)} + D_{12}^{(\pm j)} - L_{21} - l_1)/2}$, $L^{2-} = e^{(l_3 - ((L_{21} + l_1)/2))} - 2$, $P_{\pm j}^{2-} = e^{(2W_{21}^{(\pm j)} - 2D_{12}^{(\pm j)} - L_{21} + l_1)/4}$, and $\theta_{2R} = \theta_{2R} + ((L_{21} - l_1)/4)$, with

$$\begin{aligned} k_2 &= k_{2R} + ik_{2I}, \\ \theta_{2I} &= k_{2I}x + (k_{2R}^2 - k_{2I}^2)t, \\ \theta_{2R} &= k_{2R}x - 2k_{2R}k_{2I}t. \end{aligned} \quad (29)$$

(B) After the collision.

(1) Soliton S^{1+} :

$$\Phi_{\pm j} \rightarrow A_{\pm j}^{1+} e^{i\theta_{1I}} \text{sech}\left(\theta_{1R} + \frac{L_{12} - c_{22}}{2}\right), \quad (30)$$

where $A_{\pm j}^{1+} = (1/2)e^{(2h_{12}^{(\pm j)} - L_{12} - c_{22})/2}$.

(2) Soliton S^{2+} :

$$\Phi_{\pm j} \rightarrow A_{\pm j}^{2+} e^{i\theta_{2I}} \frac{\cosh(\theta_{2R} + P_{\pm j}^{2+})}{4 \cosh^2(\theta_{2R}) + L^{2+}}, \quad (31)$$

where $A_{\pm j}^{2-} = 2e^{(d_2^{(\pm j)} + \ln b_2^{(\pm j)} - c_{22})/2}$, $L^{2-} = e^{(l_2 - c_{22})/2} - 2$, $P_{\pm j}^{2-} = e^{(2d_2^{(\pm j)} - 2\ln b_2^{(\pm j)} - c_{22})/4}$, and $\theta_{2R} = \theta_{2R} + (c_{22}/4)$. It is noted that equations (32) and (34) have the similar forms as equation (14). Choosing suitable parameters, we give Figure 5 illustrating the collision between one-peak solitons in the ferromagnetic and polar states. The solitons S_1 in the ferromagnetic state retain their shapes invariant after the collision, and only has a phase shift at the moment of collision in all components, due to the total spin conservation. However, the amplitudes of one-peak soliton S_2 in the polar state increase in components $\Phi_{\pm 2}$ after the collision, accompanied by the decrease of amplitude for S_2 in component Φ_0 . For the case of collision between one-peak soliton in the ferromagnetic state and two-peak soliton in the polar state, the collision occurs without change of intensities for the solitons in the ferromagnetic state, while with an intensity redistribution among the two-peak solitons in the polar state in all components, as shown in Figure 6. Therefore, we can conclude that there exists no mixing collision in the internal states for the solitons in the ferromagnetic state outside the collision region.

4.3. Collisions between Two Solitons in the Polar State. In this section, we will consider the collisions between the solitons

FIGURE 4: Collision between two solitons in the ferromagnetic state for the components $\Phi_{\pm j}$ via solution (16). Parameters are $k_1 = 1.2 + i$, $k_2 = 1.2 - 0.6i$, $b_1^{(\pm 2)} = 0.6$, $b_1^{(0)} = \sqrt{0.9}$, $b_1^{(\pm 1)} = 0.9$, $b_2^{(\pm 2)} = 1.1$, $b_2^{(0)} = \sqrt{0.08}$, and $b_2^{(\pm 1)} = \sqrt{1.25}$.

FIGURE 5: Collision between one-peak solitons in the ferromagnetic and polar states for the components $\Phi_{\pm j}$ via solution (16). Parameters are the same as those in Figure 4 except $b_2^{(2)} = b_2^{(-2)*} = 0.8 + 0.5i$, $b_2^{(0)} = 0.3$, and $b_2^{(1)} = -b_2^{(-1)*} = 0.4 - 0.9i$.

FIGURE 6: Collision between one-peak soliton in the ferromagnetic state and two-peak soliton in the polar state for the components $\Phi_{\pm j}$ via solutions (16). Parameters are $k_1 = 1.2 + i$, $k_2 = 1.2 - 0.6i$, $b_1^{(\pm 2)} = 0.5$, $b_1^{(0)} = 1$, $b_1^{(\pm 1)} = \sqrt{0.75}$, $b_2^{(\pm 2)} = 0.8$, $b_2^{(0)} = 0.4$, and $b_2^{(\pm 1)} = \sqrt{0.65}$.

FIGURE 7: Collision between one-peak solitons in the polar state for the components $\Phi_{\pm j}$ via solution (16). Parameters are $k_1 = 1.2 + i$, $k_2 = 1.2 - 0.6i$, $b_1^{(2)} = b_1^{(-2)*} = 0.5 + 0.8i$, $b_1^{(0)} = 1.01$, $b_1^{(1)} = -b_1^{(-1)*} = 0.75 - 0.6i$, $b_2^{(2)} = b_2^{(-2)*} = 0.8 + 0.5i$, $b_2^{(0)} = 0.3$, and $b_2^{(1)} = -b_2^{(-1)*} = 0.4 - 0.9i$.

FIGURE 8: Collision between two-peak solitons in the polar state for the components $\Phi_{\pm j}$ via solution (16). Parameters are $k_1 = 1.2 + i$, $k_2 = 1.2 - 0.6i$, $b_1^{(\pm 2)} = 0.5$, $b_1^{(0)} = 1.1$, $b_1^{(\pm 1)} = \sqrt{0.75}$, $b_2^{(\pm 2)} = 0.8$, $b_2^{(0)} = 0.3$, and $b_2^{(\pm 1)} = \sqrt{0.66}$.

FIGURE 9: Collision between one- and two-peak solitons in the polar state for the components $\Phi_{\pm j}$ via solution (16). Parameters are the same as those in Figure 7 except $b_1^{(\pm 2)} = 0.5$, $b_1^{(0)} = 1.01$, and $b_1^{(\pm 1)} = \sqrt{0.75}$.

in the polar state ($\Lambda_m \neq 0$). In this case, the asymptotic forms are given as

(A) Before the collision.

(1) Soliton S^{1-} :

$$\Phi_{\pm j} \longrightarrow A_{\pm j}^{1-} e^{i\theta_{1l}} \frac{\cosh(\theta_{1R}^- + P_{\pm j}^{1-})}{4 \cosh^2(\theta_{1R}^-) + L^{1-}}, \quad (32)$$

where $A_{\pm j}^{1-} = 2e^{(d_1^{(\pm j)} + \ln b_1^{(\pm j)} - c_{11})/2}$, $L^{1-} = e^{(l_1 - (c_{11}/2))} - 2$, $P_{\pm j}^{1-} = e^{(2d_1^{(\pm j)} - 2\ln b_1^{(\pm j)} - c_{11})/4}$, and $\theta_{1R}^- = \theta_{1R} + (c_{11}/4)$.

(2) Soliton S^{2-} :

$$\Phi_{\pm j} \longrightarrow 2A_{\pm j}^{2-} e^{i\theta_{2l}} \frac{\cosh(\theta_{2R}^- + P_{\pm j}^{2-})}{4 \cosh^2(\theta_{2R}^-) + L^{2-}}, \quad (33)$$

where $A_{\pm j}^{2-} = 2e^{(W_{12}^{(\pm j)} + h_{12}^{(\pm j)} - l_5 - c_{11})/2}$, $L^{2-} = e^{(L_{12} - (l_5 + c_{11}/2))} - 2$, $P_{\pm j}^{2-} = e^{(2W_{12}^{(\pm j)} - 2h_{12}^{(\pm j)} - l_5 + c_{11})/4}$, and $\theta_{2R}^- = \theta_{2R} + ((l_5 - c_{11})/4)$.

(B) After the collision.

(1) Soliton S^{1+} :

$$\Phi_{\pm j} \longrightarrow 2A_{\pm j}^{1+} e^{i\theta_{1l}} \frac{\cosh(\theta_{1R}^+ + P_{\pm j}^{1+})}{4 \cosh^2(\theta_{1R}^+) + L^{1+}}, \quad (34)$$

where $A_{\pm j}^{1+} = 2e^{(W_{21}^{(\pm j)} + h_{21}^{(\pm j)} - l_5 - c_{22})/2} = (((k_1^* - k_2)(k_1^* + k_2)) / ((k_1^* - k_2^*)(k_1 + k_2^*))) A_{\pm j}^{1-}$, $L^{1+} = e^{L_{21} - ((l_5 + c_{22})/2)} - 2$, $\theta_{1R}^+ = \theta_{1R} + ((l_5 - c_{22})/4)$, and $P_{\pm j}^{1+} = e^{(2W_{21}^{(\pm j)} - 2h_{21}^{(\pm j)} - l_5 + c_{22})/4}$.

(2) Soliton S^{2+} :

$$\Phi_{\pm j} \longrightarrow 2A_{\pm j}^{2+} e^{i\theta_{2l}} \frac{\cosh(\theta_{2R}^+ + P_{\pm j}^{2+})}{4 \cosh^2(\theta_{2R}^+) + L^{2+}}, \quad (35)$$

where $A_{\pm j}^{2+} = 2e^{(d_2^{(\pm j)} + \ln b_2^{(\pm j)} - c_{22})/2} = (((k_1^* - k_2^*)(k_1^* + k_2)) / ((k_1 - k_2)(k_1 + k_2^*))) A_{\pm j}^{2-}$, $L^{2+} = e^{(l_2 - (c_{22}/2))} - 2$, $P_{\pm j}^{2+} = e^{(2d_2^{(\pm j)} - 2\ln b_2^{(\pm j)} - c_{22})/4}$, and $\theta_{2R}^+ = \theta_{2R} + (c_{22}/4)$. Figures 7–9, respectively, describe the collisions between two one-peak solitons, two two-peak solitons, and one-peak and two-peak solitons in the polar state. From equations (32)–(35), one can find that the collisions are elastic in this case. As shown in Figures 7–9, solitons with one or two peaks collide with each other and the intensity and shape of each soliton keeps invariant after the collision.

5. Conclusions

In this paper, we have investigated the solitons and their collisions for the five-component GP equations, i.e., equation (1), which can describe the dynamics of an $F = 2$ spinor BECs in one dimension. Through the Hirota method and symbolic computation, we have derived one-soliton solutions (8) and two-soliton solutions (16) for equation (1). With different polarization parameters and spin densities, one-soliton solutions (8) have been classified as (a) the one-peak soliton in the

ferromagnetic and cyclic states (as seen in Figures 1 and 2); (b) one- and two-peak solitons in the polar state (as seen in Figure 3). For the two-peak soliton in the polar state, the distance between the two peaks increases inversely with the polarization parameter Λ_1 (as seen in Figure 3). Performing the asymptotic analysis on two-soliton solutions (16), we have analyzed the collisions between two solitons in the same and different states. For the collisions between two solitons in the ferromagnetic state, we have shown the constrained condition, under which the collision is elastic. In addition, we have given the example of the inelastic collision with intensity redistribution (as seen in Figure 4), and it is feasible to exploit the inelastic collision in the design of soliton switching architectures. For the collisions between soliton in the ferromagnetic state and solitons in the polar state, the solitons in the ferromagnetic state keep their characteristics invariant while the shapes and intensities change for those in the polar state after the collision (as seen in Figures 5 and 6). For the collisions between two solitons in the polar state, the procedure occurs only with a phase shift for each soliton at the moment of collision, and the collisions are inelastic (as seen in Figures 7–9). We hope that the results given in this paper will be of value for the applications in the atom laser and coherent atom transport, which can contribute to the realization of quantum information processors or computation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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