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Research Article

A Reinsurance and Investment Game between Two Insurers under the CEV Model

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In this paper, the problem of nonzero-sum stochastic differential game between two competing insurance companies is considered, i.e., the relative performance concerns. A certain proportion of reinsurance can be taken out by each insurer to control his own risk. Moreover, each insurer can invest in a risk-free asset and risk asset with the price dramatically following the constant elasticity of variance (CEV) model. Based on the principle of dynamic programming, a general framework regarding Nash equilibrium for nonzero-sum games is established. For the typical case of exponential utilization, we, respectively, give the explicit solutions of the equilibrium strategy as well as the equilibrium function. Some numerical studies are provided at last which assist in obtaining some economic explanations.

1. Introduction

As the insurers must invest their wealth in the financial market for wealth management, the most proper investment in the financial market is one of the main problems faced by actuarial practitioners and researchers. In addition, risk management is also an important issue faced by insurance companies. In practice, reinsurance is a significant tool for insurers to diversify risk. Therefore, the optimal reinsurance problem has been widely concerned in the financial and actuarial literature. More and more scholars have studied these problems from different perspectives. Aiming at maximizing the expected terminal wealth utility, Bai and Guo [1] investigated the optimal proportional reinsurance as well as the investment problem of insurers, obtaining the explicit expression of the optimal strategy with the constraint of no-shorting. Mean-variance criterion is also a significant objective function in addition to the minimization of ruin probability as well as utility maximization. Thus, the stochastic dynamic programming method assists in studying the problem of reinsurance and investment with robust optimal excess loss of fuzzy aversion insurer, with jump, obtaining the optimal strategy together with the optimal value function, Li et al. [2].

The studies above assume that risk asset prices are subjected to the geometric Brownian motion (GBM), demonstrating the deterministic volatility exhibited by risk asset prices. However, as mentioned by much empirical evidence, risky assets prices usually exhibit random volatility and the volatility varies with the price of risky assets, Beckers [3] and Campbell [4]. Cox and Ross [5] proposed a constant variance elastic (CEV) process for modeling the underlying stock price about European options to overcome the shortcomings of GBM. The random volatility CEV process of this model naturally expands the GBM and can capture the volatility skewness. At present, the CEV model is generally advocated in the actuarial literature. Li et al. [6] investigated the problem of optimal proportional reinsurance and investment to maximize the product utility of the insurers and the reinsurers, the CEV model. As assumed by Lin and Qian [7], the risky asset price obeys the CEV model. They focused on studying the selection of the optimal time-

consistent reinsurance-investment strategy in the compound Poisson risk model. In order to solve the portfolio problem, Zhao and Rong [8] used the CEV process for describing relevant risk assets price. Wang et al. [9] applied a jump diffusion risk process to solve the optimal investment problem for an insurer and a reinsurer under the CEV model.

Besides, nonzero-sum stochastic difference game has many insightful applications in insurance field. Bensoussan et al. [10] established a nonzero-sum stochastic differential game of reinsurance and investment between two competitive insurers affected by systematic risks in a compound Poisson risk model. Deng et al. [11] considered a nonzero-sum stochastic differential game of reinsurance and investment between two CARA insurers with regard to a financier market which involves a defaultable corporate bond. Based on Chen et al. [12], the competition between them is considered as a nonzero-sum stochastic differential game. Hu and Wang [13] investigated the optimal time-consistent investment as well as reinsurance regarding 2 insurance managers with mean variance, and the Nash equilibrium policies and value functions were derived. The study is the first one that focuses on investigating the optimal reinsurance and investment problem with the CEV model under the nonzero-sum stochastic differential game framework.

In this paper, following the framework of Bensoussan et al. [10], a stochastic differential reinsurance and investment game is proposed between two insurers under the CEV model. In our problem setting, the insurers with the classical Cramér–Lundberg diffusion-approximated model are capable of purchasing proportional reinsurance as well as investing in an asset without risk and a risky asset governed by the CEV model. Each insurer aims at finding the equilibrium investment-reinsurance strategy, meanwhile pursuing maximization of expected utility. In line with the setup described above, stochastic control theory assists in formulating, a nonzero-sum game problem between two competitive insurers and obtaining the Hamilton–Jacobi–Bellman (HJB) equation, and the closed-form expressions for equilibrium strategies are obtained. Finally, sensitivity analysis is given in many numerical examples or illustrating the finding.

Our paper contributes to the literature in that the CEV model is introduced into the problem of reinsurance and investment facing two competitive insurers with relative performance concerns. Moreover, this paper has obtained the analytical solutions for the balanced investment-reinsurance strategies; thus, we can analytically examine the influences of the volatility skew on each insurer's equilibrium strategies.

The rest of the paper is divided into five sections. The basic model of the two competitive insurers is introduced in Section 2. The optimal reinsurance-investment problem affected by relative performance concern is formulated in Section 3. We show that this problem is equivalent to the nonzero-sum game between two insurers. Section 4 focuses on deriving the HJB equation under general utility function, followed by obtaining the explicit solutions to Nash equilibrium strategies and the corresponding value functions when both insurers have exponential utilities. Numerical studies are provided in Section 5 in detail for discussing how model parameters affect the equilibrium strategies. Section 6 is the conclusion, together with providing useful reference to future research.

2. Financial Market Model and Assumptions

Models as well as some related basic assumptions are given in the section. As supposed, the financial market harbors the continuous trading of all assets with time, which do not involve transaction costs or taxes. $(\Omega, \mathcal{F}, P)$ refers to an intact probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. T denotes a positive finite constant that represents the terminal time. $\mathcal{F}_t$ affects the decision that is made at t, which represents the information available until t, thereby regarding $T - t$ as the horizon at t.

2.1. Surplus Process. Consider a market with two competitive insurers. Following Bensoussan et al. [10], the surplus process of the insurer $k \in \{1, 2\}$ is modeled by the standard Cramér–Lundberg diffusion approximation, expressed as $\{R_k(t)\}_{t\geq 0}$. Klugman et al. [14] discussed the diffusion approximation in the insurance models. To be specific, $R_k(t)$ meets the stochastic differential equation (SDE):

$$\mathrm{d}R_k(t) = [p_k - \lambda_k \mathbb{E}[\xi_k]]\mathrm{d}t + \sqrt{\lambda_k \mathbb{E}\left[\xi_k^2\right]} dB_k(t), \tag{1}$$

where $p_k > 0$ and $\lambda_k > 0$ represent the premium rate and arrival rate of claims, respectively. $\xi_k \neq 0$ refers to a random variable that represents the claim size. $\mathbb{E}[\xi_k^2] < \infty$ and $\{B_k(t)\}_{t\geq 0}$ form a standard Brownian motion, $k = 1, 2$. The correlation between $\{B_1(t)\}_{t\geq 0}$ and $\{B_2(t)\}_{t\geq 0}$ demonstrates the dependence exhibited by two insurers, and $E[\mathrm{d}B_1(t)\mathrm{d}B_2(t)] = \rho\,\mathrm{d}t$, for $\rho \in [-1, 1]$.

2.2. Reinsurance and Investment Opportunities. Supposing we have a reinsurance firm, insurer $k \in \{1, 2\}$ is capable of managing his insurance risks via buying proportional reinsurance protection at a premium rate $\iota_k > p_k > 0$. $1 - a_k(t)$ refers to the reinsurance proportion regarding insurer $k \in \{1, 2\}$ at time t. $(1 - a_k(t))100\%$ of the claims are born by the reinsurance company and the rest $a_k(t)\%$ comes to insurer k. The reinsurance strategy adopted by insurer $k \in \{1, 2\}$ is expressed by $\{a_k(t)\}_{t\geq 0}$, a process of $\mathcal{F}_t$-which can be measured progressively, with value in $[0, 1]$. $A_k \triangleq \{a_k(t) : 0 \leq a_k(t) \leq 1, t \geq 0\}$ refers to the set of convex reinsurance strategies adopted by insurer k. With reinsurance, the surplus process $\left\{R_k^{a_k}(t)\right\}_{t\geq 0}$ regarding insurer $k \in \{1, 2\}$ is

$$\begin{aligned}\mathrm{d}R_k^{a_k}(t) &= [p_k - (1 - a_k(t))\iota_k - a_k(t)\lambda_k \mathbb{E}[\xi_k]]\mathrm{d}t \\ &\quad + a_k(t)\sqrt{\lambda_k \mathbb{E}\left[\xi_k^2\right]}\mathrm{d}B_k(t), \\ &=: [m_k + \theta_k a_k(t)]\mathrm{d}t + \delta_k a_k(t)\mathrm{d}B_k(t),\end{aligned} \tag{2}$$

where $m_k = p_k - \iota_k < 0$ is the premium difference, $\theta_k = \iota_k - \lambda_k \mathbb{E}[\xi_k]$ is the relative safety loading, and $\delta_k = \sqrt{\lambda_k \mathbb{E}[\xi_k^2]}$ is the volatility of the claim process.

Each insurer, besides buying reinsurance protection, can invest in one bond and one stock. The evolution of price process $S_0(t)$ regarding the bond is based on the ordinary differential equation (ODE):

$$\begin{aligned} \mathrm{d}S_0(t) &= rS_0(t)\mathrm{d}t, \\ S_0(0) &= 1, \end{aligned} \tag{3}$$

where $r > 0$ represents the risk-free interest rate. The price process $S(t)$ of the stock meets the CEV model:

$$\begin{aligned} \mathrm{d}S(t) &= S(t)\left[\mu \mathrm{d}t + \sigma S(t)^{\beta} \mathrm{d}W(t)\right], \\ S(0) &= s_0 > 0. \end{aligned} \tag{4}$$

Thereinto, $\mu > r > 0$ means the proper stock rate, β refers to the elasticity parameter, and $\beta \le 0$ based on Gao [15]. $\sigma S(t)^{\beta}$ refers to the instantaneous stock volatility. $\{W(t)\}_{t\ge 0}$ is a standard Brownian motion. For notational simplicity, $\{B_k(t) : t \ge 0, k = 1, 2\}$ is assumed to be independent of $\{W(t)\}_{t\ge 0}$.

$\{b_k(t)\}_{t\ge 0}$ refers to the $\mathcal{F}_t$-progress which can be measured progressively, and $\{X_k^{\pi_k}(t)\}_{t\ge 0}$ refers to the surplus process regarding insurer k, for $k = 1, 2$, where $b_k(t)$ and $X_k^{\pi_k}(t) - b_k(t)$ denote the amount of money invested in the stock and in the bond, respectively. $B_k \triangleq \{b_k(t) : t \ge 0\}$ as the set of investment strategies adopted by insurer k. A reinsurance-investment strategy of insurer k is then denoted as $\{\pi_k(t)\}_{t\ge 0} \triangleq \{(a_k(t), b_k(t))\}_{t\ge 0} \in A_k \times B_k$. In line with a reinsurance-investment strategy $\pi_k(t)$, the surplus process of insurer k, for $k = 1, 2$, is expressed as

$$\begin{cases} \mathrm{d}X_k^{\pi_k}(t) = \left[X_k^{\pi_k}(t) - b_k(t)\right]\dfrac{\mathrm{d}S_0(t)}{S_0(t)} + b_k(t)\dfrac{\mathrm{d}S(t)}{S(t)} + \mathrm{d}R_k^{a_k}(t) \\ \qquad\qquad = \left[rX_k^{\pi_k}(t) + m_k + \theta_k a_k(t) + (\mu - r)b_k(t)\right]\mathrm{d}t + \delta_k a_k(t)\mathrm{d}B_k(t) + \sigma b_k(t)S(t)^{\beta}\mathrm{d}W(t), \\ X_k^{\pi_k}(0) = X_k > 0. \end{cases} \tag{5}$$

We denote by $\Pi_k \triangleq \mathbb{A}_k \times \mathbb{B}_k \subset \mathbb{A}_k \times \mathbb{B}_k$ the set of convex strategies $\pi_k = (a_k, b_k) \in \Pi_k$ of insurer $k \in \{1, 2\}$ satisfying the condition that $0 \le a_k(t) \le 1$ and $\mathrm{E}[\int_0^T (b_k(t))^2 \mathrm{d}t] < \infty$. We shall refer a strategy $\pi_k \in \Pi_k$ to be an *admissible strategy*. Based on standard stochastic control theory [16], for all $\pi_k = (a_k, b_k) \in \Pi_k$, and for any initial condition $(t, x) \in [0, T] \times R$, the SDE in (5) has a unique strong solution. In this case, we also have

$$E\left[\sup_{0\le t\le T}\left|X_k^{\pi_k}(t)\right|^2\right] < \infty, \quad k = 1, 2. \tag{6}$$

3. Optimal Strategies Affected by the Relative Performance Concerns

Suppose that insurer $k \in \{1, 2\}$ involves a utility function U_k which is strictly with a strict concave and a continuous differentiability on $(-\infty, \infty)$. The optimization problem of each insurer lies in optimally selecting a reinsurance-investment strategy $\pi_k \in \Pi_k$, for $k = 1, 2$, for maximizing the expected relative performance utility compared with competitors at the terminal time T. More precisely, we have the following.

Problem 1. The problem of optimal reinsurance as well as investment affected by two competitive insurance companies under the expected utility framework is actually a coupled stochastic optimization problem.

Given the strategy of their competitor $j \in \{1, 2\}$, that is, $\pi_j = (a_j, b_j)$, the objective function, denote as $J_k(t, x_k, x_j, \pi_k(t), \pi_j(t))$, of insurer $k \ne j$ aims at finding an optimal strategy $\pi_k^* \triangleq (a_k^*, b_k^*)$, that is,

$$\max_{\pi_k \in \Pi_k} J_k\left(t, x_k, x_j, \pi_k(t), \pi_j(t)\right) = J_k\left(t, x_k, x_j, \pi_k^*(t), \pi_j^*(t)\right), \tag{7}$$

$$\begin{aligned} &J_k\left(t, x_k, x_j, \pi_k(t), \pi_j(t)\right) \\ &\quad \triangleq E\left[U_k\left((1 - \kappa_k)X_k^{\pi_k}(T) + \kappa_k\left(X_k^{\pi_k}(T) - X_j^{\pi_j}(T)\right)\right)\right. \\ &\qquad \left. \cdot \left| X_k^{\pi_k}(t) - \kappa_k X_j^{\pi_j}(t) = x_k - \kappa_k x_j\right.\right] \\ &\quad = E\left[U_k\left(X_k^{\pi_k}(T) - \kappa_k X_j^{\pi_j}(T)\right)\right| X_k^{\pi_k}(t) - \kappa_k X_j^{\pi_j} \\ &\qquad \left. \cdot (t) = x_k - \kappa_k x_j\right]. \end{aligned} \tag{8}$$

The parameter $\kappa_k \in [0, 1]$ in (8), for $k = 1, 2$, denotes the sensitivity parameter exhibited by insurer k to his competitor's performance as different insurer generally has different perception of the degree of influence of their competitor's surplus on his strategy, that is, $\kappa_1 \ne \kappa_2$. Problem 1 is a typical example of the nonzero-sum games between two insurers. Consequently, the solution of Problem 1 is the Nash equilibrium regarding the nonzero-sum game between two insurers, where the Nash equilibrium is the strategy $\pi_k^* \triangleq (a_k^*, b_k^*) \in \Pi_k$, such that, for all $\widehat{\pi}_1 \triangleq (\widehat{a}_1, \widehat{b}_1)$ and $\widehat{\pi}_2 \triangleq (\widehat{a}_2, \widehat{b}_2)$,

$$\begin{cases} J_1\left(t, x_1, x_2, \widehat{\pi}_1, \pi_2^*\right) \le J_1\left(t, x_1, x_2, \pi_1^*, \pi_2^*\right), \\ J_2\left(t, x_1, x_2, \pi_1^*, \widehat{\pi}_2\right) \le J_2\left(t, x_1, x_2, \pi_1^*, \pi_2^*\right). \end{cases} \tag{9}$$

4. Nash Equilibrium

4.1. General Case. The section focuses on using dynamic programming principle to solve the Nash equilibrium of Problem 1. To this end, the relative surplus process of insurer k is denoted by $Z_k^{\pi_k}(t) \triangleq X_k^{\pi_k}(t) - \kappa_k X_j^{\pi_j}(t)$, for $k = 1, 2$; hence it follows that

$$\begin{cases} \mathrm{d}Z_k^{\pi_k}(t) &= \left[rZ_k^{\pi_k}(t) + m_k - \kappa_k m_j + \theta_k a_k(t) - \kappa_k \theta_j a_j(t) + (\mu - r)\left(b_k(t) - \kappa_k b_j(t)\right)\right]\mathrm{d}t \\ &\quad + \delta_k a_k(t)\mathrm{d}B_k(t) - \kappa_k \delta_j a_j(t)\mathrm{d}B_j(t) + \sigma S(t)^{\beta}\left[b_k(t) - \kappa_k b_j(t)\right]\mathrm{d}W(t), \\ Z_k^{\pi_k}(0) &= X_k - \kappa_k X_j, \end{cases} \tag{10}$$

where $k \neq j \in \{1, 2\}$. Define $V^k(t, z, s)$, for $k = 1, 2$, to be the value function as follows:

$$V^k(t, z, s) \triangleq \sup_{\pi_k \in \Pi_k} E\left[U_k\left(Z_k^{\pi_k}(T)\right) \,\middle|\, Z_k^{\pi_k}(t) = z, S(t) = s\right], \tag{11}$$

for $k, j = 1, 2$ with $k \neq j$. In addition, let $\Gamma = [0, T) \times \mathbb{R} \times \mathbb{R}$ and $\overline{\Gamma} = [0, T] \times \mathbb{R} \times \mathbb{R}$; for $W^k \in \mathscr{C}^{1,2,2}(\Gamma) \cap \mathscr{C}^0(\overline{\Gamma})$, where $k \neq j \in \{1, 2\}$, denote $\mathscr{L}^{\pi_k}$ to be the infinitesimal generator of $Z_k^{\pi_k}$ with an admissible strategy of insurer k, $\pi_k \in \Pi_k$:

$$\begin{aligned} \mathscr{L}^{\pi_k} W^k(t, z, s) \triangleq{}& \mu s W_s^k(t, z, s) + \frac{1}{2}\sigma^2 s^{2\beta+2} W_{ss}^k(t, z, s) + \left[rz + m_k - \kappa_k m_j + \theta_k a_k(t) - \kappa_k \theta_j a_j^*(t)\right. \\ &\left. + (\mu - r)\left(b_k(t) - \kappa_k b_j^*(t)\right)\right] W_z^k(t, z, s) + \frac{1}{2}\left[\delta_k^2 a_k(t)^2 - 2\rho\kappa_k\delta_k\delta_j a_k(t) a_j^*(t)\right] \\ &+ \kappa_k^2 \delta_j^2 a_j^*(t)^2 + \sigma^2 s^{2\beta}\left(b_k(t)^2 - 2\kappa_k b_k(t) b_j^*(t) + \kappa_k^2 b_j^*(t)^2\right) W_{zz}^k(t, z, s) \\ &+ \sigma^2 s^{2\beta+1}\left[b_k(t) - \kappa_k b_j^*(t)\right] W_{zs}^k(t, z, s). \end{aligned} \tag{12}$$

Similar to Theorem 1 proposed by Siu et al. [17] and Theorem 2 proposed by Bensoussan et al. [10], the verification theorem is set forth as follows. Here, the proof is omitted.

Theorem 1. *Let $W^k \in \mathscr{C}^{1,2,2}(\Gamma) \cap \mathscr{C}^0(\overline{\Gamma})$, for $k = 1, 2$, be the solution of the following Hamilton–Jacobi–Bellman (HJB) equation:*

$$\begin{cases} 0 = W_t^k(t, z, s) + \sup\limits_{\pi_k \in \Pi_k} \mathscr{L}^{\pi_k} W^k(t, z, s), \quad \text{for } 0 \le t < T, \\ W^k(T, z, s) = U_k(z), \end{cases} \tag{13}$$

and, with regard to the relative surplus process $Z_k^{\pi_k}$ associated with an admissible strategy $\pi_k \in \Pi_k = \mathscr{A}_k \times \mathscr{B}_k$, we have

$$E\left[U_k\left(Z_k^{\pi_k}(T)\right) \,\middle|\, Z_k^{\pi_k}(t) = z, S(t) = s\right] \le W^k(t, z, s). \tag{14}$$

Denote $\pi_k^* = (a_k^*, b_k^*)$, where $a_k^* \in \mathscr{A}_k$ and $b_k^* \in \mathscr{B}_k$ are given as follows:

$$\begin{aligned} a_k^*(t) \triangleq \arg\max_{a_k \in \mathscr{A}_k} \Big\{ &\left[\theta_k a_k(t)\right] W_z^k(t, z, s) + \frac{1}{2} \\ &\cdot \left[\delta_k^2 a_k(t)^2 - 2\rho\kappa_k\delta_k\delta_j a_k(t) a_j^*(t)\right] W_{zz}^k(t, z, s)\Big\}, \end{aligned} \tag{15}$$

$$\begin{aligned} b_k^*(t) \triangleq \arg\max_{b_k \in \mathscr{B}_k} \Big\{ &\left[(\mu - r) b_k(t)\right] W_z^k(t, z, s) + \frac{1}{2}\sigma^2 s^{2\beta} \\ &\cdot \left[b_k(t)^2 - 2\kappa_k b_k(t) b_j^*(t)\right] W_{zz}^k(t, z, s) \\ &+ \left[\sigma^2 s^{2\beta+1} b_k(t)\right] W_{zs}^k(t, z, s)\Big\}, \end{aligned} \tag{16}$$

and let $Z_k^{\pi_k^*}$ be the corresponding relative surplus process of insurer k. Then, we have

$$V^k(t,z,s) = \mathrm{E}\left[U_k\left(Z_k^{\pi_k^*}(T)\right) \middle| Z_k^{\pi_k^*}(t) = z, S(t) = s\right] = W^k(t,z,s). \tag{17}$$

Theorem 1 gives the necessary condition for the equilibrium strategy to exist in Problem 1. In the following, we proceed to analyze the sufficient condition which enables the equilibrium strategy to exist.

Assuming that $W_{zz}^k(t,z,s) \neq 0$, for $(t,z,s) \in \Gamma$, the first-order conditions of (15) and (16) give

$$\begin{cases} a_k^*(t) = \left[\left(\rho\kappa_k \frac{\delta_j}{\delta_k} a_j^*(t) - \frac{\theta_k W_z^k(t,z,s)}{\delta_k^2 W_{zz}^k(t,z,s)}\right) \wedge 1\right]^+, \\ b_k^*(t) = \kappa_k b_j^*(t) - \dfrac{(\mu - r) W_z^k(t,z,s)}{\sigma^2 s^{2\beta} W_{zz}^k(t,z,s)} - \dfrac{s W_{zs}^k(t,z,s)}{W_{zz}^k(t,z,s)}. \end{cases} \tag{18}$$

with $a_k^*(t)$, $b_k^*(t)$ expressed in (18), corresponding to HJB equation specific to insurer $k \neq j \in \{1,2\}$ being

$$\begin{aligned} 0 = {} & W_t^k(t,z,s) + \mu s W_s^k(t,z,s) + \left[rz + m_k - \kappa_k m_j - \kappa_k\left(\theta_j - \rho\frac{\delta_j}{\delta_k}\theta_k\right)a_j^*(t)\right] W_z^k(t,z,s) \\ & + \frac{1}{2}\sigma^2 s^{2\beta+2} W_{ss}^k(t,z,s) + \frac{1}{2}\left[\left(1-\rho^2\right)\kappa_k^2\delta_j^2 a_j^*(t)^2\right] W_{zz}^k(t,z,s) - \frac{1}{2}\left[\frac{\theta_k^2}{\delta_k^2} + \frac{(\mu-r)^2}{\sigma^2 s^{2\beta}}\right]\frac{\left(W_z^k(t,z,s)\right)^2}{W_{zz}^k(t,z,s)} \\ & - \frac{\sigma^2 s^{2\beta+2}\left(W_{zs}^k(t,z,s)\right)^2}{2W_{zz}^k(t,z,s)} - \frac{(\mu-r) s W_z^k(t,z,s) W_{zs}^k(t,z,s)}{W_{zz}^k(t,z,s)}, \end{aligned} \tag{19}$$

and the terminal condition is

$$W^k(T,z,s) = U_k(z), \quad k = 1,2. \tag{20}$$

Theorem 2 presents the sufficient condition allowing the Nash equilibrium to exist in Problem 1.

Theorem 2. *Assume that $W_{zz}^k(t,z,s) \neq 0$, for $k = 1,2$, where W^k denotes the solution to (19). The Nash equilibrium reinsurance-investment strategy specific to Problem 1 acts as the solution of the coupled system of nonlinear equations as follows:*

$$\begin{cases} a_1^*(t) = \left[\left(\rho\kappa_1\frac{\delta_2}{\delta_1}a_2^*(t) - \dfrac{\theta_1 W_z^1(t,z,s)}{\delta_1^2 W_{zz}^1(t,z,s)}\right)\wedge 1\right]^+, \\ b_1^*(t) = \kappa_1 b_2^*(t) - \dfrac{(\mu-r)W_z^1(t,z,s)}{\sigma^2 s^{2\beta} W_{zz}^1(t,z,s)} - \dfrac{s W_{zs}^1(t,z,s)}{W_{zz}^1(t,z,s)}, \\ a_2^*(t) = \left[\left(\rho\kappa_2\frac{\delta_1}{\delta_2}a_1^*(t) - \dfrac{\theta_2 W_z^2(t,z,s)}{\delta_2^2 W_{zz}^2(t,z,s)}\right)\wedge 1\right]^+, \\ b_2^*(t) = \kappa_2 b_1^*(t) - \dfrac{(\mu-r)W_z^2(t,z,s)}{\sigma^2 s^{2\beta} W_{zz}^2(t,z,s)} - \dfrac{s W_{zs}^2(t,z,s)}{W_{zz}^2(t,z,s)}. \end{cases} \tag{21}$$

W^1 *and* W^2 denote the Nash equilibrium functions, subsequently acting as the solutions of the coupled PDEs system as follows:

$$\begin{cases} 0 = W_t^1(t,z,s) + \mu s W_s^1(t,z,s) + \left[rz + m_1 - \kappa_1 m_2 - \kappa_1\left(\theta_2 - \rho\dfrac{\delta_2}{\delta_1}\theta_1\right)a_2^*(t)\right] W_z^1(t,z,s) \\ \quad + \dfrac{1}{2}\sigma^2 s^{2\beta+2} W_{ss}^1(t,z,s) + \dfrac{1}{2}\left[\left(1-\rho^2\right)\kappa_1^2\delta_2^2 a_2^*(t)^2\right] W_{zz}^1(t,z,s) \\ \quad - \dfrac{1}{2}\left[\dfrac{\theta_1^2}{\delta_1^2} + \dfrac{(\mu-r)^2}{\sigma^2 s^{2\beta}}\right]\dfrac{\left(W_z^1(t,z,s)\right)^2}{W_{zz}^1(t,z,s)} - \dfrac{\sigma^2 s^{2\beta+2}\left(W_{zs}^1(t,z,s)\right)^2}{2W_{zz}^1(t,z,s)} - \dfrac{(\mu-r)sW_z^1(t,z,s)W_{zs}^1(t,z,s)}{W_{zz}^1(t,z,s)}, \\ 0 = W_t^2(t,z,s) + \mu s W_s^2(t,z,s) + \left[rz + m_2 - \kappa_2 m_1 - \kappa_2\left(\theta_1 - \rho\dfrac{\delta_1}{\delta_2}\theta_2\right)a_1^*(t)\right] W_z^2(t,z,s) \\ \quad + \dfrac{1}{2}\sigma^2 s^{2\beta+2} W_{ss}^2(t,z,s) + \dfrac{1}{2}\left[\left(1-\rho^2\right)\kappa_2^2\delta_1^2 a_1^*(t)^2\right] W_{zz}^2(t,z,s) \\ \quad - \dfrac{1}{2}\left[\dfrac{\theta_2^2}{\delta_2^2} + \dfrac{(\mu-r)^2}{\sigma^2 s^{2\beta}}\right]\dfrac{\left(W_z^2(t,z,s)\right)^2}{W_{zz}^2(t,z,s)} - \dfrac{\sigma^2 s^{2\beta+2}\left(W_{zs}^2(t,z,s)\right)^2}{2W_{zz}^2(t,z,s)} - \dfrac{(\mu-r)sW_z^2(t,z,s)W_{zs}^2(t,z,s)}{W_{zz}^2(t,z,s)}, \end{cases} \tag{22}$$

and the terminal condition is $W^1(T,z,s) = U_1(z)$; $W^2(T,z,s) = U_2(z)$.

Based on Theorem 2, the existence of Nash equilibrium is equivalent to the solvability of the coupled systems in (21), and the solvability equals that of the coupled PDEs in (22). Bensoussan et al. [10] and Siu et al. [17] mentioned that it was very difficult to establish the general existence of the solution to (22) for any $T > 0$. Nevertheless, for a sufficiently small time $T > 0$, Cauchy–Kowalevski theorem can assisted in establishing the local existence as well as the uniqueness regarding the solution to (22). It is interesting to find that the corresponding coupled equations in (21) and the coupled PDEs in (22) can be explicitly solved, specific to the representative case concerning CARA insurers.

4.2. CARA Insurers. The section pays attention to discussing the constant absolute risk aversion (CARA) insurer $k \in \{1,2\}$ with an exponential utility function:

$$U_k(z) = -\frac{1}{\gamma_k}\exp\left(-\gamma_k z\right), \tag{23}$$

where $\gamma_k > 0$ is the risk aversion coefficient of insurer k. Based on Theorem 3, the Nash equilibrium reinsurance-investment strategies and the corresponding equilibrium value functions in Theorem 2 have closed-form solutions specific to the situation involving two CARA insurers.

Theorem 3. *It is assumed that* $\kappa_1\kappa_2 < 1$ *and insurer* k*, for* $k = 1,2$ *involves an exponential utility function (23) and the relative surplus process* $\left\{Z_k^{\pi_k}(t)\right\}_{t\geq 0}$ *in (10). Then, the solution to the coupled PDE system in (22) is* $V^k(t,z,s)$*, for* $k = 1,2$*, which admits the explicit form as follows:*

$$V^k(t,z,s) = -\frac{1}{\gamma_k}\exp\left\{\left(-\gamma_k z - f^k(t)\right)e^{r(T-t)} + A(t) + B(t)s^{-2\beta}\right\}, \tag{24}$$

$$\begin{aligned} f^k(t) = {} & \frac{\gamma_k\left[m_k - \kappa_k m_j - \kappa_k\left(\theta_j - \rho\left(\delta_j/\delta_k\right)\theta_k\right)a_j^*(t)\right]}{r} \\ & \cdot\left[1 - e^{-r(T-t)}\right] - \frac{\left(1-\rho^2\right)\kappa_k^2\gamma_k^2\delta_j^2 a_j^*(t)^2}{4r} \\ & \times\left[e^{r(T-t)} - e^{-r(T-t)}\right] + \frac{\theta_k^2(T-t)}{2\delta_k^2}e^{-r(T-t)}, \end{aligned} \tag{25}$$

$$A(t) = -\frac{(2\beta+1)(\mu-r)^2}{4r}\left(T - t - \frac{1 - e^{-2r\beta(T-t)}}{2r\beta}\right), \tag{26}$$

$$B(t) = -\frac{(\mu-r)^2}{4\sigma^2 r\beta}\left(1 - e^{-2r\beta(T-t)}\right). \tag{27}$$

For $0 \leq t \leq T$, define

$$\begin{cases} \widehat{a}_1(t) \triangleq \dfrac{e^{-r(T-t)}}{1-\rho^2\kappa_1\kappa_2}\left(\dfrac{\rho\kappa_1\theta_2}{\delta_1\delta_2\gamma_2} + \dfrac{\theta_1}{\delta_1^2\gamma_1}\right), \\ \widehat{a}_2(t) \triangleq \dfrac{e^{-r(T-t)}}{1-\rho^2\kappa_1\kappa_2}\left(\dfrac{\rho\kappa_2\theta_1}{\delta_1\delta_2\gamma_1} + \dfrac{\theta_2}{\delta_2^2\gamma_2}\right), \end{cases} \tag{28}$$

and then the reinsurance strategy (a_1^*, a_2^*) at equilibrium admits one of the following forms, for $k \neq j \in \{1,2\}$,

(i) If $\widehat{a}_1(t) \geq 0$ and $\widehat{a}_2(t) \geq 0$, then

(a) If $\widehat{a}_1(t) \leq 1$ and $\widehat{a}_2(t) \leq 1$, then $(a_1^*(t), a_2^*(t)) = (\widehat{a}_1(t), \widehat{a}_2(t))$.

(b) If $\widehat{a}_1(t) \leq 1$ and $\widehat{a}_2(t) > 1$, then

$$\left(a_1^*(t), a_2^*(t)\right) = \left(\rho\kappa_1\frac{\delta_2}{\delta_1} + \frac{e^{-r(T-t)}\theta_1}{\delta_1^2\gamma_1}, 1\right). \tag{29}$$

(c) If $\widehat{a}_1(t) > 1$ and $\widehat{a}_2(t) \leq 1$, then

$$\left(a_1^*(t), a_2^*(t)\right) = \left(1, \rho\kappa_2\frac{\delta_1}{\delta_2} + \frac{e^{-r(T-t)}\theta_2}{\delta_2^2\gamma_2}\right). \tag{30}$$

(d) If $\widehat{a}_1(t) > 1$ and $\widehat{a}_2(t) > 1$, then $(a_1^*(t), a_2^*(t)) = (1,1)$.

(ii) In other cases, the following statements are true:

$$\eta \triangleq \frac{\theta_1\delta_2\gamma_2}{\theta_2\delta_1\gamma_1}, \tag{31}$$

(a) If $\kappa_2\eta \geq -1/\rho$ and $1/\kappa_1\eta \geq -\rho$, then

$$\left(a_1^*(t), a_2^*(t)\right) = \left(\left(\frac{e^{-r(T-t)}\theta_1}{\delta_1^2\gamma_1}\right)\wedge 1, \left(\rho\kappa_2\frac{\delta_1}{\delta_2} + \frac{e^{-r(T-t)}\theta_2}{\delta_2^2\gamma_2}\right)^+\right). \tag{32}$$

(b) If $\kappa_2\eta < -1/\rho$ and $1/\kappa_1\eta < -\rho$, then

$$\left(a_1^*(t), a_2^*(t)\right) = \left(\left(\rho\kappa_1\frac{\delta_2}{\delta_1} + \frac{e^{-r(T-t)}\theta_1}{\delta_1^2\gamma_1}\right)^+, \frac{e^{-r(T-t)}\theta_2}{\delta_2^2\gamma_2}\wedge 1\right), \tag{33}$$

and the investment amount (b_1^*, b_2^*) at equilibrium is

$$\begin{cases} b_1^*(t) = \dfrac{e^{-r(T-t)}}{1-\kappa_1\kappa_2}\left[\left(\dfrac{1}{\gamma_1} + \dfrac{\kappa_1}{\gamma_2}\right)\left(\dfrac{\mu-r}{\sigma^2 s^{2\beta}} - 2\beta s^{-2\beta}B(t)\right)\right], \\ b_2^*(t) = \dfrac{e^{-r(T-t)}}{1-\kappa_1\kappa_2}\left[\left(\dfrac{1}{\gamma_2} + \dfrac{\kappa_2}{\gamma_1}\right)\left(\dfrac{\mu-r}{\sigma^2 s^{2\beta}} - 2\beta s^{-2\beta}B(t)\right)\right]. \end{cases} \tag{34}$$

Proof. Consider the following Ansatz:

$$W^k(t,z,s) = -\frac{1}{\gamma_k}\exp\left\{\left(-\gamma_k z - f^k(t)\right)e^{r(T-t)} + g(t,s)\right\}, \tag{35}$$

and the boundary condition is $f^k(T) = 0$ and $g(T,s) = 0$.

From fd35(35), we have

$$\begin{aligned} W_t^k &= \left[\left(rz\gamma_k + rf^k\right)e^{r(T-t)} - f_t^k e^{r(T-t)} + g_t\right]W^k, \\ W_s^k &= g_s W^k, \\ W_{ss} &= \left(g_s^2 + g_{ss}\right)W^k, \\ W_z^k &= -\gamma_k e^{r(T-t)}W^k, \\ W_{zz}^k &= \gamma_k^2 e^{2r(T-t)}W^k, \\ W_{zs} &= -\gamma_k e^{r(T-t)}g_s W^k. \end{aligned} \tag{36}$$

Under the notations of (35) and (36), we now derive the corresponding $a_k^*(t)$ and $b_k^*(t)$, for $k = 1,2$. From (15), we can find that the minimizer $a_k(t)$ satisfies

$$\begin{aligned} 0 &= \theta_k W_z^k(t,z,s) + \left[\delta_k^2 a_k - \rho\kappa_k\delta_k\delta_j\widehat{a}_j\right]W_{zz}^k(t,z,s) \\ &= -\theta_k\gamma_k e^{r(T-t)}W^k(t,z,s) \\ &\quad + \left[\delta_k^2 a_k - \rho\kappa_k\delta_k\delta_j\widehat{a}_j\right]\gamma_k^2 e^{2r(T-t)}W^k(t,z,s), \end{aligned} \tag{37}$$

$$\widehat{a}_k = \left(\rho\kappa_k\frac{\delta_j}{\delta_k}\right)\widehat{a}_j + e^{-r(T-t)}\frac{\theta_k}{\delta_k^2\gamma_k}. \tag{38}$$

With (38) in mind, by using the similar method presented by Bensoussan et al. [10], we can derive the corresponding $a_k^*(t)$, for $k = 1,2$; here the details are omitted.

On the other hand, the minimizers b_k^* in (15) are

$$b_k^*(t) = \kappa_k b_j^*(t) + e^{-r(T-t)}\frac{\mu-r}{\sigma^2 s^{2\beta}\gamma_k} + e^{-r(T-t)}\frac{sg_s}{\gamma_k}, \tag{39}$$

yielding (26), where $B(t)$ shall be determined in the following.

Substituting (35) and (36) into (19) yields

$$\begin{aligned} 0 &= rf^k e^{r(T-t)} - f_t^k e^{r(T-t)} + g_t + \frac{1}{2}\sigma^2 s^{2\beta+2}g_{ss} - e^{r(T-t)}\gamma_k \\ &\quad \cdot\left[m_k - \kappa_k m_j - \kappa_k\left(\theta_j - \rho\frac{\delta_j}{\delta_k}\theta_k\right)a_j^*(t)\right] \\ &\quad + \frac{1}{2}e^{2r(T-t)}\left(1-\rho^2\right)\kappa_k^2\gamma_k^2\delta_j^2 a_j^*(t)^2 - \frac{\theta_k^2}{2\delta_k^2} - \frac{(\mu-r)^2}{2\sigma^2 s^{2\beta}} + \mathrm{rsg}_s, \end{aligned} \tag{40}$$

for $j = 1,2$ with $j \neq k$, which can be decomposed into two equations:

$$0 = g_t + \frac{1}{2}\sigma^2 s^{2\beta+2}g_{ss} - \frac{(\mu-r)^2}{2\sigma^2 s^{2\beta}} + \mathrm{rsg}_s, \tag{41}$$

$$\begin{aligned} 0 &= -f_t^k + rf^k - \gamma_k\left[m_k - \kappa_k m_j - \kappa_k\left(\theta_j - \rho\frac{\delta_j}{\delta_k}\theta_k\right)a_j^*(t)\right] \\ &\quad + \frac{1}{2}e^{r(T-t)}\left(1-\rho^2\right)\kappa_k^2\gamma_k^2\delta_j^2 a_j^*(t)^2 - e^{-r(T-t)}\frac{\theta_k^2}{2\delta_k^2}. \end{aligned} \tag{42}$$

To solve (41), we assume that $g(t,s)$ is expressed as

$$g(t,s) = A(t) + B(t)s^{-2\beta}, \tag{43}$$

with the boundary conditions $A(T) = 0$ and $B(T) = 0$. Substituting (43) into (41), we have

$$A_t + \beta(2\beta+1)\sigma^2 B(t) + s^{-2\beta}\left[B_t - 2r\beta B(t) - \frac{(\mu-r)^2}{2\sigma^2}\right] = 0. \tag{44}$$

By matching coefficients, we derive

$$\begin{aligned} A_t + \beta(2\beta+1)\sigma^2 B(t) &= 0, \\ A(T) &= 0, \end{aligned} \tag{45}$$

$$\begin{aligned} B_t - 2r\beta B(t) - \frac{(\mu-r)^2}{2\sigma^2} &= 0, \\ B(T) &= 0. \end{aligned} \tag{46}$$

Solving (45) and (46), we can find that the solutions, $A(t)$ and $B(t)$, admit the forms in (26) and (27), respectively.

Finally, from (42), obviously, $f^k(t)$ is a solution of the linear ODE, which admits the form in (25).

Proof of Theorem 3 is completed. □

5. Numerical Studies

The section focuses on conducting many numerical studies to investigate how model parameters affect the equilibrium reinsurance-investment strategy adopted by the CARA insurers as mentioned in Section 4.2. Except as otherwise specified, the numerical studies will be conducted using the model parameters listed in Table 1.

5.1. Optimal Reinsurance Proportion at Equilibrium. Figures 1 and 2 give a diagram regarding Theorem 3 of $a_k^*(0)$, when $k = 1, 2X \sim f_Z$. Figure 1 illustrates how k' (a parameter of risk aversion) owned by insurer γ_k affects the adopted proportional reinsurance strategy $a_k^*(0)$ in the equilibrium state when the insurers are positively correlated ($\rho = 0.5$) while Figure 2 shows the same influence in the negative correlation ($\rho = -0.5$). Firstly, it is observed that with the increase of γ_k, the insurer + tends to show stronger risk-aversiveness. Hence, a smaller retention level $a_k^*(0)$ is selected for transferring a larger amount of risks to the reinsurer. This is consistent with the strategy of optimal proportional reinsurance without competition. For $a_k^*(0)$ represents the proportion retained by the insurer k during the purchase of proportional reinsurance, the γ_k increase will increase the risk that the insurer k transfers to the reinsurance company. In addition, the condition of $a_k^*(0)$% in Theorem 3 guarantees that $a_k^*(0) \in [0,1]$. This is especially obvious in the influence of η_2 on $a_2^*(0)$. As shown in Figures 1 and 2 when $\rho = 0.5$ and $\rho = -0.5$, respectively.

Table 1: Model parameters.

Base parameters						
r	μ	σ	β	ρ	T	s
0.05	0.2	0.4	−1	−0.5/0.5	4	10
Insurer 1						
p_1	ι_1	λ_1	$E[\xi_1]$	$E[\xi_1^2]$	γ_1	κ_1
5	7	0.8	2.5	80	0.1	0.7
Insurer 2						
p_2	ι_2	λ_2	$E[\xi_2]$	$E[\xi_2^2]$	γ_2	κ_2
3	4	0.5	2	50	0.3	0.5

For illustrating the impact brought about by competition, Figures 1 and 2 also show the influence of the sensitivity parameter $k\imath$ of the insurer κ_k on its proportional reinsurance strategy $a_k^*(0)$, for $k = 1, 2$, when $\kappa_k = 0$, $\kappa_k = 0.3$, and $\kappa_k = 0.7$. The parameter κ_k reflects the dependence gradation of competitors of insurance company k' on the terminal wealth; the higher κ_k causes k to pay more attention than that of its competitors to its performance during the terminal period T. Although purchasing proportional reinsurance is capable of reducing the risk facing insurer k, insurer k is required to pay $(1 - a_k^*(0))\iota_k$ to the reinsurance company for the reinsurance protection fee (see (2)), thereby it has a high cost and declines the terminal value of the insurer compared with the competitor, $X_k^{\pi_k}(T) - \kappa_k X_m^{\pi_m^*}(T)$, for $k \neq m \in \{1,2\}$. As shown in Figure 1, if there is a positive relation between the insurance companies, that is, $\rho = 0.5$, insurer k will prefer to pay less to the reinsurance company, that is, smaller $(1 - a_k^*(0))\iota_k$; thus, the dependence parameter κ_k is increased, also meaning an increase in $a_k^*(0)$. The opposite result appears in the case of negative correlation of insurance companies, that is, $\rho = -0.5$. In this case, based on Figure 2, an increase in the dependency parameter κ_k will cause $a_k^*(0)$ to decrease.

5.2. Optimal Investment at Equilibrium. In Figure 3, the optimal investment strategy adopted by insurer k is positively affected at equilibrium, relying on the expected instantaneous rate of the stocks return μ, which can also be explained as a larger μ leads to a higher expected return of the stock. Thus, $k \in \{1,2\}$ will focus on increasing the investment in the stock.

Figure 4 displays how the elasticity coefficient β affects the optimal investment strategy of insurance company k. As can be seen from the figure, there is an optimistic connection between b_k^* and β. With the decrease of beta coefficient, the insurance company $k \in \{1,2\}$ will reduce the investment in risk assets.

As shown in Figure 5, the optimal investment strategy b_k^* acts as a decreasing function of the interest rate r. The increase in the interest rate r will add the attractiveness of the

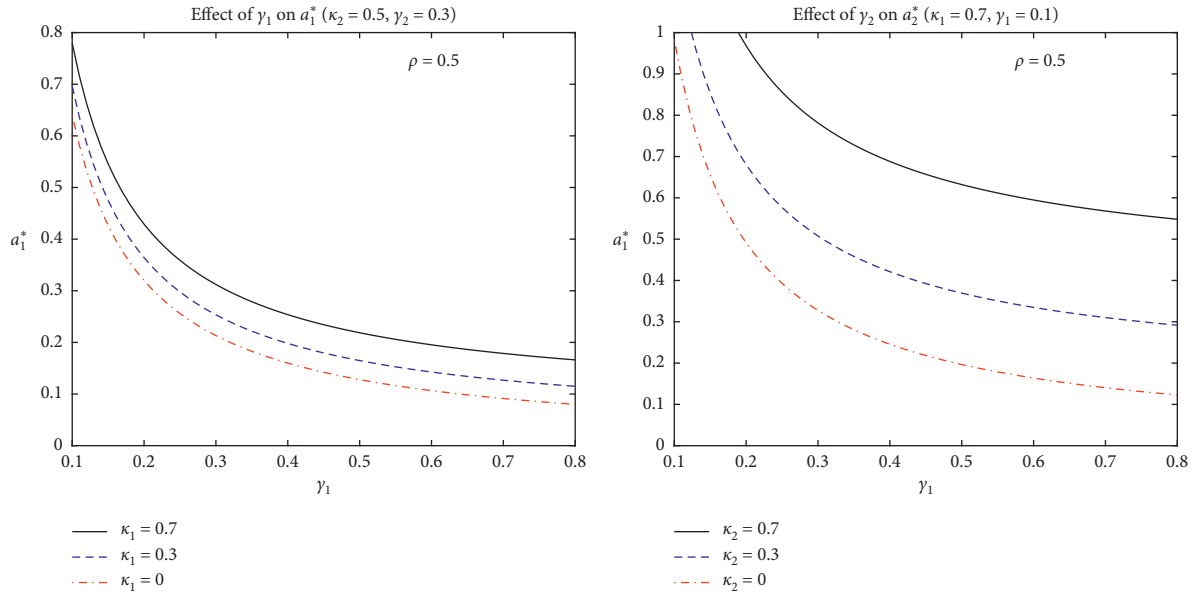


Figure 1: Effects of γ_k on $a_k^*(0)$, when $\rho = 0.5$, for $k = 1, 2$.

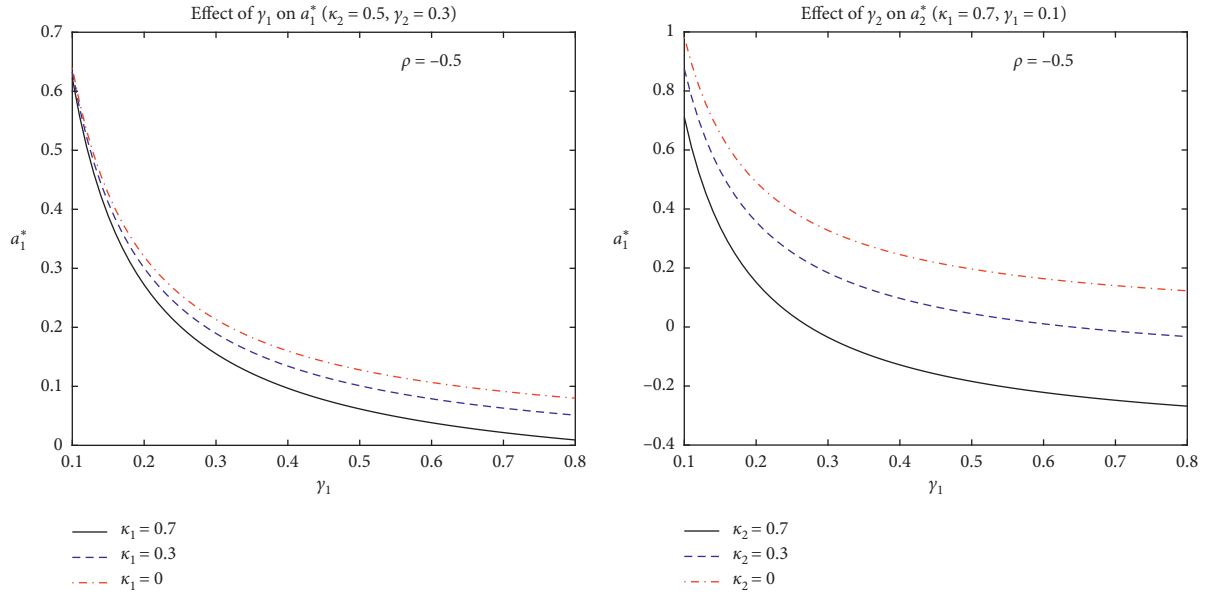
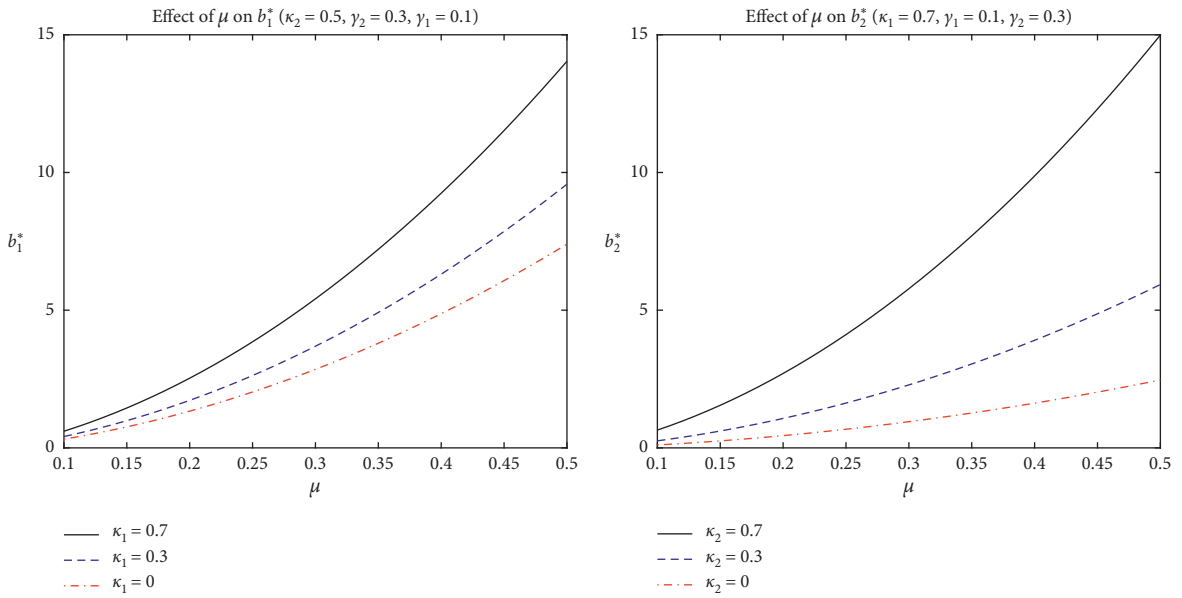


Figure 2: Effects of γ_k on $a_k^*(0)$, when $\rho = -0.5$, for $k = 1, 2$.

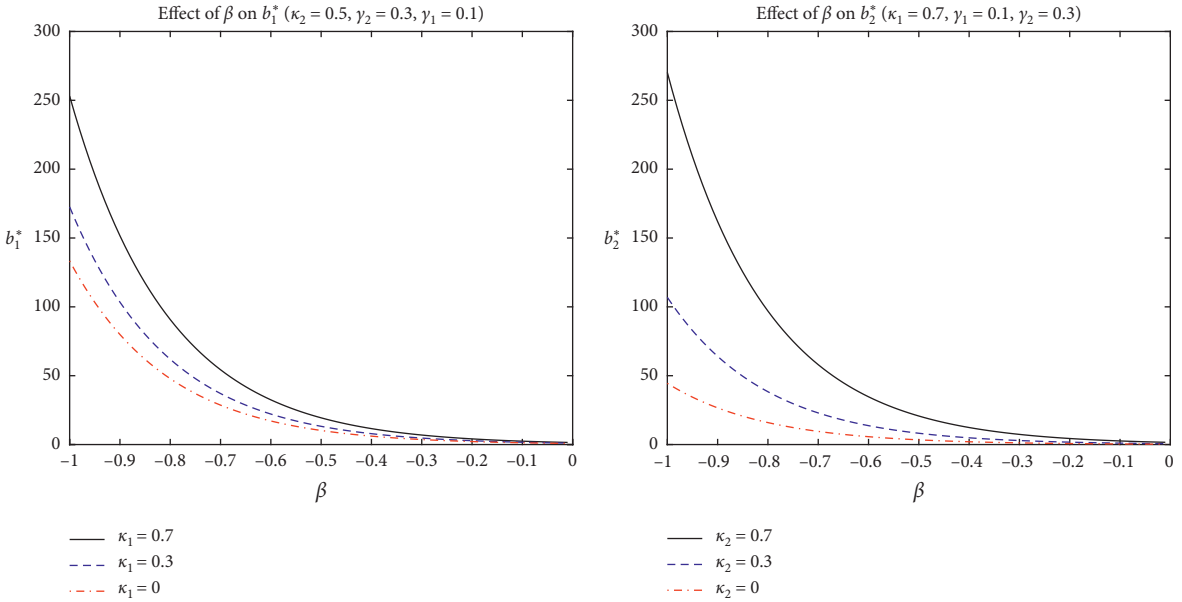
bond. On that account, insurer $k \in \{1, 2\}$ will pay attention to investing more in the bond.

In addition, Figures 3–5 also capture the effect of competition on the equilibrium investment strategies. In contrast to reinsurance protection, investment in stocks is likely to generate income, so CARA insurance company will increase its exposure to stock S at the terminal time T. Moreover, the presence of competition, which is captured by the sensitivity parameter κ_k, for $k = 1, 2$, induces both CARA insurers to increase their exposure on the stock S.



(a) (b)

Figure 3: Effects of μ on b_k^* (0), for $k = 1, 2$.



(a) (b)

Figure 4: Effects of β on b_k^* (0), for $k = 1, 2$.

6. Conclusion

The paper takes into account the relative performance exhibited by two competitive insurers relying on a nonzero-sum stochastic differential game framework under the CEV model. The dynamic programming principle is applied to solve the Nash equilibrium game, obtaining the optimal reinsurance and investment strategies capable of maximizing the expected utility regarding the relative performance of insurers. We consider two CARA insurers who both have an exponential utility function and give the necessary and sufficient conditions of the equilibrium strategy adaption at the same time. Finally, some numerical studies demonstrate how model parameters affect the equilibrium reinsurance-investment strategies of the CARA insurers.

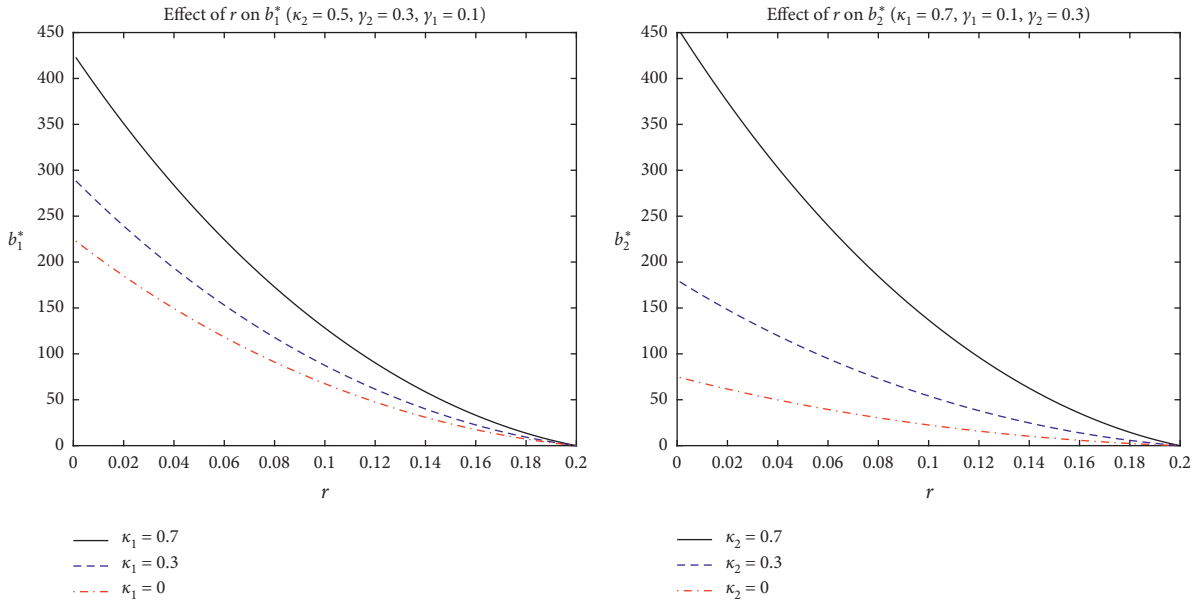


Figure 5: Effects of r on b_k^* (0), for $k = 1, 2$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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