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Research Article

Practical Stability and Integral Stability for Singular Differential Systems with Maxima

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In this paper, we introduce various definitions of practical stability and integral stability for nonlinear singular differential systems with maxima and give criteria of stability for such systems via the Lyapunov method and comparison principle.

1. Introduction and Preliminaries

Differential equations with maxima are a special type of differential equations that contain the maximum of the unknown function over a previous interval, of which many examples are found in the fields of application such as automatic control, population dynamics, disease control, and so on. Recently, the research interest in differential equations with maxima has increased exponentially. Some stability results for such equations can be found in the monographs [1, 2], the papers [3–9], and references cited therein.

In practical applications, many problems can be described by singular system models, such as optimal control problems and constrained control problems, which can be found in the monographs of Campbell [10] and Dai [11]. Singular system is a type of dynamic system which is more complicated than the ordinary one. Owing to its complicated structure and many other factors, the study of stability for singular systems involves greater difficulty than that of nonsingular systems. Till now, various types of stability for singular systems have been investigated via Lyapunov functions. However, most previous studies focused on the singular systems described by ordinary differential equations [10–13], difference equations [14–17], and delay differential equations [18–20], and there are a few results for singular differential systems with maxima. In addition, differential equations with maxima have some different properties from the well-known differential equations and delay differential equations.

The purpose of this paper is to integrate these two areas and analyze the practical stability and integral stability of nonlinear singular systems with maxima. To extend Lyapunov's stability and support the specific needs of singular systems, we introduce the function $q(t,x)$ and obtain some different types of stability criteria by using the Lyapunov function method and comparison principle.

2. Practical Stability

The practical stability, being quite different from the stability in the sense of Lyapunov, is neither weaker nor stronger than the usual stability. It is significant from the perspective of engineering application (see [21–25]). In this section, by using Lyapunov functions and the comparison principle, we study some practical stability for the following singular differential systems.

Consider the singular differential systems with maxima

$$\begin{cases} E\dot{x} = f\left(t, x, \max_{s\in[t-\tau,t]} x(s)\right), & \text{for } t \geq t_0 \geq 0, \\ x_{t_0} = \varphi(t), & t \in [-\tau, 0], \end{cases} \quad (1)$$

where $E \in R^{n\times n}$ with rank $(E) < n$ is a singular constant matrix, $x \in R^n$, $f \in C(R_+ \times R^n \times R^n, R^n)$, $f(t,0,0) \equiv 0$, $\tau > 0$ is a constant, and $\varphi \in C([-\tau, 0], R^n)$.

Firstly, we introduce the following notations and sets for convenience.

Let $T_k = [0, t_k)$, where $0 < t_k \leq +\infty$; $q(t,x) \in C^1(J \times R^n, R^m)$, $q(t,0) \equiv 0$, $J \subseteq R_+$. $S_k(t_0)$ is a set of all consistent initial functions at initial time t_0. Then, for any $\varphi \in S_k(t_0)$, there exists at least one continuous solution of systems (1) in $[t_0 - \tau, +\infty)$ through (t_0, φ) (see [20]).

$K = \{a(t) \in C(T_k, R_+) \mid a(t);$ is strictly increasing and $a(0) = 0\}$

$K^* = \{a(t_0, r) \in C(T_k \times R_+, R_+) \mid a(t_0, r);$ is strictly increasing in r and $a(t_0, 0) = 0\}$

$Q(\varepsilon) = \{x \in R^n \mid \|q(t,x)\| < \varepsilon, t \in T_k, \varepsilon > 0$ is a constant$\}$

$D(A) = \{x \in R^n \mid \|q(t,x)\| < A, A > 0$ is a constant$\}$

$\Lambda = \{V \mid V \in C(R \times D(A), R_+)$ and V islocallyLipschitzianin $x\}$

$G \Longrightarrow H$ means that G implies H

We denote by $x(t) \equiv x(t; t_0, \varphi)$ the solution of the initial value problems (1).

Definition 1. Let $V \in \Lambda, t \in T_k$, and we define the derivative of the function *V (t; x)* along the trajectory of solution of the singular systems (1) as follows:

$$D^+_{(2.1)} V(t, x(t)) = \limsup_{h \longrightarrow 0} \frac{1}{h} \left\{ V\left(t + h, x(t) + hf\left(t, x(t), \max_{s \in [-\tau, 0]} x(t+s) \right) \right) - V(t, x(t)) \right\}. \quad (2)$$

Definition 2. Let $\varphi \in S_k(t_0)$. The singular systems (1) is said to be

(S_1) stable with respect to $(q(t,x), T_k)$ if for any $\varepsilon > 0$, and some $t_0 \in T_k$, there exists $\delta(t_0, \varepsilon) > 0$, such that

$$\max_{s \in [-\tau,0]} \|\varphi(s)\| < \delta(t_0, \varepsilon) \Longrightarrow \|q(t, x(t; t_0, \varphi))\| < \varepsilon, \quad \text{for } t \geq t_0. \quad (3)$$

Definition 3. Let $\varphi \in S_k(t_0)$. The singular systems (1) is said to be (PS_1) practically stable for given (λ, A) with $0 < \lambda < A$ and some $t_0 \in T_k$, such that

$$\max_{s \in [-\tau,0]} \|\varphi(s)\| < \lambda \Longrightarrow \|q(t, x(t; t_0, \varphi))\| < A, \quad \text{for } t \geq t_0, \quad (4)$$

(PS_2) uniformly practically stable if (PS_1) holds for all $t_0 \in T_k$

(PS_3) practically quasistable for given (λ, B, T) with $\lambda, B, T > 0$, and some $t_0 \in T_k$, we have

$$\max_{s \in [-\tau,0]} \|\varphi(s)\| < \lambda \Longrightarrow \|q(t, x(t; t_0, \varphi))\| < B, \quad \text{for } t \geq t_0 + T, \quad (5)$$

(PS_4) uniformly practically quasistable if (PS_1) holds for all $t_0 \in T_k$

(PS_5) strongly practically stable if (PS_1) and (PS_3) hold simultaneously

(PS_6) strongly uniformly practically stable if (PS_2) and (PS_4) hold simultaneously

Remark 1. If $q(t,x) = x, t_k = +\infty$, and $S_k(t_0, t_k) = C([-\tau, 0], R^n)$, then Definitions 2 and 3 reduce to the concepts of classic Lyapunov stability.

It is well known that the comparison principle plays an important role in the development of stability theory. By the comparison principle, we can reduce the study of a given complicated differential system to that of a relatively simpler differential equation. For this purpose, we give the following lemma and definition.

Lemma 1 (See [1]). *Assume that the following conditions hold*

(A_1) $m(t) \in C(R_+, R_+)$, $g(t,u) \in C(R_+ \times R_+, R)$ *and for any* $t \in T_k$ *such that* $m(t) > m(t+s)$ *for* $s \in [-\tau, 0)$, *the inequality*

$$D^+ m(t) \leq g(t, m(t)), \quad (6)$$

holds, where $D^+ m(t) = \limsup_{h \longrightarrow 0^+} (1/h)[m(t+h) - m(t)]$, $g(t,0) \equiv 0$

(A_2) *the maximal solution* $r(t) \equiv r(t; t_0, u_0)$ *of the scalar equation*

$$\dot{u} = g(t,u), u(t_0) = u_0, \quad (7)$$

exists, on $[t_0, +\infty)$. *Then,* $m(t) \leq r(t), t \geq t_0$, *provided* $\max_{s \in [-\tau,0]} m(t_0 + s) \leq u_0$.

Definition 4. Comparison equation (7) is said to be (PS_7) practically stable if for given (λ, A) with $0 < \lambda < A$ and some $t_0 \in R_+$, we have $u_0 < \lambda$ implies $u(t) < A$, for $t \geq t_0$

(PS_8) uniformly practically stable if (PS_7) holds for all $t_0 \in R_+$

(PS_9) practically quasistable if for given (λ, B, T) with $0 < \lambda < A$, $B > 0$, $T > 0$, and some $t_0 \in R_+$, we have that $u_0 < \lambda$ implies $u(t) < B$, for $t \geq t_0 + T$

(PS_{10}) uniformly practically quasistable if (PS_9) holds for all $t_0 \in R_+$

Theorem 1. *Assume that the following conditions hold*

(A_3) $(q(t,x), T_k, \lambda, A)$ *with* $0 < \lambda < A$ *are given*

(A_4) *there exists a function* $V \in C(T_k \times D(A), R_+)$ *and* $V \in \Lambda$ *such that*

(i) for any $t > t_0$, $V(t, x(t)) > V(t+s, x(t+s))$, $s \in [-\tau, 0)$, *the inequality*

$$D^+_{(2.1)} V(t, x(t)) \le g(t, V(t, x(t))), \tag{8}$$

holds, where $g \in C(T_k \times R_+, R)$ *and* $g(t, 0) \equiv 0$

(ii) $b(\|q(t,x)\|) \le V(t,x) \le a(\|x\|)$, *where* $a(\cdot)$, $b(\cdot) \in K$ *and* $a(\lambda) < b(A)$

Then, equation (7) is (uniformly) practically stable with respect to $(a(\lambda), b(A))$ *implies that system (1) is (uniformly) practically stable with respect to* $(q(t,x), T_k, \lambda, A)$.

Proof. Assume that $u(t; t_0, u_0)$ is a solution of the equation (7), and is practically stable with respect to $(a(\lambda), b(A))$ for given $0 < \lambda < A$. Let $m(t) = V(t, x(t))$, where $x(t)$ is a solution of the systems (1). From the condition (i) of (A_4), it follows that $D^+ m(t) \le g(t, m(t))$, for $t \ge t_0$. Let

$$u(t_0) = \max_{s \in [-\tau, 0]} V(t_0 + s, \varphi(s)) = \max_{s \in [-\tau, 0]} m(t_0 + s). \tag{9}$$

By Lemma 1, we know that the inequality $V(t, x(t)) = m(t) \le r(t)$, for $t \ge t_0$, holds, where $r(t)$ is the maximal solution of comparison equation (7) existing on T_k. Assume that $\max_{s \in [-\tau,0]} \|\varphi(s)\| < \lambda$, then, we have

$$\begin{aligned} u(t_0) &= \max_{s \in [-\tau,0]} V(t_0 + s, \varphi(s)) \le \max_{s \in [-\tau,0]} a(\|\varphi(s)\|) \\ &= a\left(\max_{s \in [-\tau,0]} \|\varphi(s)\|\right) < a(\lambda). \end{aligned} \tag{10}$$

Furthermore, from the condition (ii) of (A_4) and Lemma 1, we get $b(\|q(t,x)\|) \le V(t,x) = m(t) \le r(t) < b(A)$. Thus, $\max_{s \in [-\tau,0]} \|\varphi(s)\| < \lambda$ implies $\|q(t,x)\| < A$, $t \ge t_0$, that is, system (1) is practically stable with respect to $(q(t,x), T_k, \lambda, A)$.

Similarly, we can prove that equation (7) is uniformly practically stable with respect to $(a(\lambda), b(A))$ implies that the systems (1) is uniformly practically stable with respect to $(q(t,x), T_k, \lambda, A)$. The proof is completed.

By Theorem 1, we can obtain the following corollaries. □

Corollary 1. *Assume that the conditions* (A_3) *and (ii) of* (A_4) *hold in Theorem 1, and*

(A_5) *there exists a function* $V \in C(T_k \times D(A), R_+)$ *and* $V \in \Lambda$ *such that for any* $t > t_0$, $V(t, x(t)) > V(t+s, x(t+s))$, $s \in [-\tau, 0)$, *the inequality*

$$D^+_{(2.1)} V \le 0 \tag{11}$$

holds. Then, system (1) is uniformly practically stable with respect to $(q(t,x), T_k, \lambda, A)$.

The conclusion of Corollary 1 can be obtained by considering the case of $g(t,u) \equiv 0$ and $\dot{u} = 0$ is uniformly practically stable with respect to $(a(\lambda), b(A))$ for given $0 < \lambda < A$.

Corollary 2. *Assume that the conditions* (A_3) *and (ii) of* (A_4) *hold in Theorem 1, and*

(A_6) *there exists a function* $V(t,x) \in C(T_k \times D(A), R_+)$ *and* $V \in \Lambda$ *such that for any* $t > t_0$, $V(t, x(t)) > V(t+s, x(t+s))$, $s \in [-\tau, 0)$, *the inequality*

$$D^+_{(2.1)} V \le \alpha(t) F(V(t,x)), \tag{12}$$

holds, where $F \in C(R_+, R_+)$ *and* $0 < F(V) \le V$

(A_7) *the inequalities*

$$\int_{t_0}^{t} \alpha(t) \mathrm{d}t \le M < +\infty, \quad M \le \ln \frac{b(A)}{a(\lambda)}, t \in [t_0, +\infty), \tag{13}$$

hold

Then, system (1) is uniformly practically stable with respect to $(q(t,x), T_k, \lambda, A)$.

Proof. By Theorem 1, we only prove that the system $\dot{u} = (\alpha(t) F(u))$ is uniformly practically stable with respect to $(a(\lambda), b(A))$. In fact, let $u(t_0) = u(t_0; t_0, u_0) = \max_{s \in [-\tau,0]} V(t_0 + s, \varphi(s))$. Assume that $\max_{s \in [-\tau,0]} \|\varphi(s)\| < \lambda$, it follows from the condition (ii) of (A_4) that

$$u(t_0; t_0, u_0) \le \max_{s \in [-\tau,0]} a(\|\varphi(s)\|) = a\left(\max_{s \in [-\tau,0]} \|\varphi(s)\|\right) < a(\lambda). \tag{14}$$

Furthermore, by condition (A_7), the inequality

$$u(t; t_0, u_0) \le u_0 \exp\left\{\int_{t_0}^{t} \alpha(s) \mathrm{d}s\right\} < a(\lambda) e^M < b(A), \tag{15}$$

holds. Then, system (1) is uniformly practically stable with respect to $(q(t,x), T_k, \lambda, A)$. The proof is completed. □

Corollary 3. *Assume that the conditions* (A_3) *and (ii) of* (A_4) *hold in Theorem 1, and*

(A_8) *there exists a function* $V \in C(T_k \times D(A), R_+)$ *and* $V \in \Lambda$ *such that for any* $t > t_0$, $V(t, x(t)) > V(t+s, x(t+s))$ *for* $s \in [-\tau, 0)$, *the inequality*

$$D^{+}_{(2.1)}V \leq -\alpha F(V(t,x)) + \beta, \tag{16}$$

holds, in which α and β are positive constants, $0 < F(V) \leq V$

(A_9) *the inequalities*

$$a(\lambda) < b(A),$$
$$a(\lambda) + \frac{\beta}{\alpha} \leq b(A), \tag{17}$$

hold

Then, system (1) is uniformly practically stable with respect to $(q(t,x), T_k, \lambda, A)$.

Proof. In fact, we only need to prove that the system $\dot{u} = -\alpha F(u) + \beta$ is uniformly practically stable with respect to $(a(\lambda), b(A))$. Let $u(t_0) = u(t_0; t_0, u_0) = \max_{s\in[-\tau,0]} V(t_0 + s, \varphi(s))$; then, we have

$$u(t_0; t_0, u_0) \leq \max_{s\in[-\tau,0]} a(\|\varphi(s)\|) = a\left(\max_{s\in[-\tau,0]} \|\varphi(s)\|\right) < a(\lambda). \tag{18}$$

Furthermore, by condition (A_8), the inequality

$$u(t; t_0, u_0) \leq u_0 e^{-\alpha(t-t_0)} + \frac{\beta}{\alpha} < a(\lambda) + \frac{\beta}{\alpha} \leq b(A) \tag{19}$$

holds. Then, system (1) is uniformly practically stable with respect to $(q(t,x), T_k, \lambda, A)$. □

Theorem 2. *Assume that the conditions (A_3) and (ii) of (A_4) hold in Theorem 1, and (iii) $V(t,x) \geq b(\|q(t,x)\|)$, $V(t_0,x) \leq a(t_0, \|x\|)$, where $b(\cdot) \in K, a(t_0,\cdot) \in K^*$.*

Then, equation (7) is (uniformly) practically stable with respect to $(a(t_0,\lambda), b(A))$ implies that the systems (1) is (uniformly) practically stable with respect to $(q(t,x), T_k, \lambda, A)$.

Proof. In fact, by the condition (iii) of (A_4), we have

$$u(t_0) = \max_{s\in[-\tau,0]} V(t_0 + s, \varphi(s)) \leq \max_{s\in[-\tau,0]} a(t_0, \|\varphi(s)\|)$$
$$= a\left(t_0, \max_{s\in[-\tau,0]} \|\varphi(s)\|\right). \tag{20}$$

Then, we can get the result by using a method similar to Theorem 1. We omit its details. □

Theorem 3. *Assume that the following conditions hold*

(A_{10}) *$(q(t,x), T_k, \lambda, A, B, T)$ with $0 < \lambda < A$, $0 < B < A$ and $T > 0$ are given*

(A_4) *there exists a function $V \in C(T_k \times D(A), R_+)$ and $V \in \Lambda$ such that*

(i) for any $t > t_0$, $V(t, x(t)) > V(t+s, x(t+s))$, $s \in [-\tau, 0)$, the inequality

$$D^{+}_{(2.1)}V(t,x(t)) \leq g(t, V(t,x(t))), \tag{21}$$

holds, where $g \in C(T_k \times R_+, R)$, $g(t,0) \equiv 0$

(ii) $b(\|q(t,x)\|) \leq V(t,x) \leq a(\|x\|)$, where $a(\cdot), b(\cdot) \in K$ and $a(\lambda) < b(A)$

Then, equation (7) is (uniformly) practically quasistable with respect to $(a(\lambda), b(B), T)$ implies that system (1) is (uniformly) practically quasistable with respect to $(q(t,x), T_k, \lambda, B, T)$.

Proof. Assume that $u(t; t_0, u_0)$ is a solution of equation (7) and is practically quasistable with respect to $(a(\lambda), b(B), T)$ for given $0 < \lambda < A$, $0 < B < A$ and $T > 0$. Let $m(t) = V(t, x(t))$, where $x(t)$ is a solution of system (1). It follows from the condition (A_4) that

$$D^{+}m(t) \leq g(t, m(t)), \quad \text{for } t \geq t_0. \tag{22}$$

Let $u(t_0) = \max_{s\in[-\tau,0]} V(t_0 + s, \varphi(s)) = \max_{s\in[-\tau,0]} m(t_0 + s)$. Then, by Lemma 1, we know that the inequality

$$V(t,x) = m(t) \leq r(t), \quad \text{for } t \geq t_0, \tag{23}$$

holds, where $r(t)$ is the maximal solution of comparison equation (7) existing on T_k. Assume that $\varphi(s) \in C([-\tau,0], R^n)$ and $\max_{s\in[-\tau,0]} \|\varphi(s)\| < \lambda$. Then, we obtain

$$u(t_0) = \max_{s\in[-\tau,0]} V(t_0 + s, \varphi(s)) \leq \max_{s\in[-\tau,0]} a(\|\varphi(s)\|)$$
$$= a\left(\max_{s\in[-\tau,0]} \|\varphi(s)\|\right) < a(\lambda). \tag{24}$$

Furthermore, by comparison equation (7) which is practically quasistable with respect to $(a(\lambda), b(B), T)$, the condition (A_4), and Lemma 1, we get

$$b(\|q(t,x)\|) \leq V(t,x) = m(t) \leq r(t) < b(B), \quad \text{for } t \geq t_0 + T. \tag{25}$$

Thus, $\max_{s\in[-\tau,0]} \|\varphi(s)\| < \lambda$ implies $\|q(t,x)\| < B$, $t \geq t_0 + T$, that is, system (1) is practically quasistable with respect to $(q(t,x), T_k, \lambda, B, T)$.

Similarly, we can prove that equation (7) is uniformly practically stable with respect to $(a(\lambda), b(B), T)$ implies that the systems (1) is uniformly practically stable with respect to $(q(t,x), T_k, \lambda, B, T)$. □

Theorem 4. *Assume that the conditions (A_{10}) and (i) of (A_4) hold in Theorem 3, and the condition (ii) of (A_4) is replaced by*

(iv) $V(t,x) \geq b(\|q(t,x)\|)$, $V(t_0,x) \leq a(t_0, \|x\|)$, where $b(\cdot) \in K, a(t_0,\cdot) \in K^*$

Then, equation (7) is (uniformly) practically quasistable with respect to $(a(\lambda), b(B), T)$ implies that the systems (1) is (uniformly) practically quasistable with respect to $(q(t,x), T_k, \lambda, B, T)$.

The proof of Theorem 4 is similar to that of Theorem 3, so we omit its details.

3. Integral Stability

The concept of integral stability, which was introduced for ordinary differential equations by Vrhoc in 1959 [26] and Lakshmikantham in 1969 [27], enlarges the collection of dynamical properties of solutions which can be investigated by the direct Lyapunov method. The integral stability theory has been rapidly developed recently. For example, Martynyuk [28], Salvadori and Visentin [29], Soliman and Abdalla [30] obtained the integral stability criteria for nonlinear differential equations, respectively; Hristova [31] obtained the integral stability in terms of two measures for impulsive differential equations; and Sood and Srivastava [32] gave the φ_0-integral stability criteria for impulsive differential equations. The main purpose of this section is to discuss the integral stability of singular differential systems with maxima and its perturbed systems.

Consider singular differential system (1) and its perturbed systems

$$\begin{cases} E\dot{x} = f\left(t, x, \max_{s\in[t-\tau,t]} x(s)\right) + h\left(t, x, \max_{s\in[t-\tau,t]} x(s)\right), & \text{for } t \ge t_0 \ge 0, \\ x_{t_0} = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{26}$$

where $h \in C(R_+ \times R^n \times R^n, R^n)$, $h(t, 0, 0) \equiv 0$.

Let $S_{pk}(t_0)$ be a set of all consistent initial functions of (1) and (26) in $[t_0, t_k)$ through (t_0, φ). For any $\varphi \in S_{pk}(t_0)$, assume that there exists a continuous solution of (1) and (26) in $[t_0, t_k)$ through (t_0, φ) at least.

$$B(\varphi, \delta) = \Big\{\psi \in C([-\tau, 0], R^n) \mid \max_{s\in[-\tau,0]} \|\psi(s) - \varphi(s)\| < \delta, \varphi \in C([-\tau, 0], R^n), \delta \in R_+\Big\}. \tag{27}$$

Definition 5. Let $\varphi \in S_{pk}(t_0)$. Singular system (1) is said to be

(IS_1) equi-integrally stable on $\{q(t, x), T_k\}$, if for given $\alpha \ge 0$ and $t_0 \in T_k$, there exists a positive function $\beta = \beta(t_0, \alpha)$, which is continuous in t_0 for each α and $\beta \in K$, such that, for every solution $x(t; t_0, \varphi)$ of the perturbed systems (26),

$$q(t, x) \in D(\beta), \quad t \ge t_0, \tag{28}$$

holds, provided that $\varphi \in B(0, \alpha) \cap S_{pk}(t_0)$ and

$$\int_{t_0}^{t_0+T} \sup_{\|q(t,x)\|\le\beta} \left\| h\left(s, x, \max_{u\in[s-\tau,s]} x(u)\right)\right\| ds \le \alpha, \quad \text{for } T > 0; \tag{29}$$

(IS_2) uniformly integrally stable on $\{q(t, x), T_k\}$, if the β in (IS_1) is independent of t_0

(IS_3) equiasymptotically integrally stable on $\{q(t, x), T_k\}$, if (IS_1) holds and for every $\epsilon > 0$, $\alpha \ge 0$ and $t_0 \in T_k$ $(t_k = +\infty)$, there exist positive functions $T = T(t_0, \alpha, \epsilon)$ and $\gamma = \gamma(t_0, \alpha, \epsilon)$, which are continuous in t_0 for each α and ϵ, and for every solution $x(t; t_0, \varphi)$ of the perturbed systems (26),

$$q(t, x) \in D(\epsilon), \quad t \ge t_0 + T, \tag{30}$$

holds, provided that $\varphi \in B(0, \alpha) \cap S_p(t_0)$, and

$$\int_{t_0}^{+\infty} \sup_{\|q(t,x)\|\le\beta} \left\| h\left(s, x, \max_{u\in[s-\tau,s]} x(u)\right)\right\| ds \le \gamma, \tag{31}$$

(IS_4) uniformly asymptotically integrally stable on $\{q(t, x), T_k\}$ if the T and γ in (IS_3) are independent of t_0 and (IS_2) holds

Now, we consider comparison scalar differential equation (7) and its perturbed equation

$$\dot{u} = g(t, u) + p(t), \quad u(t_0) = u_0, \tag{32}$$

where $g(t, 0) \equiv 0$, $g \in C[T_k \times R_+, R]$, $p(t) \in C[T_k, R_+]$.

Definition 6. Equation (7) is said to be equi-integrally stable, if for given $\alpha_1 \ge 0$, $t_0 \in T_k$, there exists a positive function $\beta_1 = \beta_1(t_0, \alpha_1)$ that is continuous in t_0 for each α_1 and $\beta_1 \in K$, such that, for every solution $u(t; t_0, u_0)$ of the perturbed differential equation (32), the inequality

$$|u(t; t_0, u_0)| < \beta_1, \tag{33}$$

holds, provided that $|u_0| \le \alpha_1$ and $\int_{t_0}^{t_0+T} p(s)ds \le \alpha_1$ for every $T > 0$.

Remark 2. Similar to Definition 5, we can give the corresponding concepts of stability of equation (7).

Next, we investigate the integral stability of system (1) via the Lyapunov function method and comparison principle.

Theorem 5. *Assume that the condition (i) of (A_4) holds in Theorem 1, and condition (ii) of (A_4) is replaced by*

$$(A_4')\, b(\|q(t, x)\|) \le V(t, x), \quad t \ge t_0, \text{ where } b \in K, \text{ and } b(r) \longrightarrow +\infty \text{ as } r \longrightarrow +\infty. \tag{34}$$

Then, equation (7) which is equi-integrally stable implies that system (1) is equi-integrally stable on $\{q(t, x), T_k\}$.

Proof. Let $\varphi \in B(0, \alpha) \cap S_{pk}(t_0)$ for every $\alpha \ge 0$, $t_0 \in T_k$. Since $V(t, x)$ is Lipschitzian in x, we have

$$|V(t, x) - V(t, y)| \le L\|x - y\|, \quad L > 0 \text{ is a constant}, \quad \max_{s\in[-\tau,0]} V(t_0 + s, \varphi(s)) \le L \max_{s\in[-\tau,0]} \|\varphi(s)\| \le L\alpha = \alpha_1. \tag{35}$$

Let $x(t) = x(t; t_0, \varphi)$ be any solution of (26). Thus, by the condition (i) of (A_4) and (29), we get

$$D^+_{(21)} V(t, x) \le g(t, V(t, x)) + L\left\| h\left(t, x, \max_{s\in[t-\tau,t]} x(s)\right)\right\|. \tag{36}$$

Defining $p(t) = L\|h(t, x, \max_{s\in[t-\tau,t]} x(s))\|$ and choosing $u_0 = \max_{s\in[-\tau,0]} V(t_0 + s, \varphi(s))$, by Lemma 1, we have

$$V(t, x) \le r(t; t_0, u_0), \tag{37}$$

where $r(t; t_0, u_0)$ is the maximal solution of (32).

If equation (7) is equi-integrally stable, then for $\alpha_1 \ge 0$ and $t_0 \in T_k$, there exists a $\beta_1 = \beta_1(t_0, \alpha_1)$, which is continuous in t_0 for each α_1 and $\beta_1 \in K$, such that, for every solution $u(t; t_0, u_0)$ of (32), the inequality

$$u(t; t_0, u_0) < \beta_1, \quad t \ge t_0 \tag{38}$$

holds, whenever $u_0 \le \alpha_1$ and $\int_{t_0}^{t_0+T} p(s)\mathrm{d}s \le \alpha_1$ for any $T > 0$.

By the condition (A_4'), it is possible to choose a $\beta = \beta(t_0, \alpha)$ satisfying

$$b(\beta) \ge \beta_1. \tag{39}$$

It is easily shown that β is continuous in t_0 for each α and $\beta \in K$ for each $t_0 \in T_k$. Moreover, we claim that system (1) is equi-integrally stable on $\{q(t, x), T_k\}$. In fact, if this is not true, there exists a $t_1 > t_0$ such that

$$\|q(t_1, x)\| = \beta, \text{ and } \|q(t, x)\| < \beta,\ t \in [t_0, t_1). \tag{40}$$

From (37)–(40), we have

$$\beta_1 \le b(\beta) \le V(t_1, x(t_1; t_0, \varphi)) \le r(t_1; t_0, u_0) < \beta_1, \tag{41}$$

which is a contradiction. Thus, system (1) is equi-integrally stable on $\{q(t, x), T_k\}$. □

Corollary 4. *Assume that the conditions of Theorem 5 hold. Then, equation (7) which is uniformly integrally stable implies that system (1) is uniformly integrally stable on {q(t; x); T_k}.*

The detailed proof of Corollary 4 is similar to the proof in Theorem 5, so we omit it.

Theorem 6. *Assume that the conditions of Theorem 5 hold and $t_k = +\infty$ Then, equation (7) which is equiasymptotically integrally stable implies that system (1) is equiasymptotically integrally stable on {q(t. x), R_+}.*

Proof. It can be known from the proof of Theorem 5 that system (1) is equi-integrally stable on $\{q(t, x), R_+\}$. Let $\varphi \in B(0, \alpha) \cap S_p(t_0, +\infty)$ and $\alpha_1 = L\alpha$, for given $\alpha \ge 0$ and $t_0 \in [0, +\infty)$. For given .., $\alpha_1 \ge 0$ and $t_0 \in R_+$, there exist $\gamma_1 = \gamma_1(t_0, \alpha_1, \varepsilon)$ and $T = T(t_0, \alpha_1, \varepsilon)$, such that for every solution $u(t; t_0, u_0)$ of (32),

$$u(t; t_0, u_0) < b(\varepsilon),\ t \ge t_0 + T, \tag{42}$$

holds, whenever $u_0 = \max_{s\in[-\tau,0]} V(t_0 + s, \varphi(s)) \le \alpha_1$ and $\int_{t_0}^{\infty} p(s)\mathrm{d}s < \gamma_1$.

Choosing a positive number $\gamma = \gamma(t_0, \alpha, \epsilon)$ such that $L\gamma = \gamma_1$, for given γ and T, system (1) satisfies (IS_3) of Definition 5. In fact, suppose that the conclusion is not true, then $q(t, x) \in D(\epsilon)$, $t \ge t_0 + T$ cannot be satisfied when

$$\begin{gathered}\varphi \in B(0, \alpha) \cap S_p(t_0, +\infty) \\ \text{and } \int_{t_0}^{+\infty} \sup_{\|q(t,x)\|\le\beta} \left\| h\left(s, x, \max_{u\in[s-\tau,s]} x(u)\right)\right\| \mathrm{d}s \le \gamma.\end{gathered} \tag{43}$$

Let $\{t_k\}$ be a sequence such that $t_k \ge t_0 + T$ and $\lim_{k\longrightarrow+\infty tk} = +\infty$. Suppose that there is a solution $x(t) = x(t; t_0, \varphi)$ of system (26), such that for every k,

$$\|q(t_k, x(t_k; t_0, \varphi))\| \ge \varepsilon. \tag{44}$$

By the condition (A_4') and (44), we obtain

$$b(\varepsilon) \le b\left(\|q(t_k, x(t_k; t_0, \varphi))\|\right) \le V(t_k, x(t_k; t_0, \varphi)). \tag{45}$$

Furthermore, by the equiasymptotical integral stability of equations (7), (37), and (42)–(59), we can get

$$b(\epsilon) \le V(t_k, x(t_k; t_0, \varphi)) \le r(t_k; t_0, u_0) < b(\epsilon). \tag{46}$$

which is a contradiction. Thus, system (1) is equiasymptotically integrally stable on $\{q(t, x), [0, +\infty)\}$. □

Corollary 5. *Assume that the conditions of Theorem 6 hold. Then, equation (7) which is uniformly asymptotically integrally stable implies that system (1) is uniformly asymptotically integrally stable on {q(t; x); T_k}.*

In fact, we can show that the positive numbers T and γ_1 in proof of Theorem 6 are independent of t_0; therefore, (44) implies that γ is independent of t_0. The rest of the proof is similar to that of Theorem 6, so we omit the details here.

For the comparison equation (7), if we suppose that the function $g(t, u)$ is nonincreasing in u for $t \in [t_0, +\infty)$; then, we can get the uniform asymptotic integral stability of system (1) by the uniform asymptotic stability of the comparison equation (7). Therefore, we firstly give the following definition and Lemmas, which can be found in [27].

Definition 7. Comparison equation (7) is said to be

(S_1) equistable, if for each $\epsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \epsilon)$, such that

$$|u_0| \le \delta \Longrightarrow |u(t; t_0, u_0)| < \varepsilon, \quad \text{for } t \ge t_0, \tag{47}$$

(S_2) uniformly stable if δ in (S_1) is independent of t_0

(S_3) equiasymptotically stable, if it is equistable and for each $\epsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta_0 = \delta_0(t_0)$, $T = T(t_0, \epsilon)$, such that

$$|u_0| \le \delta_0 \Longrightarrow |u(t; t_0, u_0)| < \epsilon, \text{for } t \ge t_0 + T, \tag{48}$$

(S_4) uniformly asymptotically stable if (S_2) holds and the numbers δ_0 and T in (S_3) are independent of t_0

Lemma 2 (see [27]). *Equation (7) is uniformly stable if and only if there exists a function $a(r) \in K$, such that*

$$|u_0| \le \delta \Longrightarrow |u(t; t_0, u_0)| \le a(|u_0|), \quad \text{for } t \ge t_0. \tag{49}$$

Lemma 3 (see [27]). *Equation (7) is uniformly asymptotically stable if and only if there exist functions $a(r) \in K$ and $\sigma(r) \in \mathcal{S}$, such that*

$$|u_0| \leq \delta \Longrightarrow |u(t;t_0,u_0)| \leq a(|u_0|)\sigma(t-t_0), \quad \text{for } t \geq t_0, \tag{50}$$

where $\mathcal{F} = \{\sigma(r) \in C[R_+, R_+]\sigma(r)$ is monotone decreasing in r, and $\sigma(r) \longrightarrow 0$ as $r \longrightarrow +\infty\}$.

Theorem 7. *Assume that the conditions of Theorem 6 hold, the function $g(t;u)$ be nonincreasing in u for any $t \in [t_0, +\infty)$, and*

$$V(t,x(t)) > V(t+s, x(t+s)) - L\int_t^{t+s} \left\| h\left(v, x, \max_{u\in[v-\tau,v]} x(u)\right)\right\| dv, \quad \text{for } s \in [-\tau, 0). \tag{51}$$

Then, equation (7) which is uniformly asymptotically stable implies that system (1) is uniformly asymptotically integrally stable on $\{q(t,x), T_k\}$.

Proof. Firstly, we prove system (1) is uniformly integrally stable. Because equation (7) is uniformly stable, by Lemma 2, there exists a function $\beta_1 \in K$, such that

$$0 < u_0 \leq \alpha_1 \text{ implies } u(t;t_0,u_0) < \beta_1(u_0), \quad t \geq t_0. \tag{52}$$

For given $\alpha > 0$ and $t_0 \in T_k$, where $L\alpha = \alpha_1$, let $\varphi \in B(0,\alpha) \cap S_p(t_0)$. Since $V(t,x)$ is Lipschitzian in x, we have

$$\max_{s\in[-\tau,0]} V(t_0+s, \varphi(s)) \leq L \max_{s\in[-\tau,0]} \|\varphi(s)\| \leq L\alpha = \alpha_1. \tag{53}$$

Let $x(t) = x(t;t_0,\varphi)$ be any solution of (26) with $\varphi \in B(0,\alpha) \cap S_p(t_0)$, and

$$V(t, x(t;t_0,\varphi)) \equiv m(t) + p(t), \tag{54}$$

where $p(t) = L\int_{t_0}^t \|h(s,x,\max_{u\in[s-\tau,s]} x(u))\| ds$. According to the condition (i) of (A_4), we get

$$\begin{aligned} D^+ m(t) &\leq D^+_{(3.1)} V(t,x(t)) - L\left\| h\left(t, x, \max_{s\in[t-\tau,t]} x(s)\right)\right\| \\ &\leq D^+_{(2.1)} V(t,x(t)) \leq g(t, m(t)). \end{aligned} \tag{55}$$

By (51) and Lemma 1, we have

$$m(t) \leq r(t;t_0,u_0), \tag{56}$$

where $r(t;t_0,u_0)$ is the maximal solution of (7) with $u_0 = \max_{s\in[-\tau,0]} m(t_0+s)$.

We choosing $\beta > 0$, such that

$$b(\beta) > \beta_1(L\alpha) + L\alpha. \tag{57}$$

In view of the condition $b(r) \longrightarrow +\infty$ as $r \longrightarrow +\infty$, the choice of β is reasonable. It is obvious that $\beta = \beta(\alpha)$ and $\beta \in K$. At the same time, we can claim that system (1) is uniformly integrally stable. In other words, the solution of system (26) satisfies $q(t,x) \in D(\beta)$, $t \geq t_0$, whenever $\varphi \in B(0,\alpha) \cap S_p(t_0)$, and

$$\int_{t_0}^{t_0+T} \sup_{\|q(t,x)\|\leq\beta} \left\| h\left(s, x, \max_{u\in[s-\tau,s]} x(u)\right)\right\| ds \leq \alpha, \quad \text{for } T > 0. \tag{58}$$

Suppose that this is not true; there exists a $t_1 > t_0$ such that

$$\|q(t_1, x(t_1))\| = \beta \text{ and } \|q(t,x(t))\| \leq \beta,\ t \in [t_0, t_1]. \tag{59}$$

From the condition (A_4') and (54)–(58), we get

$$\begin{aligned} b(\beta) &\leq V(t_1, x(t_1)) \leq r(t_1;t_0,u_0) + p(t_1) \\ &\leq \beta_1(L\alpha) + L\int_{t_0}^{t_1} \sup_{\|q(t,x)\|\leq\beta} \left\| h\left(s, x, \max_{u\in[s-\tau,s]} x(u)\right)\right\| ds \\ &\leq \beta_1(L\alpha) + L\alpha < b(\beta). \end{aligned} \tag{60}$$

This is a contradiction, and then system (1) is uniformly integrally stable.

Secondly, we prove that system (1) is uniformly asymptotically integrally stable. By the uniform asymptotic stability of equation (7) and Lemma 3, we have

$$u(t;t_0,u_0) \leq \beta_1(u_0)\sigma(t-t_0), \quad t \geq t_0, \tag{61}$$

where $\beta_1 \in K$ and $\sigma \in \mathcal{F}$. For given $\epsilon > 0$, $\alpha \geq 0$ and $t_0 \in T_k$, let $\varphi \in B(0,\alpha) \cap S_p(t_0)$, and

$$L\int_{t_0}^{+\infty} \sup_{\|q(t,x)\|\leq\beta} \left\| h\left(s, x, \max_{u\in[s-\tau,s]} x(u)\right)\right\| ds \leq L\gamma_1 < b(\epsilon), \tag{62}$$

where $\gamma = \min(\gamma_1, \alpha)$. For any solution $x(t) = x(t;t_0,\varphi)$ of (26) and (56), holds whenever $u_0 = \max_{s\in[-\tau,0]} V(t_0+s, \varphi(s))$. By (54), (56), and (61), together with (A_4'), we can obtain the inequality

$$\begin{aligned} b(\|q(t,x(t))\|) &\leq V(t,x(t)) \leq r(t;t_0,u_0) \\ &\quad + L\left\|\int_{t_0}^t h\left(s, x, \max_{u\in[s-\tau,s]} x(u)\right) ds\right\| \\ &\leq \beta_1(L\alpha)\sigma(t-t_0) \\ &\quad + L\int_{t_0}^t \sup_{\|q(t,x)\|\leq\beta} \left\| h\left(s, x, \max_{u\in[s-\tau,s]} x(u)\right)\right\| ds \\ &< \beta_1(L\alpha)\sigma(t-t_0) + L\gamma. \end{aligned} \tag{63}$$

Since $\sigma \in \mathcal{F}$, then there exists a $T = T(\alpha, \epsilon)$, such that

$$\sigma(t-t_0) < \frac{b(\epsilon) - L\gamma}{\beta_1(L\alpha)}, \quad t \geq t_0 + T. \tag{64}$$

Furthermore, we have

$$b(\|q(t,x(t))\|) < b(\epsilon), \quad t \geq t_0 + T, \tag{65}$$

which implies that

$$\|q(t, x(t))\| < \epsilon, \quad t \geq t_0 + T, \tag{66}$$

provided $\varphi \in B(0, \alpha) \cap S_p(t_0)$, and (62) is satisfied. Therefore, system (1) is uniformly asymptotically integrally stable. □

4. Conclusion

This paper discussed a class of nonlinear singular differential systems with maxima. Some notions of practical stability and integral stability for such systems were introduced, and various stability criteria were obtained by using the Lyapunov method and comparison principle.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors read and approved the final manuscript.

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