Research Article

Nontrivial Solutions for a System of Second-Order Discrete Boundary Value Problems

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In this work, we shall study the existence of nontrivial solutions for a system of second-order discrete boundary value problems. Under some conditions concerning the eigenvalues of relevant linear operator, we use the topological degree theory to obtain our main results.

1. Introduction

Nonlinear discrete problems appear in many mathematical models, such as computer science, mechanical engineering, control systems, economics, and fluid mechanics (see [1–4]). Owing to the wide applications, in recent years, there are a large number of researchers paying special attention in this direction (we refer to some results [5–15] and the references therein). For example, in [5], the authors used the Guo–Krasnosel’skii fixed point theorem to study the existence of positive solutions for the following second-order discrete boundary value problem:

$$\begin{cases} \Delta^2 x_{i-1} + f(x_i) = 0, & i \in [1, n], \\ x_0 = 0 = x_{n+1}, \end{cases} \quad (1)$$

and the following discrete second-order system:

$$\begin{cases} \Delta^2 x_{i-1} + f(x_i, y_i) = 0, & i \in [1, n], \\ \Delta^2 y_{i-1} + g(x_i, y_i) = 0, & i \in [1, n], \\ x_0 = x_{n+1} = y_0 = y_{n+1} = 0, \end{cases} \quad (2)$$

where n is a positive integer, $[1, n] = \{1, 2, \dots, n\}$, Δ is the forward difference operator, i.e., $\Delta x_{i-1} = x_i - x_{i-1}$, and $\Delta^2 x_{i-1} = \Delta(\Delta x_{i-1})$.

In [6], the authors used the monotone iterative technique to investigate the existence and uniqueness of positive solutions for the following discrete p -Laplacian fractional boundary value problem:

$$\begin{cases} \Delta_{\nu-1}^\nu (\phi_p(\Delta_{\nu-1}^\nu y(t))) = f(y(t+\nu-1)), & t \in [0, T]_{\mathbb{Z}}, \\ y(\nu-1) = y(\nu+T), \Delta_{\nu-1}^\nu y(\nu-1) = \Delta_{\nu-1}^\nu y(\nu+T), \end{cases} \quad (3)$$

where $\nu \in (0, 1)$ is a real number, $\Delta_{\nu-1}^\nu$ is a discrete fractional operator, and $\phi_p(s) = |s|^{p-2}s$ is the p -Laplacian with $s \in \mathbb{R}$, $p > 1$.

Coupled systems of discrete problems have also been investigated by many authors; some results can be found in a series of papers [11–15] and the references cited therein (also see some results on differential systems [16–24]). For example, in [11], the authors used the Guo–Krasnosel’skii fixed point theorem to study the following systems of three-point discrete boundary value problems:

$$\begin{cases} \Delta^2 u(n-1) + \lambda a(n)f(u(n), v(n)) = 0, & n \in \{1, 2, \dots, N-1\}, \\ \Delta^2 v(n-1) + \mu b(n)g(u(n), v(n)) = 0, \\ u(0) = \beta u(\eta), u(N) = \alpha u(\eta), v(0) = \beta v(\eta), v(N) = \alpha v(\eta), \end{cases} \quad (4)$$

where $N \geq 4$, $\eta \in \{1, 2, \dots, N-1\}$, $\alpha > 0, \beta > 0, \lambda, \mu > 0$. They offered some values for the parameters λ, μ to yield a positive solution for the above system.

In [12], the authors used the fixed point index to study the positive solutions for the following system of first-order discrete fractional boundary value problems:

$$\begin{cases} \Delta_{v-1}^v x(t) = f_1(t+v-1, x(t+v-1), y(t+v-1)), & t \in [0, T]_{\mathbb{Z}}, \\ \Delta_{v-1}^v y(t) = f_2(t+v-1, x(t+v-1), y(t+v-1)), & t \in [0, T]_{\mathbb{Z}}, \\ x(v-1) = x(v+T), y(v-1) = y(v+T). \end{cases} \quad (5)$$

By discrete Jensen's inequality, the authors adopted some appropriate nonnegative concave and convex functions to characterize the coupling behavior of the nonlinearities f_i ($i = 1, 2$).

Motivated by the aforementioned works, in this paper, by means of the topological degree theory, we study the existence of nontrivial solutions for the following system of second-order discrete boundary value problems:

$$\begin{cases} \Delta^2 u(k-1) + f(k, v(k)) = 0, & k \in \{1, 2, \dots, T\}, \\ \Delta^2 v(k-1) + g(k, u(k)) = 0, \\ u(0) = u(T+1) = v(0) = v(T+1) = 0, \end{cases} \quad (6)$$

where $T > 2$ is a fixed positive integer number, $\Delta u(k) = u(k+1) - u(k)$, $\Delta^2 u(k) = \Delta(\Delta u(k))$, and $f, g: \{1, 2, \dots, T\} \times \mathbb{R} \rightarrow \mathbb{R}$ ($\mathbb{R} := (-\infty, +\infty)$) are continuous and satisfy the following conditions:

(H1) There exist three nonnegative functions $a_i(k), b_i(k)$ ($b_i(k) \neq 0, k \in \mathbb{T}_1$) and β_i ($i = 1, 2$) on $\mathbb{R}^+$ such that

$$\begin{aligned} f(k, v) &\geq -a_1(k) - b_1(k)\beta_1(v), g(k, u) \\ &\geq -a_2(k) - b_2(k)\beta_2(u), \quad \forall u, v \in \mathbb{R}, t \in \mathbb{T}_1, \end{aligned} \quad (7)$$

where $\mathbb{T}_1 := \{1, 2, \dots, T\}$.

(H2) $\lim_{|v| \rightarrow +\infty} \beta_1(v)/|v| = 0$, $\lim_{|u| \rightarrow +\infty} \beta_2(u)/|u| = 0$.

(H3) $\liminf_{|v| \rightarrow +\infty} f(k, v)/|v| > \lambda_1$, $\liminf_{|u| \rightarrow +\infty} g(k, u)/|u| > \lambda_1$ uniformly on $k \in \mathbb{T}_1$, where $\lambda_1 = 4\sin^2(\pi/(2T+2))$.

(H4) $\limsup_{|v| \rightarrow 0} |f(k, v)|/|v| < \lambda_1$, $\limsup_{|u| \rightarrow 0} |g(k, u)|/|u| < \lambda_1$ uniformly on $k \in \mathbb{T}_1$.

Now, we state our main result here.

Theorem 1. Suppose that (H1)–(H4) hold. Then, (6) has at least one nontrivial solution.

2. Preliminaries

Let E be the Banach space of real valued functions defined on the discrete interval $\mathbb{T}_2$ with the norm $\|u\| = \max_{k \in \mathbb{T}_2} |u(k)|$, where $\mathbb{T}_2 := \{0, 1, 2, \dots, T+1\}$. Define the following sets:

$$\begin{aligned} P &= \{u \in E: u(k) \geq 0, \quad \forall k \in \mathbb{T}_2\}, \\ P_0 &= \left\{u \in E: \min_{k \in \mathbb{T}_1} u(k) \geq \frac{1}{T} \|u\|\right\}, \end{aligned} \quad (8)$$

and $B_r = \{x \in E: \|x\| < r\}$ for $r > 0$. Then, P, P_0 are cones on E , and B_r is an open ball in E .

Lemma 1 (see [11, 15]). Let $h(k) \in C(\mathbb{T}_1)$. Then, the discrete boundary value problem

$$\begin{cases} \Delta^2 u(k-1) + h(k) = 0, & k \in \mathbb{T}_1, \\ u(0) = u(T+1) = 0, \end{cases} \quad (9)$$

has a solution with the form

$$u(k) = \sum_{l=1}^T G(k, l)h(l), \quad k \in \mathbb{T}_2, \quad (10)$$

where

$$G(k, l) = \frac{1}{T+1} \begin{cases} l(T+1-k), & 1 \leq l \leq k-1 \leq T, \\ k(T+1-k), & 0 \leq k \leq l \leq T. \end{cases} \quad (11)$$

Furthermore, $G(k, l)$ has the following properties (see [13, 15]):

- (i) $G(k, l) > 0$ and $G(k, l) = G(l, k)$, for $(k, l) \in \mathbb{T}_1 \times \mathbb{T}_1$.
- (ii) $G(l, l)/T \leq G(k, l) \leq G(l, l)$, for $(k, l) \in \mathbb{T}_1 \times \mathbb{T}_1$.

By Lemma 1, system (6) is equivalent to

$$\begin{cases} u(k) = \sum_{l=1}^T G(k, l)f(l, v(l)), & k \in \mathbb{T}_2, \\ v(k) = \sum_{l=1}^T G(k, l)g(l, u(l)), & k \in \mathbb{T}_2. \end{cases} \quad (12)$$

Then, we can define operators $\mathcal{T}, \mathcal{S}: E \rightarrow E$ by

$$\begin{aligned} (\mathcal{T}v)(k) &= \sum_{l=1}^T G(k, l)f(l, v(l)), (\mathcal{S}u)(k) \\ &= \sum_{l=1}^T G(k, l)g(l, u(l)), \end{aligned} \quad (13)$$

and operator $\mathcal{A}: E \times E \rightarrow E \times E$ by

$$\mathcal{A}(u, v)(k) = ((\mathcal{T}v), (\mathcal{S}u))(k). \quad (14)$$

Note that $\mathcal{T}, \mathcal{S}, \mathcal{A}$ are completely continuous operators (see [11]), and (u, v) solves (6) if and only if (u, v) is a fixed point of the operator $\mathcal{A}$.

Lemma 2 (see [7, 15]). Let $\phi(k) = \sin(k\pi)/(T+1)$, $k \in \mathbb{T}_2$. Then, $\lambda_1 \sum_{l=1}^T G(k, l)\phi(l) = \phi(k)$, $\forall k \in \mathbb{T}_1$.

Define a linear operator as follows:

$$(Lx)(k) = \sum_{l=1}^T G(k, l)x(l), \quad \forall k \in \mathbb{T}_2. \quad (15)$$

Then, we have

$$(L\phi)(k) = \frac{1}{\lambda_1} \phi(k), \quad (16)$$

and we have the following lemma.

Lemma 3. *If $x \in P$, then $Lx \in P_0$.*

This is a direct result by Lemma 1 (ii), so we omit the proof.

Remark 1. $\phi \in P_0$ in Lemma 2.

Lemma 4 (see [25, Theorem A.3.3]). *Let Ω be a bounded open set in a Banach space E and $T: \Omega \rightarrow E$ be a continuous compact operator. If there exists $x_0 \in E \setminus \{0\}$ such that*

$$x - Tx \neq \mu x_0, \quad \forall x \in \partial\Omega, \quad \mu \geq 0, \quad (17)$$

then the topological degree $\deg(I - T, \Omega, 0) = 0$.

Lemma 5 (see [25, Lemma 2.5.1]). *Let Ω be a bounded open set in a Banach space E with $0 \in \Omega$ and $T: \Omega \rightarrow E$ be a continuous compact operator. If*

$$Tx \neq \mu x, \quad \forall x \in \partial\Omega, \quad \mu \geq 1, \quad (18)$$

then the topological degree $\deg(I - T, \Omega, 0) = 1$.

3. Main Results

In order to obtain the Proof of Theorem 1, we first provide a lemma.

Lemma 6. *There exists a sufficiently large $R > 0$ such that*

$$\deg(I - \mathcal{A}, B_R, 0) = 0. \quad (19)$$

Proof. By (H3), there exist $\varepsilon_1 > 0$ and $X_1 > 0$ such that

$$\begin{aligned} f(k, v) &\geq (\lambda_1 + \varepsilon_1)|v|, \quad g(k, u) \geq (\lambda_1 + \varepsilon_1)|u|, \\ &\forall k \in \mathbb{T}_1, |u|, |v| > X_1. \end{aligned} \quad (20)$$

Note that when $k \in \mathbb{T}_1, |u|, |v| \leq X_1$, the functions $|f(k, v)|$ and $|g(k, u)|$ are bounded, so we can choose some appropriate positive numbers M_1, M_2 such that

$$\begin{aligned} f(k, v) &\geq (\lambda_1 + \varepsilon_1)|v| - M_1, \quad g(k, u) \geq (\lambda_1 + \varepsilon_1)|u| - M_2, \\ &\forall k \in \mathbb{T}_1, u, v \in \mathbb{R}, \end{aligned} \quad (21)$$

where

$$\begin{aligned} M_1 &= \max_{k \in \mathbb{T}_1, |u|, |v| \leq X_1} |f(k, v)| + (\lambda_1 + \varepsilon_1)X_1, \\ M_2 &= \max_{k \in \mathbb{T}_1, |u|, |v| \leq X_1} |g(k, u)| + (\lambda_1 + \varepsilon_1)X_1. \end{aligned} \quad (22)$$

From (H2), for any given $\varepsilon, \tilde{\varepsilon} > 0$ with $\varepsilon_1 - \varepsilon\|b_1\| > 0, \varepsilon_1 - \tilde{\varepsilon}\|b_2\| > 0$, there is $X_2 > X_1$ such that

$$\beta_1(v) \leq \varepsilon|v|, \quad \beta_2(u) \leq \tilde{\varepsilon}|u|, \quad \forall |u|, |v| > X_2. \quad (23)$$

Let $\beta_1^* = \max_{|v| \leq X_2} \beta_1(v)$ and $\beta_2^* = \max_{|u| \leq X_2} \beta_2(u)$. Then,

$$\beta_1(v) \leq \varepsilon|v| + \beta_1^*, \quad \beta_2(u) \leq \tilde{\varepsilon}|u| + \beta_2^*, \quad u, v \in \mathbb{R}. \quad (24)$$

Thus, we have

$$\begin{aligned} f(k, v) &\geq (\lambda_1 + \varepsilon_1)|v| - a_1(k) - b_1(k)\beta_1(v) - M_1 \\ &\geq (\lambda_1 + \varepsilon_1)|v| - a_1(k) - b_1(k)[\varepsilon|v| + \beta_1^*] - M_1 \\ &\geq (\lambda_1 + \varepsilon_1 - \varepsilon\|b_1\|)|v| - a_1(k) - \beta_1^*b_1(k) - M_1, \\ &\forall k \in \mathbb{T}_1, v \in \mathbb{R}, \end{aligned} \quad (25)$$

$$g(k, u) \geq (\lambda_1 + \varepsilon_1 - \tilde{\varepsilon}\|b_2\|)|u| - a_2(k) - \beta_2^*b_2(k) - M_2,$$

$$\forall k \in \mathbb{T}_1, u \in \mathbb{R}. \quad (26)$$

Note that $\varepsilon, \tilde{\varepsilon}$ can be chosen arbitrarily small, so we can let

$$R > \max\{N_1, N_2, N_3, N_4\}, \quad (27)$$

where

$$\begin{aligned} N_1 &= \frac{2 \sum_{l=1}^T G(l, l)[a_1(l) + \beta_1^*b_1(l) + M_1]}{1 - 2\varepsilon \sum_{l=1}^T G(l, l)b_1(l)}, \\ N_2 &= \frac{2 \sum_{l=1}^T G(l, l)[a_2(l) + \beta_2^*b_2(l) + M_2]}{1 - 2\tilde{\varepsilon} \sum_{l=1}^T G(l, l)b_2(l)}, \\ N_3 &= \frac{(\lambda_1 T + (1 + T)(\varepsilon_1 - \varepsilon\|b_1\|)) \sum_{l=1}^T G(l, l)[a_1(l) + a_2(l) + \beta_1^*b_1(l) + \beta_2^*b_2(l) + M_1 + M_2]}{(\varepsilon_1 - \varepsilon\|b_1\|) - (\lambda_1 T + (1 + T)(\varepsilon_1 - \varepsilon\|b_1\|)) [\varepsilon \sum_{l=1}^T G(l, l)b_1(l) + \tilde{\varepsilon} \sum_{l=1}^T G(l, l)b_2(l)]}, \\ N_4 &= \frac{(\lambda_1 T + (1 + T)(\varepsilon_1 - \varepsilon\|b_2\|)) \sum_{l=1}^T G(l, l)[a_1(l) + a_2(l) + \beta_1^*b_1(l) + \beta_2^*b_2(l) + M_1 + M_2]}{(\varepsilon_1 - \varepsilon\|b_2\|) - (\lambda_1 T + (1 + T)(\varepsilon_1 - \varepsilon\|b_2\|)) [\varepsilon \sum_{l=1}^T G(l, l)b_1(l) + \tilde{\varepsilon} \sum_{l=1}^T G(l, l)b_2(l)]}. \end{aligned} \quad (28)$$

Now, we prove

$$(u, v) - \mathcal{A}(u, v) \neq \mu(\phi, \phi), \quad \forall u, v \in \partial B_R, \mu \geq 0, \quad (29)$$

where $\phi(k) = \sin(k\pi)/(T+1)$, $k \in \mathbb{T}_2$. We argue this claim by indirection. Suppose that there exist $u, v \in \partial B_R, \mu \geq 0$ such that

$$(u, v) - \mathcal{A}(u, v) = \mu(\phi, \phi). \quad (30)$$

$$u(k) = (\mathcal{T}v)(k) + \mu\phi(k) = \sum_{l=1}^T G(k, l) f(l, v(l)) + \mu\phi(k), \quad (31)$$

$$v(k) = (\mathcal{S}u)(k) + \mu\phi(k) = \sum_{l=1}^T G(k, l) g(l, u(l)) + \mu\phi(k). \quad (32)$$

$$\begin{aligned} \tilde{v}(k) &= \sum_{l=1}^T G(k, l) [a_1(l) + b_1(l)\beta_1(v(l)) + M_1], \\ \tilde{u}(k) &= \sum_{l=1}^T G(k, l) [a_2(l) + b_2(l)\beta_2(u(l)) + M_2]. \end{aligned} \quad (33)$$

Then by Lemma 3, $\tilde{u}, \tilde{v} \in P_0$, and we also have

$$\begin{aligned} u(k) + \tilde{v}(k) &= \sum_{l=1}^T G(k, l) [f(l, v(l)) + a_1(l) \\ &\quad + b_1(l)\beta_1(v(l)) + M_1] + \mu\phi(k), \\ v(k) + \tilde{u}(k) &= \sum_{l=1}^T G(k, l) [g(l, u(l)) + a_2(l) \\ &\quad + b_2(l)\beta_2(u(l)) + M_2] + \mu\phi(k). \end{aligned} \quad (34)$$

Using (24) and (25), we have

$$\begin{aligned} f(l, v(l)) + a_1(l) + b_1(l)\beta_1(v(l)) + M_1 &\in P, \\ g(l, u(l)) + a_2(l) + b_2(l)\beta_2(u(l)) + M_2 &\in P. \end{aligned} \quad (35)$$

So, from Lemma 3 and Remark 1, we have

$$v + \tilde{u}, u + \tilde{v} \in P_0. \quad (36)$$

Note that $u, v \in \partial B_R$, and using (24), $R > N_1$, and $R > N_2$, we have

$$\begin{aligned} \|\tilde{v}\| &\leq \sum_{l=1}^T G(l, l) [a_1(l) + b_1(l)\beta_1(v(l)) + M_1] \\ &\leq \sum_{l=1}^T G(l, l) [a_1(l) + b_1(l)(\varepsilon\|v\| + \beta_1^*) + M_1] < \frac{R}{2}, \\ \|\tilde{u}\| &\leq \sum_{l=1}^T G(l, l) [a_2(l) + b_2(l)(\tilde{\varepsilon}\|u\| + \beta_2^*) + M_2] < \frac{R}{2}. \end{aligned} \quad (37)$$

It is noted that $\|u\| = \|v\| = R$, $u + \tilde{u} + \tilde{v} \in P_0$, and $v + \tilde{u} + \tilde{v} \in P_0$. Therefore, we get

$$\begin{aligned} u(k) + \tilde{u}(k) + \tilde{v}(k) &\geq \frac{1}{T} \|u + \tilde{u} + \tilde{v}\| \geq \frac{1}{T} (\|u\| - \|\tilde{u} + \tilde{v}\|) \\ &\geq \frac{1}{T} [\|u\| - (\|\tilde{u}\| + \|\tilde{v}\|)], \\ v(k) + \tilde{u}(k) + \tilde{v}(k) &\geq \frac{1}{T} \|v + \tilde{u} + \tilde{v}\| \geq \frac{1}{T} (\|v\| - \|\tilde{u} + \tilde{v}\|) \\ &\geq \frac{1}{T} [\|v\| - (\|\tilde{u}\| + \|\tilde{v}\|)]. \end{aligned} \quad (38)$$

Using $R > N_3$, we have

$$\begin{aligned} &(\varepsilon_1 - \varepsilon\|b_1\|) \sum_{l=1}^T G(k, l) [v(l) + \tilde{u}(l) + \tilde{v}(l)] - (\lambda_1 + \varepsilon_1 - \varepsilon\|b_1\|) \sum_{l=1}^T G(k, l) [\tilde{u}(l) + \tilde{v}(l)] \\ &\geq \frac{\varepsilon_1 - \varepsilon\|b_1\|}{T} \sum_{l=1}^T G(k, l) [R - (\|\tilde{u}\| + \|\tilde{v}\|)] - (\lambda_1 + \varepsilon_1 - \varepsilon\|b_1\|) \sum_{l=1}^T G(k, l) [(\|\tilde{u}\| + \|\tilde{v}\|)] \geq 0, \end{aligned} \quad (39)$$

and $R > N_4$ implies that

$$(\varepsilon_1 - \tilde{\varepsilon}\|b_2\|) \sum_{l=1}^T G(k, l) [u(l) + \tilde{u}(l) + \tilde{v}(l)] - (\lambda_1 + \varepsilon_1 - \tilde{\varepsilon}\|b_2\|) \sum_{l=1}^T G(k, l) [\tilde{u}(l) + \tilde{v}(l)] \geq 0. \quad (40)$$

Consequently, we obtain

$$\begin{aligned}
(\mathcal{T}v)(k) + \tilde{v}(k) &= \sum_{l=1}^T G(k, l) [f(l, v(l)) + a_1(l) + b_1(l)\beta_1(v(l)) + M_1] \\
&\geq \sum_{l=1}^T G(k, l) [(\lambda_1 + \varepsilon_1 - \varepsilon\|b_1\|)|v(l)| - a_1(l) - \beta_1^* b_1(l) - M_1 + a_1(l) + b_1(l)\beta_1(v(l)) + M_1] \\
&\geq \sum_{l=1}^T G(k, l) [(\lambda_1 + \varepsilon_1 - \varepsilon\|b_1\|)|v(l)| - \beta_1^* b_1(l) + b_1(l)(\varepsilon|v(l)| + \beta_1^*)] \geq (\lambda_1 + \varepsilon_1 - \varepsilon\|b_1\|) \sum_{l=1}^T G(k, l)|v(l)| \\
&\geq (\lambda_1 + \varepsilon_1 - \varepsilon\|b_1\|) \sum_{l=1}^T G(k, l)[v(l) + \tilde{u}(l) + \tilde{v}(l)] - (\lambda_1 + \varepsilon_1 - \varepsilon\|b_1\|) \sum_{l=1}^T G(k, l)[\tilde{u}(l) + \tilde{v}(l)] \\
&\geq \lambda_1 \sum_{l=1}^T G(k, l)[v(l) + \tilde{u}(l) + \tilde{v}(l)] \geq \lambda_1 \sum_{l=1}^T G(k, l)[v(l) + \tilde{u}(l)], \\
(\mathcal{S}u)(k) + \tilde{u}(k) &= \sum_{l=1}^T g(k, l) [g(l, v(l)) + a_2(l) + b_2(l)\beta_2(u(l)) + M_2] \\
&\geq \sum_{l=1}^T G(k, l) [(\lambda_1 + \varepsilon_1 - \tilde{\varepsilon}\|b_2\|)|u(l)| - a_2(l) - \beta_2^* b_2(l) - M_2 + a_2(l) + b_2(l)\beta_2(u(l)) + M_2] \\
&\geq (\lambda_1 + \varepsilon_1 - \tilde{\varepsilon}\|b_2\|) \sum_{l=1}^T G(k, l)|u(l)| \\
&\geq (\lambda_1 + \varepsilon_1 - \tilde{\varepsilon}\|b_2\|) \sum_{l=1}^T G(k, l)[u(l) + \tilde{u}(l) + \tilde{v}(l)] - (\lambda_1 + \varepsilon_1 - \tilde{\varepsilon}\|b_2\|) \sum_{l=1}^T G(k, l)[\tilde{u}(l) + \tilde{v}(l)] \\
&\geq \lambda_1 \sum_{l=1}^T G(k, l)[u(l) + \tilde{u}(l) + \tilde{v}(l)] \\
&\geq \lambda_1 \sum_{l=1}^T G(k, l)[u(l) + \tilde{v}(l)].
\end{aligned} \tag{41}$$

As a result, we get

$$(\mathcal{T}v)(k) + (\mathcal{S}u)(k) + \tilde{u}(k) + \tilde{v}(k) \geq \lambda_1 (L(u + v + \tilde{u} + \tilde{v}))(k). \tag{42}$$

In view of (31) and (32), we see

$$\begin{aligned}
u(k) + v(k) + \tilde{u}(k) + \tilde{v}(k) &= (\mathcal{T}v)(k) + (\mathcal{S}u)(k) + \tilde{u}(k) \\
&\quad + \tilde{v}(k) + 2\mu\phi(k) \\
&\geq \lambda_1 (L(u + v + \tilde{u} + \tilde{v}))(k) \\
&\quad + 2\mu\phi(k) \geq 2\mu\phi(k).
\end{aligned} \tag{43}$$

Define $\mu^* = \sup S_\mu := \sup\{\mu > 0 : u + v + \tilde{u} + \tilde{v} \geq 2\mu\phi\}$. Then, $S_\mu \neq \emptyset$, $\mu^* \geq \mu$ and $u + v + \tilde{u} + \tilde{v} \geq 2\mu^*\phi$. From $\phi = \lambda_1 L\phi$, we obtain

$$\lambda_1 L(u + v + \tilde{u} + \tilde{v}) \geq \lambda_1 L(2\mu^*\phi) = 2\mu^*\lambda_1 L\phi = 2\mu^*\phi. \tag{44}$$

Hence,

$$u + v + \tilde{u} + \tilde{v} \geq \lambda_1 L(u + v + \tilde{u} + \tilde{v}) + 2\mu\phi \geq 2(\mu^* + \mu)\phi, \tag{45}$$

which contradicts the definition of μ^* . Therefore, (29) holds, and from Lemma 4, we obtain

$$\deg(I - \mathcal{A}, B_R, 0) = 0. \tag{46}$$

This completes the proof. $\square$

Proof of Theorem 1. From (H4), there exist $\varepsilon_2 \in (0, \lambda_1)$ and $r \in (0, R)$ such that

$$\begin{aligned}
|f(k, v)| &\leq (\lambda_1 - \varepsilon_2)|v|, |g(k, u)| \leq (\lambda_1 - \varepsilon_2)|u|, \\
\forall k \in \mathbb{T}_1, u, v \in \mathbb{R} \text{ with } |u|, |v| &\leq r.
\end{aligned} \tag{47}$$

This implies that

$$\begin{aligned}
|(\mathcal{T}v)(k)| &= \left| \sum_{l=1}^T G(k, l) f(l, v(l)) \right| \leq \sum_{l=1}^T G(k, l) |f(l, v(l))| \leq (\lambda_1 - \varepsilon_2) \sum_{l=1}^T G(k, l) |v(l)|, \\
|(\mathcal{S}u)(k)| &= \left| \sum_{l=1}^T G(k, l) g(l, u(l)) \right| \leq \sum_{l=1}^T G(k, l) |g(l, u(l))| \leq (\lambda_1 - \varepsilon_2) \sum_{l=1}^T G(k, l) |u(l)|.
\end{aligned} \tag{48}$$

Consequently, we have

$$|(\mathcal{T}v)(k)| + |(\mathcal{S}u)(k)| \leq (\lambda_1 - \varepsilon_2) \sum_{l=1}^T G(k, l) [|u(l)| + |v(l)|]. \tag{49}$$

Now, we prove that

$$(u, v) \neq \mu \mathcal{A}(u, v), \tag{50}$$

for all $u, v \in \partial B_r$ and $\mu \in [0, 1]$. We argue by contradiction. Suppose that there exist $u, v \in \partial B_r$ and $\mu \in [0, 1]$ such that

$$(u, v) = \mu \mathcal{A}(u, v). \tag{51}$$

Therefore,

$$u(k) = \mu(\mathcal{T}v)(k), \text{ and } v(k) = \mu(\mathcal{S}u)(k), \quad k \in \mathbb{T}_1. \tag{52}$$

Hence, we have

$$\begin{aligned}
|u(k)| + |v(k)| &\leq |(\mathcal{T}v)(k)| + |(\mathcal{S}u)(k)| \\
&\leq (\lambda_1 - \varepsilon_2) \sum_{l=1}^T G(k, l) [|u(l)| + |v(l)|].
\end{aligned} \tag{53}$$

From Lemma 1 (i) and Lemma 2, we have

$$\lambda_1 \sum_{k=1}^T G(k, l) \phi(k) = \phi(l), \quad \forall l \in \mathbb{T}_1. \tag{54}$$

Multiplying both sides of (53) by $\sin(k\pi)/(T+1)$, then summing from 1 to T , and using (54), we obtain

$$\sum_{k=1}^T [|u(k)| + |v(k)|] \frac{\sin(k\pi)}{(T+1)} \leq (\lambda_1 - \varepsilon_2) \sum_{k=1}^T \frac{\sin(k\pi)}{(T+1)} \sum_{l=1}^T G(k, l) [|u(l)| + |v(l)|] = \frac{\lambda_1 - \varepsilon_2}{\lambda_1} \sum_{l=1}^T [|u(l)| + |v(l)|] \frac{\sin(l\pi)}{(T+1)}. \tag{55}$$

This implies that

$$\sum_{k=1}^T [|u(k)| + |v(k)|] \frac{\sin(k\pi)}{(T+1)} = 0. \tag{56}$$

Because $\sin(k\pi)/(T+1) \geq 0 (\neq 0)$ for $k \in \mathbb{T}_1$, we have $|u(k)| + |v(k)| \equiv 0$, $k \in \mathbb{T}_1$. This contradicts $u, v \in \partial B_r$. Therefore, (50) holds, and Lemma 5 implies that

$$\deg(I - \mathcal{A}, B_r, 0) = 1. \tag{57}$$

Combining this with Lemma 6, we have

$$\deg\left(I - \mathcal{A}, \frac{B_R}{B_r}, 0\right) = \deg(I - \mathcal{A}, B_R, 0) - \deg(I - \mathcal{A}, B_r, 0) = -1. \tag{58}$$

Therefore, the operator $\mathcal{A}$ has at least one fixed point in $B_R/\overline{B_r}$, and (6) has at least one nontrivial solution. This completes the proof. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

This study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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