Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 5136104, 5 pages
https://doi.org/10.1155/2020/5136104

Research Article

Anti-Ramsey Numbers in Complete k-Partite Graphs

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Received 22 March 2020; Revised 29 July 2020; Accepted 17 August 2020; Published 7 September 2020

Academic Editor: Haipeng Peng

The anti-Ramsey number $AR(G, H)$ is the maximum number of colors in an edge-coloring of G such that G contains no rainbow subgraphs isomorphic to H. In this paper, we discuss the anti-Ramsey numbers $AR(K_{p_1,p_2,\ldots,p_k}, \mathcal{T}_n)$, $AR(K_{p_1,p_2,\ldots,p_k}, \mathcal{M})$, and $AR(K_{p_1,p_2,\ldots,p_k}, \mathcal{C})$ of $K_{p_1,p_2,\ldots,p_k}$, where $\mathcal{T}_n$, $\mathcal{M}$, and $\mathcal{C}$ denote the family of all spanning trees, the family of all perfect matchings, and the family of all Hamilton cycles in $K_{p_1,p_2,\ldots,p_k}$, respectively.

1. Introduction

Let G be a graph, a k-edge-coloring of a graph $G = (V, E)$ is a mapping c: $E \longrightarrow C$, where C is a set of colors, namely, $C = \{1, 2, \ldots, k\}$ [1]. A subgraph H of an edge-colored graph G is rainbow if all of its edges have different colors. An edge-colored graph is called rainbow graph if all the colors on the edges are distinct. A representing subgraph in an edge-coloring of G is a spanning subgraph obtained by taking one edge of each color. The anti-Ramsey number $AR(G, H)$ is the maximum number of colors in an edge-coloring of G with no rainbow copy of H. Rainbow coloring of graphs also has its application in practice. It comes from the secure communication of information between agencies of government. The anti-Ramsey number was introduced by Erdös, Simonovits, and Sos in 1973 [2]. It has been shown that the anti-Ramsey number $AR(G, H)$ is closely related to Turan number. The Turan number $ex(n, H)$ is the maximum number of edges in a graph G on n vertices which does not contain any subgraph isomorphic to H. Erdös et al. conjectured that $AR(K_n, C_k) = n((k - 2/2) + (1/k - 1)) + O(1)$, for every fixed $k \geq 3$ [2]. The conjecture is proved completely for all $k \geq 3$ in [3] by Montellano-Ballesteros and Neumann-Lara. The anti-Ramsey numbers for some other special graph classes in complete graphs have also been studied, including independent cycles [4], stars [5], spanning trees [6], and matchings [7, 8]. The anti-Ramsey problems for rainbow matchings, cycles, and trees in complete bipartite graphs have been studied in [9–11]. Some other graphs were also considered as the host graphs in anti-Ramsey problems, such as hypergraphs [12], hypecubes [13], plane triangulations [14], and planar graphs [15].

It is natural to consider that the anti-Ramsey problems for rainbow matchings, cycles, and trees in complete k-partite graphs. In this paper, we are interested in the anti-Ramsey numbers for spanning trees, perfect matchings, and Hamilton cycles in complete k-partite graphs. A complete k-partite graph is a graph whose vertices can be partitioned into k different independent sets, and any two vertices from different independent sets are connected by an edge. A complete k-partite graph, with partitions $S_1, S_2, \ldots, S_k$, $k \geq 2$, is denoted by $K_{p_1,p_2,\ldots,p_k}$, without loss of generality; in the following, we always assume that $|S_1| = p_1, |S_2| = p_2, \ldots, |S_k| = p_k$, $p_1 \geq p_2 \geq \ldots \geq p_k \geq 2$, $p_1 + p_2 + \ldots + p_k = \sum_{i=1}^{n} p_i = n$. If $p_1 = p_2 = \ldots = p_k = 1$, $G = K_k$ is a complete graph. Bialostocki and Voxman proved that $AR(K_n, T_n) = \binom{n-2}{2} + 1$, where T_n denotes the family of all spanning trees in K_n [6]. The maximum number of colors in an edge-coloring of K_n with no rainbow perfect matching (for even n) is $\binom{n-3}{2} + 2$, when $n \geq 14$ [8].

2. Main Result

The family of all spanning trees in $K_{p_1,p_2,\ldots,p_k}$ is denoted by $\mathcal{T}_n$. The maximum number of colors in an edge-coloring of $K_{p_1,p_2,\ldots,p_k}$ not containing any rainbow spanning tree is denoted by $AR(K_{p_1,p_2,\ldots,p_k},\mathcal{T}_n)$.

Theorem 1. *If* $|S_1| = p_1, |S_2| = p_2, \ldots, |S_k| = p_k$, $p_1 \geq p_2 \geq \ldots \geq p_k \geq 2$, $k \geq 2$, *then*

$$AR\left(K_{p_1,p_2,\ldots,p_k},\mathcal{T}_n\right) \geq \begin{cases} \sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i + 1, & p_1 > p_2, \\ \sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i + 2, & p_1 = p_2. \end{cases} \quad (1)$$

Proof. Let $K_{p_1,p_2,\ldots,p_k}$ be a complete k-partite graph with vertex set $S_1 \cup S_2 \cup \ldots \cup S_k$, $k \geq 2$, $|S_1| = p_1, |S_2| = p_2, \ldots, |S_k| = p_k$, $p_1 \geq p_2 \geq \ldots \geq p_k \geq 2$.

The proof of the theorem is distinguished into the following two cases (see Figure 1):

Case 1: $p_1 > p_2$.

There is an edge-coloring of $K_{p_1,p_2,\ldots,p_k}$ using $\sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i + 1$ colors such that $K_{p_1,p_2,\ldots,p_k}$ does not contain any rainbow spanning tree T_n.

Firstly, fix two vertices v_1 and v_2 from S_1 and color all edges incident with v_1 and v_2 by some color, say c_1, that is, $c(v_1u_i) = c_1$ and $c(v_2u_i) = c_1$, for all vertices $u_i \in V(K_{p_1,p_2,\ldots,p_k}) - S_1$. Since $|E(K_{p_1,p_2,\ldots,p_k})| = \sum_{1\leq i<j\leq k} p_i p_j$, the number of remaining edges which are not colored is $\sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i$. Then, color all other edges of $K_{p_1,p_2,\ldots,p_k}$ using $\sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i$ colors such that each appears on one edge. Assume that there is a rainbow spanning tree T_n of $K_{p_1,p_2,\ldots,p_k}$ in this coloring, and then the spanning tree T_n must contain two edges with the same color c_1, one incident with v_1 and the other incident with v_2, a contradiction. Thus,

$$AR\left(K_{p_1,p_2,\ldots,p_k},\mathcal{T}_n\right) \geq \sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i + 1. \quad (2)$$

Case 2: $p_1 = p_2$.

If we use $\sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i + 2$ different colors to color the edges of $K_{p_1,p_2,\ldots,p_k}$, then the $K_{p_1,p_2,\ldots,p_k}$ does not contain any rainbow spanning tree T_n.

Fix vertices v_1 from S_1 and u_1 from S_2. Firstly, color the edges incident with v_1 and u_1 by color c_1, that is, $c(v_1w_1) = c_1$, for all vertices $w_1 \in V(K_{p_1,p_2,\ldots,p_k}) - S_1$ and $c(u_1w_2) = c_1$, for all vertices $w_2 \in V(K_{p_1,p_2,\ldots,p_k}) - S_2$, then color the remaining edges of $K_{p_1,p_2,\ldots,p_k}$ using $\sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i + 1$ colors such that each appears on one edge, and the number of colors is $\sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i + 2$. Now, every spanning tree T_n of $K_{p_1,p_2,\ldots,p_k}$ has at least two edges of the same color c_1. Thus,

$$AR\left(K_{p_1,p_2,\ldots,p_k},\mathcal{T}_n\right) \geq \sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i + 2. \quad (3)$$

□

Theorem 2. *If* $|S_1| = p_1, |S_2| = p_2, \ldots, |S_k| = p_k$, $p_1 \geq p_2 \geq \ldots \geq p_k \geq 2$, $k \geq 2$, *then*

$$AR\left(K_{p_1,p_2,\ldots,p_k},\mathcal{T}_n\right) \leq \sum_{1\leq i<j\leq k} p_i p_j - \sum_{i=2}^{k} p_i. \quad (4)$$

Proof. We consider an arbitrary edge-coloring of $K_{p_1,p_2,\ldots,p_k}$ using $\sum_{1\leq i<j\leq k} p_i p_j - \sum_{i=2}^{k} p_i + 1$ different colors. We only show that there is a spanning tree T_n of $K_{p_1,p_2,\ldots,p_k}$. We choose a representing subgraph G from $K_{p_1,p_2,\ldots,p_k}$ with $|E(G)| = \sum_{1\leq i<j\leq k} p_i p_j - \sum_{i=2}^{k} p_i + 1$. Note that $K_{p_1,p_2,\ldots,p_k}$ is disconnected by deleting at least $\sum_{i=2}^{k} p_i$ edges. Thus, G is connected. G contains a rainbow spanning tree T_n since every connected graph has a spanning tree.

The family of all Hamilton cycles in $K_{p_1,p_2,\ldots,p_k}$ is denoted by $\mathcal{C}$. $AR(K_{p_1,p_2,\ldots,p_k},\mathcal{C})$ is the maximum number of colors in an edge-coloring $K_{p_1,p_2,\ldots,p_k}$ not containing any rainbow Hamilton cycle.

In order to prove our main result, we need the following lemma. □

Lemma 1 *(Dirac's theorem, see [1]). If G is a graph on $n \geq 3$ vertices such that $\delta(G) \geq (n/2)$, then G is Hamiltonian.*

Theorem 3. *Let $K_{p_1,p_2,\ldots,p_k}$ be a complete k-partite graph with $p_1 \geq p_2 \geq \ldots \geq p_k \geq 2$ and $k \geq 2$; if $\sum_{i=2}^{k} p_i \geq p_1$, then $K_{p_1,p_2,\ldots,p_k}$ must have a Hamilton cycle.*

Proof. By assumption and the structure of $K_{p_1,p_2,\ldots,p_k}$, it is clear that $\delta(K_{p_1,p_2,\ldots,p_k}) = \sum_{i=2}^{k} p_i$, and according to Dirac's Theorem, $\sum_{i=2}^{k} p_i \geq (|V(K_{p_1,p_2,\ldots,p_k})|/2) = (\sum_{i=1}^{k} p_i/2)$, $K_{p_1,p_2,\ldots,p_k}$ have a Hamilton cycle. In fact, $\sum_{i=2}^{k} p_i \geq (\sum_{i=1}^{k} p_i/2)$, namely, $\sum_{i=2}^{k} p_i \geq p_1$. The proof is finished. □

Theorem 4. *If* $p_1 \geq p_2 \geq \ldots \geq p_k \geq 2$, $k \geq 2$, $\sum_{i=2}^{k} p_i \geq p_1$, *then*

$$AR\left(K_{p_1,p_2,\ldots,p_k},\mathcal{C}\right) \geq \sum_{1\leq i<j\leq k} p_i p_j - \sum_{i=2}^{k} p_i + 1. \quad (5)$$

Proof. By assumption and Theorem 4, $K_{p_1,p_2,\ldots,p_k}$ is clear Hamiltonian. Now, we show that there is an edge-coloring of $K_{p_1,p_2,\ldots,p_k}$ using $\sum_{1\leq i<j\leq k} p_i p_j - \sum_{i=2}^{k} p_i + 1$ colors such that $K_{p_1,p_2,\ldots,p_k}$ does not contain rainbow Hamilton cycle. Firstly, fix any one vertex v from S_1, color all the edges incident with v by color c_1, and then color all other edges of $K_{p_1,p_2,\ldots,p_k}$ using $\sum_{1\leq i<j\leq k} p_i p_j - \sum_{i=2}^{k} p_i$ colors such that each appears on one edge. Note that every Hamilton cycle of $K_{p_1,p_2,\ldots,p_k}$ must contain two edges incident with v, and the

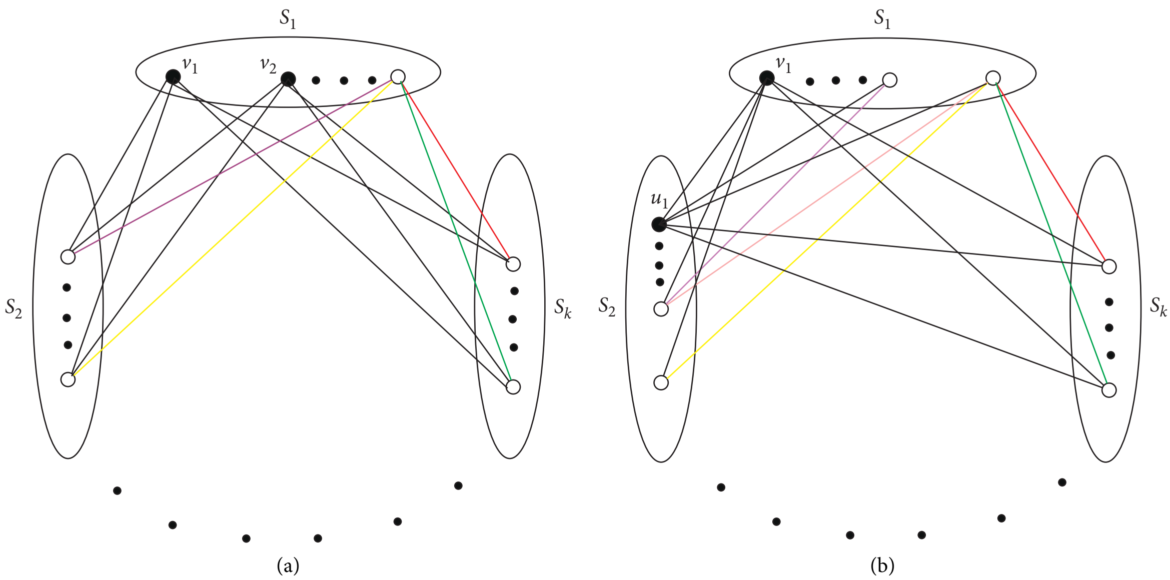

Figure 1: Two ways of coloring in case 1 and case 2.

two edges have the same color c_1. So, $K_{p_1,p_2,\ldots,p_k}$ has no rainbow Hamilton cycle. Thus,

$$AR\left(K_{p_1,p_2,\ldots,p_k},\mathscr{C}\right) \geq \sum_{1\leq i<j\leq k} p_i p_j - \sum_{i=2}^{k} p_i + 1. \tag{6}$$

In order to prove the next main theorem, we need the following definition.

Let G be a k-partite graph with vertex set $S_1 \cup S_2 \cup \ldots \cup S_k$, if there are two vertices $u \in S_i$ and $v \in S_j$, $i \neq j$, $i, j \in \{1, 2, \ldots, k\}$, $uv \notin E(G)$, with $d_G(u) + d_G(v) \geq (k-1)p + 1$, then add an edge uv to $E(G)$. The closure of G is the graph obtained from G by repeating this step until there are no such pair of vertices, denoted by $c(G)$. □

Lemma 2 (see [1]). *G is a simple graph, and then G contains a Hamilton cycle if and only if its closure $c(G)$ is Hamiltonian.*

Theorem 5. *Let G be a k-partite graph with $|S_1| = |S_2| = \ldots = |S_k| = p \geq 2$. Suppose $d_1^j, d_2^j, \ldots, d_p^j$ are the vertex degrees of S_j of G, all in nondecreasing order, where $1 \leq j \leq k, k \geq 2$. If $d_i^j > i$ for each $i \leq ((k-1)p/2)$, then G contains a Hamilton cycle.*

Proof. By Lemma 2, we only need to prove that $c(G)$ is Hamiltonian. $c(G)$ is a k-partite graph with vertex set $S_1 \cup S_2 \cup \ldots \cup S_k$. Suppose $d_1^{(j)}, d_2^{(j)}, \ldots, d_p^{(j)}$ are the vertex degrees of S_j of $c(G)$, all in nondecreasing order, where $1 \leq j \leq k, k \geq 2, p \geq 2$. $d_i^{(j)} \geq d_i^j$ for $1 \leq i \leq p, 1 \leq j \leq k$. To the contrary, suppose $c(G)$ contains no Hamilton cycle, then $c(G)$ is not a complete k-partite graph. Let u and v be two vertices in $c(G)$, $u \in S_i$, $v \in S_j$, $i \neq j$, $uv \notin E(G)$, and $d_{c(G)}(u) + d_{c(G)}(v) \leq (k-1)p$. If $d_{c(G)}(u) \leq d_{c(G)}(v)$, set $i = d_{c(G)}(u) \leq ((k-1)p/2)$. $(k-1)p - d_{c(G)}(v) \geq d_{c(G)}(u) = i$, $c(G)$ contains at least $(k-1)p - d_{c(G)}(v) \geq i$ vertices that are not adjacent to v, and each of which has degree at most $d_{c(G)}(u) = i$. Thus, we can find some vertex w whose degree is at most $d_{c(G)}(u) = i$ in $c(G)$, which implies that $i \geq d_{c(G)}(w) \geq d_G(w)$, that is $i \geq d_i^{(j)} \geq d_i^j$, a contradiction. □

Theorem 6. *If $p_1 \geq p_2 \geq \ldots \geq p_k = p \geq 2$, $k \geq 2$, then*

$$AR\left(K_{p_1,p_2,\ldots,p_k},\mathscr{C}\right) = \left(\frac{k(k-1)}{2}\right)p^2 - (k-1)p + 1. \tag{7}$$

Proof. By Theorem 4, we can easily prove the lower bound. We consider an arbitrary edge-coloring of $K_{p_1,p_2,\ldots,p_k}$ using $(k(k-1)/2)p^2 - (k-1)p + 2$ different colors, and we will find a rainbow Hamilton cycle in $K_{p_1,p_2,\ldots,p_k}$. We choose a representing subgraph G with $|E(G)| = (k(k-1)/2)p^2 - (k-1)p + 2$. Let $d_1^j, d_2^j, \ldots, d_p^j$ be the vertex degrees of S_j of G, all in nondecreasing order. If $d_i^j > i$ for each $i \leq ((k-1)p/2)$, $1 \leq j \leq k$, then G must contain a Hamilton cycle. If not, we assume that there exists i^* such that $d_{i^*}^j \leq i^*$, $i^* \leq ((k-1)p/2)$. Without loss of generality, we assume that $d_1^1 \leq d_1^2 \leq \ldots \leq d_1^k$ and $d_{i^*}^1 \leq i^*$. We have $i^* \cdot i^* + (p - i^*)(k-1)p + ((k-1)(k-2)/2)p^2 \geq (k(k-1)/2)p^2 - (k-1)p + 2 = |E(G)|$, that is, $(i^*)^2 - p(k-1)i^* + p(k-1) - 2 \geq 0$. Set $f(i) = i^2 - p(k-1)i + p(k-1) - 2 =](i - (p(k-1) - 1))(i-1) - 1$, $f(1) = -1 < 0$, $f(2) = 2 - p(k-1) < 0$, $f((k-1)p/2) \leq f(p(k-1) - 1) = -1 < 0$. We conclude that $f(i) < 0$ for each $1 \leq i \leq ((k-1)p/2)$. Thus, we have $d_i^j > i$ for each $i \leq ((k-1)p/2)$, $1 \leq j \leq k$. By Theorem 7, G must have a rainbow Hamilton cycle.

A matching in a graph is a set of nonadjacent edges. A perfect matching M is a matching which saturates every vertex of the graph. The family of all perfect matchings

$K_{p_1,p_2,\ldots,p_k}$ is denoted by $\mathscr{M}$. $AR(K_{p_1,p_2,\ldots,p_k},\mathscr{M})$ is the maximum number of colors in an edge-coloring of $K_{p_1,p_2,\ldots,p_k}$ not containing any rainbow perfect matching.

In [9], it has been shown that $AR(K_{m,n},kK_2) = m(k-2)+1, m \geq n \geq k \geq 3$, which is the maximum numbers of colors in an edge-coloring of $K_{m,n}$ that contains no rainbow kK_2. Now, we consider the maximum numbers of colors in an edge-coloring of $K_{p_1,p_2,\ldots,p_k}$ not containing any rainbow perfect matching.

Tutte gives the sufficient and necessary condition of a graph with perfect matchings. □

Lemma 3 (Tutte's theorem, see [1]). *A graph G has a perfect matching if and only if* $C_0(G-S) \leq |S|$, *for all* $S \subseteq V(G)$, *where* $C_0(G-S)$ *is the number of odd components of* $G-S$.

According to Tutte's theorem, we give the following sufficient condition that completes k-partite graph $K_{p_1,p_2,\ldots,p_k}$ have a perfect matching.

Theorem 7. *If* $p_1 \geq p_2 \geq \ldots \geq p_k \geq 1$, $k \geq 2$, $\sum_{i=1}^{k} p_i$ *is even and* $\sum_{i=2}^{k} p_i \geq p_1$, *then the complete k-partite graph* $K_{p_1,p_2,\ldots,p_k}$ *must have a perfect matching.*

Proof. Let S be a subset of $V(K_{p_1,p_2,\ldots,p_k})$, and we consider the following three cases according to the cardinality of S.

Case 1: $0 \leq |S| < \sum_{i=2}^{k} p_i$.

Note that $K_{p_1,p_2,\ldots,p_k}$ is disconnected by deleting at least $\sum_{i=2}^{k} p_i$ vertices. For $0 \leq |S| < \sum_{i=2}^{k} p_i$, it is clear that $K_{p_1,p_2,\ldots,p_k} - S$ is connected. If $|V(K_{p_1,p_2,\ldots,p_k} - S)|$ is even, then $C_0(K_{p_1,p_2,\ldots,p_k} - S) = 0 \leq |S|$, and if $|V(K_{p_1,p_2,\ldots,p_k} - S)|$ is odd, then $C_0(K_{p_1,p_2,\ldots,p_k} - S) = 1$, by assuming that $\sum_{i=1}^{k} p_i$ is even; thus, $|S|$ is odd and $|S| \geq 1$ by the parity. So, $C_0(K_{p_1,p_2,\ldots,p_k} - S) \leq |S|$. By Lemma 3, $K_{p_1,p_2,\ldots,p_k}$ has a perfect matching.

Case 2: $|S| = \sum_{i=2}^{k} p_i$.

If $|S| = \sum_{i=2}^{k} p_i$, $C_0(K_{p_1,p_2,\ldots,p_k} - S) = p_1 \leq \sum_{i=2}^{k} p_i = |S|$, which meets Lemma 3, then $K_{p_1,p_2,\ldots,p_k}$ has a perfect matching.

Case 3: $|S| > \sum_{i=2}^{k} p_i$.

If $S > \sum_{i=2}^{k} p_i$, $C_0(K_{p_1,p_2,\ldots,p_k} - S) = \sum_{i=1}^{k} p_i - |S| < p_1 < |S|$, which also meets Lemma 9, then $K_{p_1,p_2,\ldots,p_k}$ has a perfect matching.

Therefore, if $\sum_{i=1}^{k} p_i$ is even and $\sum_{i=2}^{k} p_i \geq p_1$, $K_{p_1,p_2,\ldots,p_k}$ must have a perfect matching.

In this section, we consider the anti-Ramsey problem of perfect matching in complete k-partite graph $K_{p_1,p_2,\ldots,p_k}$. □

Theorem 8. *If* $p_1 \geq p_2 \geq \ldots \geq p_k \geq 2$, $k \geq 2$, $\sum_{i=1}^{k} p_i$ *is even and* $\sum_{i=2}^{k} p_i \geq p_1$, *then*

$$AR\left(K_{p_1,p_2,\ldots,p_k},\mathscr{M}\right) \geq \sum_{1 \leq i < j \leq k} p_i p_j - 2\sum_{i=2}^{k} p_i + 1. \tag{8}$$

Proof. The known conditions clearly met that $K_{p_1,p_2,\ldots,p_k}$ must have a perfect matching by Theorem 9. Now, we firstly show that there is an edge-coloring of $K_{p_1,p_2,\ldots,p_k}$ using $\sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i + 1$ colors such that $K_{p_1,p_2,\ldots,p_k}$ does not contain any rainbow perfect matching M. Fix two vertices v_1 and v_2 from S_1, color the edges incident with v_1 and v_2 by coloring c_1, and color the remaining edges of $K_{p_1,p_2,\ldots,p_k}$ using $\sum_{1\leq i<j\leq k} p_i p_j - 2\sum_{i=2}^{k} p_i$ colors such that each appears on one edge. It is clear that there is no rainbow perfect matching in $K_{p_1,p_2,\ldots,p_k}$. So, we have

$$AR\left(K_{p_1,p_2,\ldots,p_k},\mathscr{M}\right) \geq \sum_{1 \leq i < j \leq k} p_i p_j - 2\sum_{i=2}^{k} p_i + 1. \tag{9}$$

□

Theorem 9. *If* $p_1 \geq p_2 \geq \ldots \geq p_k = p \geq 2$, $k \geq 2$, kp *is even, then*

$$AR\left(K_{p_1,p_2,\ldots,p_k},\mathscr{M}\right) \leq \left(\frac{k(k-1)}{2}\right)p^2 - (k-1)p + 1. \tag{10}$$

Proof. We consider an arbitrary edge-coloring of $K_{p_1,p_2,\ldots,p_k}$ using $(k(k-1)/2)p^2 - (k-1)p + 2$ different colors, and we choose a representing subgraph G from $K_{p_1,p_2,\ldots,p_k}$. By the proof of Theorem 6, we know that G must have a rainbow Hamilton cycle C. Then, we can find a rainbow perfect matching from C since the number of vertices in $K_{p_1,p_2,\ldots,p_k}$ is even. So,

$$AR\left(K_{p_1,p_2,\ldots,p_k},\mathscr{M}\right) < \left(\frac{k(k-1)}{2}\right)p^2 - (k-1)p + 2. \tag{11}$$

The proof is completed. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The research and publication of our article was funded by the National Natural Science Foundation of China (61662079). H. Bian was supported by the National Natural Science Foundation of China (11761070). H. Z. Yu was supported by the National Natural Science Foundation of China (61662079). J. L. Ding was supported by the 2020 Post-graduate Innovation Project of Xinjiang.

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