Research Article

$L(p, q)$ -Label Coloring Problem with Application to Channel Allocation

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In this paper, the $L(p, q)$ -coloring problem of the graph is studied with application to channel allocation of the wireless network. First, by introducing two new logical operators, some necessary and sufficient conditions for solving the $L(p, q)$ -coloring problem are given. Moreover, it is noted that all solutions of the obtained logical equations are corresponding to each coloring scheme. Second, by using the semitensor product, the necessary and sufficient conditions are converted to an algebraic form. Based on this, all coloring schemes can be obtained through searching all column indices of the zero columns. Finally, the obtained result is applied to analyze channel allocation of the wireless network. Furthermore, an illustration example is given to show the effectiveness of the obtained results in this paper.

1. Introduction

It is well known that the coloring problem is a basic and classical problem in graph theory. Graph coloring is originated from famous conjecture called four-colour conjecture [1] and widely used in many real-life areas [2–4], such as scheduling and timetabling in engineering, air traffic flow management, and channel allocation of mobiles. There are various forms of graph coloring, such as set coloring, list coloring, T -coloring, and $L(n_1, n_2, \dots, n_s)$ -coloring (n_i denotes a nonnegative integer, $i = 1, 2, \dots, s$). The labeling problems of graphs arise in many networking and telecommunication contexts. The channel allocation problem is first formulated as a graph coloring problem by Hale [5]. Furthermore, Griggs and Yeh formulated this problem as a graph labeling problem [6]. $L(p, q)$ -label coloring is one kind of graph labeling, which has major application in channel allocation [5, 7–10]. Thus, it is still more interesting to introduce a new method to study the coloring problem.

Recently, Cheng et al. and Li et al. provided a new mathematical method, which is called the semitensor

product with matrices [11–13] to study logical systems [14–23], probability logical networks [24, 25], game theory [26, 27], coloring problem [1, 10, 28], and some other related fields [29–31]. Wang et al. first studied the graph problem by using the semitensor product [1]. In [1], the maximum (weight) stable set and vertex coloring problems of graphs were investigated with application to the group consensus of multiagent systems, and an algorithm was established to find all the internally stable sets for any graph. In [28], Zhong et al. investigated the minimum stable set and core of the graph and established an algorithm to find all the externally stable sets.

This paper studies the $L(p, q)$ -coloring problem of the graph with application to channel allocation of the wireless network. Some necessary and sufficient conditions for solving the $L(p, q)$ -coloring problem are first made by introducing two new logical operators. Moreover, it is noted that all solutions of the obtained logical equations are corresponding to each coloring scheme. Then, by using the semitensor product, the necessary and sufficient conditions are converted to an algebraic form. Based on this, all coloring

schemes can be obtained through searching all column indices of the zero columns. Finally, the obtained results are applied to analyze channel allocation of the wireless network. Furthermore, an illustration example is given to show the effectiveness of the obtained results in this paper.

The rest of this paper is organized as follows. Section 2 gives some necessary preliminaries on the semitensor product of matrices and $L(p, q)$ -labeling. The main results are shown in Section 3. In Section 4, we apply the obtained results to the channel allocation of the wireless network, which is followed by the conclusion in Section 5.

2. Preliminaries

In this section, we give some necessary preliminaries on the semitensor product, the pseudo-Boolean function, and graph theory, which will be used in the sequel.

First, we give some notations to be used in this paper.

$\mathcal{D}_k = \{0, 1, \dots, k-1\}$, especially, $\mathcal{D} = \mathcal{D}_2 = \{0, 1\}$.

δ_n^i : the i -th column of the identity matrix I_n .

Denote by $\text{Col}_b(B)$ the b -th column of matrix B and by $\text{Col}(B)$ the set of all columns of matrix B .

$\mathbb{R}^{n \times r}$: the set of $n \times r$ real matrices, where $\mathbb{R}$ denotes the set of real numbers.

$\Delta_n = \{\delta_n^i | i = 1, 2, \dots, n\}$, and for simplicity, let $\Delta = \Delta_2$. Identify $k-i \sim \delta_k^i$, $i = 1, 2, \dots, k$, which implies $\mathcal{D}_k \sim \Delta_k$, where $p \sim q$ means they are equivalent.

A matrix $L \in \mathbb{R}^{n \times r}$ is called a logical matrix if columns of L are of the form of δ_n^i . Denote by $\mathcal{L}_{n \times r}$ the set of $n \times r$ logical matrices.

If $L \in \mathcal{L}_{n \times r}$, it can be expressed as $L = [\delta_n^{i_1} \delta_n^{i_2} \dots \delta_n^{i_r}]$. For the sake of compactness, it is briefly denoted by $L = \delta_n[i_1 i_2 \dots i_r]$.

Next, we give some definitions and results about the semitensor product.

Definition 1 (see [11]). The semitensor product of two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is

$$A \ltimes B = (A \otimes I_{\alpha/n})(B \otimes I_{\alpha/p}), \quad (1)$$

where $\alpha = \text{lcm}(n, p)$ is the least common multiple of n and p and $\otimes$ is the Kronecker product.

Throughout this paper, the default matrix product is the semitensor product. The semitensor product is a generalization of the conventional matrix product. Thus, we can simply call it “product” and omit the symbol “ $\ltimes$ ” without confusion.

Definition 2 (see [11]). A swap matrix $W_{[m,n]}$ is an $mn \times mn$ matrix defined as follows: its rows and columns are labeled by double index (i, j) , the columns are arranged by the ordered multiindex $Id(i, j; m, n)$, and the rows are arranged by the ordered multiindex $Id(j, i; n, m)$. Then, the element at the position $[(I, J), (i, j)]$ is

$$w_{(I,J)(i,j)} = \delta_{i,j}^{I,J} = \begin{cases} 1, & I = i \text{ and } J = j; \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Remark 1. When $m = n$, $W_{[m,n]}$ is briefly rewritten as $W_{[n]}$. Furthermore, from Definition 2, $W_{[m,n]}$ can be written as the following form for all m, n :

$$\begin{aligned} W_{[m,n]} &= \delta_{mn} [1 \ m+1 \ \dots \ (i-1)m+1 \ \dots \ (n-1)m+1 \\ &\quad 2 \ m+2 \ \dots \ (i-1)m+2 \ \dots \ (n-1)m+2 \\ &\quad \vdots \\ &\quad m \ m+m \ \dots \ (i-1)m+m \ \dots \ (n-1)m+m] \\ \text{or } W_{[m,n]} &= [\delta_n^1 \ltimes \delta_m^1 \ \dots \ \delta_n^n \ltimes \delta_m^1 : \delta_n^1 \ltimes \delta_m^2 \ \dots \ \delta_n^n \ltimes \delta_m^2 : \dots : \delta_n^1 \ltimes \delta_m^m \ \dots \ \delta_n^n \ltimes \delta_m^m]. \end{aligned} \quad (3)$$

Now, we list some basic properties of the semitensor product [11]:

(1) Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be column vectors. Then,

$$W_{[m,n]}XY = YX. \quad (4)$$

(2) Let $X \in \mathbb{R}^t$ be a column vector. Then,

$$XA = (I_t \otimes A)X. \quad (5)$$

(3) Let $X = \delta_t^i \in \mathbb{R}^t$ be a logical vector. Then,

$$X^2 = M_{r,t}X, \quad (6)$$

where

$$\begin{aligned} M_{r,t} &= \delta_{t^2} [1 \ t+2 \ \dots \ (i-1)t+i \ \dots \ t^2] \\ &= [\delta_t^1 \ltimes \delta_t^1 \ \delta_t^2 \ltimes \delta_t^2 \ \dots \ \delta_t^t \ltimes \delta_t^t] \in \mathcal{L}_{t^2 \times t}. \end{aligned} \quad (7)$$

(4) Let $X = \delta_t^i \in \mathbb{R}^t$ be a logical vector and $A \in \mathbb{R}^{m \times nt}$. Then,

$$AX = A_i, \quad (8)$$

where $A_i \in \mathbb{R}^{m \times n}$ is the i -th block of $A = [A_1 \ A_2 \ \cdots \ A_t]$. Especially, when $n = 1$,

$$AX = \text{Col}_i(A). \quad (9)$$

Lemma 1 (see [11]). Any logical function $y = f(x_1, x_2, \dots, x_n)$ with logical variables $x_i \in \mathcal{D}_k$, $i = 1, \dots, n$, can be expressed in a multilinear form as

$$y = f(x_1, x_2, \dots, x_n) = M_f x_1 x_2 \cdots x_n, \quad (10)$$

where $y \in \Delta_k$ and $M_f \in \mathcal{L}_{k \times k^n}$ is unique, called the structural matrix of f .

Remark 2. The first row of the structural matrix M_f corresponds to the truth value of the logical function $f(x_1, x_2, \dots, x_n)$.

Now, we list the structural matrices for some basic k -valued logical operators [11], which will be used later.

Negation ($\neg$): $x = k - 1 - x$, which has a structural matrix as $M_{\neg,k} = \delta_k[k \ k - 1 \ \cdots \ 2 \ 1]$.

Conjunction ($\wedge$): $x \wedge y = \min\{x, y\}$, which has a structural matrix as

$$M_{\wedge,k} = \delta_k[123 \cdots k : 223 \cdots k : 333 \cdots k : \cdots : kkk \cdots k]. \quad (11)$$

Disjunction ($\vee$): $x \vee y = \max\{x, y\}$, which has a structural matrix as

$$M_{\vee,k} = \delta_k[111 \cdots 1 : 122 \cdots 2 : 123 \cdots 3 : \cdots : 123 \cdots k]. \quad (12)$$

Exclusive or (∇): $x \nabla y = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$, which has a structural matrix as

$$M_{\nabla,k} = \delta_k[1kk \cdots k : k1k \cdots k : kkk1 \cdots k : \cdots : kkk \cdots 1]. \quad (13)$$

Dummy operator (σ_d): $\sigma_d x^2 = x$, which has a structural matrix as

$$E_{d,k} = \delta_k[123 \cdots k : \cdots : 123 \cdots k]. \quad (14)$$

The following concepts and properties will be used in the next section.

Definition 3 (see [12]). An n -ary pseudo-Boolean function $f(x_1, x_2, \dots, x_n)$ is a mapping from $\mathcal{D}^n$ to $\mathbb{R}$, where $\mathcal{D}^n = \mathcal{D} \times \mathcal{D} \times \cdots \times \mathcal{D}$.

A graph $\mathcal{G}$ consists of a vertex (node) set $V = \{v_1, v_2, \dots, v_n\}$ and an edge set $E \subset V \times V$ denoted by $\mathcal{G} = \{V, E\}$.

Lemma 2 (see [32]). Given a simple graph $\mathcal{G} = (V, E)$, an $L(p, q)$ -labeling f of $\mathcal{G}$ is an integer assignment $f: V \rightarrow \{1, 2, \dots, k\}$ such that

$$|f(u) - f(v)| \geq \begin{cases} p, & d(u, v) = 1, \\ q, & d(u, v) = 2, \end{cases} \quad (15)$$

where $u, v \in V$, $d(u, v)$ denotes the distance between u and v , and p, q are two given positive integers.

3. Main Results

In this section, we investigate the $L(p, q)$ -label coloring problem by the semitensor product method and present the main results of this paper.

Consider a graph $\mathcal{G}$ with n nodes $V = \{v_1, v_2, \dots, v_n\}$. Assume that the adjacency matrix, $A = [a_{ij}]$, of $\mathcal{G}$ is given as

$$a_{ij} = \begin{cases} 1, & v_j \in \mathcal{N}_i \\ 0, & v_j \notin \mathcal{N}_i, \end{cases} \quad (16)$$

where $\mathcal{N}_i$ denotes a neighbor set of node i .

It is noted that $a_{ij} = a_{ji}$ for an undirected graph and $a_{ii} = 0$ in our study since the graph G is a simple graph. Furthermore, let $B = [b_{ij}]$, where $b_{ii} = 0$ and $b_{ij} = a_{i1} \times_b b_{1j} + a_{i2} \times_b b_{2j} + \cdots + a_{in} \times_b b_{nj}$, $\times_b$ and $+_b$ are, respectively, Boolean addition and Boolean multiplication. It is easy to obtain that $d(v_i, v_j) = 1$ when $a_{ij} = 1$ and $d(v_i, v_j) = 2$ when $b_{ij} = 1$.

Let $S = \{0, 1, 2, \dots, k-1\}$. For all $v_i \in V$, assign it an integer $y_i \in S$, i.e., $f(v_i) = y_i$. We need two logical operators as

$$\begin{aligned} \phi_{k+}(x) &= \begin{cases} x+1, & x=0, 1, 2, \dots, k-2; \\ k-1, & x=k-1; \end{cases} \\ \phi_{k-}(x) &= \begin{cases} x-1, & x=1, 2, \dots, k-2, k-1; \\ 0, & x=0. \end{cases} \end{aligned} \quad (17)$$

Moreover, the structural matrices are

$$\begin{aligned} M_{\phi_{k+}} &= \delta_k[112 \cdots k-1], \\ M_{\phi_{k-}} &= \delta_k[23 \cdots kk]. \end{aligned} \quad (18)$$

Then, we have the following result to determine whether the $L(p, q)$ -label problem is solved.

Theorem 1. Consider an undirected graph $\mathcal{G}$ with n nodes $V = \{v_1, v_2, \dots, v_n\}$. Its $L(p, q)$ -label problem is solved if and only if the logical equations

$$\left\{ \sum_{i=1}^n \sum_{j>i}^n \left[a_{ij} \bigvee_{s=0}^{p-1} \left\{ \left[y_i \nabla \phi_{k+}^s(y_j) \right] \vee \left[y_i \nabla \phi_{k-}^s(y_j) \right] \right\} \right] \right\} = 0, \quad \sum_{i=1}^n \sum_{j>i}^n \left[b_{ij} \bigvee_{s=0}^{q-1} \left\{ \left[y_i \nabla \phi_{k+}^s(y_j) \right] \vee \left[y_i \nabla \phi_{k-}^s(y_j) \right] \right\} \right] = 0, \quad (19)$$

are solved. That is,

$$\sum_{i=1}^n \sum_{j>i} \left\{ a_{ij} \bigvee_{s=0}^{p-1} \left\{ \left[y_i \nabla \mathcal{O}_{k+}^s(y_j) \right] \vee \left[y_i \nabla \mathcal{O}_{k-}^s(y_j) \right] \right\} + b_{ij} \bigvee_{s=0}^{q-1} \left\{ \left[y_i \nabla \mathcal{O}_{k+}^s(y_j) \right] \vee \left[y_i \nabla \mathcal{O}_{k-}^s(y_j) \right] \right\} \right\} = 0, \quad (20)$$

is solved.

Proof. Necessity: assume that $L(p, q)$ coloring of $\mathcal{G}$ is solvable, and $A = [a_{ij}]$ is the adjacent matrix of the graph $\mathcal{G}$. For all v_i and $v_j \in V$ ($v_i \neq v_j$), when $d(v_i, v_j) = 1$, $|f(v_i) - f(v_j)| \geq p$. It is easy to see that $d(v_i, v_j) = 1 \Leftrightarrow a_{ij} = 1$ and $|f(v_i) - f(v_j)| \geq p \Leftrightarrow f(v_i) \neq f(v_j) \pm s$, $s = 0, 1, 2, \dots, p-1$. That is, $y_i \geq y_j + p$ when $y_i \geq y_j$ or $y_i \leq y_j - p$ when $y_i \leq y_j$.

Since $y_i \in \{0, 1, \dots, k-1\}$, we introduce the logical operators $\mathcal{O}_{k+}$ and $\mathcal{O}_{k-}$, and then we have $y_i \geq y_j + p$ is equivalent to $\bigvee_{s=0}^{p-1} [y_i \nabla \mathcal{O}_{k+}^s(y_j)] = 0$, and $y_i \leq y_j - p$ is equivalent to $\bigvee_{s=0}^{p-1} [y_i \nabla \mathcal{O}_{k-}^s(y_j)] = 0$. That is, $|f(v_i) - f(v_j)| \geq p$ is equivalent to

$$\bigvee_{s=0}^{p-1} \left\{ \left[y_i \nabla \mathcal{O}_{k+}^s(y_j) \right] \vee \left[y_i \nabla \mathcal{O}_{k-}^s(y_j) \right] \right\} = 0. \quad (21)$$

Then, we have $a_{ij} \bigvee_{s=0}^{p-1} \left\{ \left[y_i \nabla \mathcal{O}_{k+}^s(y_j) \right] \vee \left[y_i \nabla \mathcal{O}_{k-}^s(y_j) \right] \right\} = 0$ for all i, j . Since $a_{ij} = a_{ji}$ for the undirected graph and $|y_i - y_j| = |y_j - y_i|$, we obtain that

$$a_{ij} \cdot \left(\bigvee_{s=0}^{p-1} \left\{ \left[y_i \nabla \mathcal{O}_{k+}^s(y_j) \right] \vee \left[y_i \nabla \mathcal{O}_{k-}^s(y_j) \right] \right\} \right) = 0, b_{ij} \cdot \left(\bigvee_{s=0}^{q-1} \left\{ \left[y_i \nabla \mathcal{O}_{k+}^s(y_j) \right] \vee \left[y_i \nabla \mathcal{O}_{k-}^s(y_j) \right] \right\} \right) = 0. \quad (24)$$

From (24), if $a_{ij} = 1$, $\bigvee_{s=0}^{p-1} \left\{ \left[y_i \nabla \mathcal{O}_{k+}^s(y_j) \right] \vee \left[y_i \nabla \mathcal{O}_{k-}^s(y_j) \right] \right\} = 0$. Then, $y_i \nabla \mathcal{O}_{k+}^s(y_j) = 0$ or $y_i \nabla \mathcal{O}_{k-}^s(y_j) = 0$, i.e., $y_i \neq y_j \pm s$, $s = 0, 1, \dots, p-1$. Therefore, $|y_i - y_j| \geq p$. Similarly, if $b_{ij} = 1$, $|y_i - y_j| \geq q$ by (24).

Since $a_{ij} = 1 \Leftrightarrow d(v_i, v_j) = 1$ and $b_{ij} = 1 \Leftrightarrow d(v_i, v_j) = 2$, we have the $L(p, q)$ coloring of the graph $\mathcal{G}$ is solvable, and the proof is complete.

$$\sum_{i=1}^n \sum_{j \neq i} \left\{ a_{ij} \bigvee_{s=0}^{p-1} \left\{ \left[y_i \nabla \mathcal{O}_{k+}^s(y_j) \right] \vee \left[y_i \nabla \mathcal{O}_{k-}^s(y_j) \right] \right\} + b_{ij} \bigvee_{s=0}^{q-1} \left\{ \left[y_i \nabla \mathcal{O}_{k+}^s(y_j) \right] \vee \left[y_i \nabla \mathcal{O}_{k-}^s(y_j) \right] \right\} \right\} = 0, \quad (25)$$

is solved.

Let $x = \delta_k^{k-y}$, $y \in S = \{0, 1, 2, \dots, k-1\}$. Then, $y = [k-1, k-2, \dots, 1, 0]x$. Using the semitensor product and the vector form of logical variables, we have the following results.

$$\sum_{i=1}^n \sum_{j>i} \left\{ a_{ij} \bigvee_{s=0}^{p-1} \left\{ \left[y_i \nabla \mathcal{O}_{k+}^s(y_j) \right] \vee \left[y_i \nabla \mathcal{O}_{k-}^s(y_j) \right] \right\} \right\} = 0. \quad (22)$$

Set $B = [b_{ij}]$, where $b_{ij} = a_{i1} \times_b a_{1j} +_b a_{i2} \times_b a_{2j} +_b \dots +_b a_{in} \times_b a_{nj}$ ($i \neq j$) and $b_{ii} = 0$, $i, j = 1, 2, \dots, n$. Obviously, $d(v_i, v_j) = 2 \Leftrightarrow b_{ij} = 1$. Similarly, when $d(v_i, v_j) = 2$, we have

$$\sum_{i=1}^n \sum_{j>i} \left\{ b_{ij} \bigvee_{s=0}^{q-1} \left\{ \left[y_i \nabla \mathcal{O}_{k+}^s(y_j) \right] \vee \left[y_i \nabla \mathcal{O}_{k-}^s(y_j) \right] \right\} \right\} = 0. \quad (23)$$

Therefore, from (22) and (23), we obtain that (19) is satisfied, and the necessity is proved.

Sufficiency: suppose that (19) is satisfied. Then, for all i, j , we have

It is note that, for a directed graph $\mathcal{G}$, we have the following corollary. $\square$

Corollary 1. Consider a directed graph $\mathcal{G}$ with n nodes $V = \{v_1, v_2, \dots, v_n\}$. Its $L(p, q)$ -label problem is solved if and only if

Theorem 2. Logical equations (19) are solved if and only if there exists at least an integer v ($1 \leq v \leq k^n$) such that the v -th column of matrix M is 0, where

$$\begin{cases} M = \sum_{i=1}^n \sum_{j>i} [a_{ij}M_{ij,p} + b_{ij}M_{ij,q}], \\ M_{ij,s} = M_s M_{ij}, \quad i, j = 1, 2, \dots, n, i \neq j, s = p, q, \\ M_s = J_k M_{d,k}^{s-1} Q_0 [(I_{k^2} \otimes Q_1) M_{r,k^2}] \cdots [(I_{k^2} \otimes Q_s) M_{r,k^2}], \\ M_{ij} = E_{d,k}^{n-2} W_{[k^j, k^{n-j}]} W_{[k^i, k^{j-1-i}]} (i < j), \\ Q_l = M_{d,k} M_{\nabla,k} [I_k \otimes M_{\phi_{k+}}^l] [I_{k^2} \otimes (M_{\nabla,k} [I_k \otimes M_{\phi_{k-}}^l])] M_{r,k^2}, l = 0, 1, \dots, s-1, \end{cases} \quad (26)$$

$J_k = [k-1 \ k-2 \ \cdots \ 10]$, and the product is " $\propto$ "

Proof. Using the semitensor product and the vector form of logical variables, there exists one matrix $M \in R^{1 \times k^n}$ such that the left-hand side of equation (13) is $Mx_1 x_2 \cdots x_n$, where

$x_i = \delta_k^{k-y_i} \in \Delta_k, i = 1, 2, \dots, n$. Equation (13) is solved if and only if there exists at least an integer $i (1 \leq i \leq k^n)$ such that the i th column of matrix M is 0. Now, we only need to study matrix M .

Since the logical form of $[y_i \nabla \phi_{k+}^s(y_j)] \vee [y_i \nabla \phi_{k-}^s(y_j)]$ is

$$\begin{aligned} M_{d,k} [M_{\nabla,k} x_i M_{\phi_{k+}}^s x_j] [M_{\nabla,k} x_i M_{\phi_{k-}}^s x_j] &= M_{d,k} \{M_{\nabla,k} [I_k \otimes M_{\phi_{k+}}^s] x_i x_j\} \{M_{\nabla,k} [I_k \otimes M_{\phi_{k-}}^s] x_i x_j\} \\ &= M_{d,k} M_{\nabla,k} [I_k \otimes M_{\phi_{k+}}^s] [I_{k^2} \otimes (M_{\nabla,k} [I_k \otimes M_{\phi_{k-}}^s])] (x_i x_j)^2 \\ &= M_{d,k} M_{\nabla,k} [I_k \otimes M_{\phi_{k+}}^s] [I_{k^2} \otimes (M_{\nabla,k} [I_k \otimes M_{\phi_{k-}}^s])] M_{r,k} x_i x_j \\ &\triangleq Q_s x_i x_j, \end{aligned} \quad (27)$$

we have

$$\begin{aligned} \bigvee_{s=0}^{p-1} \{[y_i \nabla \phi_{k+}^s(y_j)] \vee [y_i \nabla \phi_{k-}^s(y_j)]\} &= J_k M_{d,k}^{p-1} (Q_0 x_i x_j) (Q_1 x_i x_j) \cdots (Q_{p-1} x_i x_j) \\ &= J_k M_{d,k}^{p-1} Q_0 [(I_{k^2} \otimes Q_1) M_{r,k^2} x_i x_j] \\ &= J_k M_{d,k}^{p-1} Q_0 [(I_{k^2} \otimes Q_1) M_{r,k^2}] \cdots [(I_{k^2} \otimes Q_{p-1}) M_{r,k^2}] x_i x_j \\ &\triangleq M_p x_i x_j (i < j), \end{aligned} \quad (28)$$

where

$$\begin{aligned} M_p &= J_k M_{d,k}^{p-1} Q_0 [(I_{k^2} \otimes Q_1) M_{r,k^2}] \cdots [(I_{k^2} \otimes Q_{p-1}) M_{r,k^2}], \\ Q_s &= M_{d,k} M_{\nabla,k} [I_k \otimes M_{\phi_{k+}}^s] [I_{k^2} \otimes (M_{\nabla,k} [I_k \otimes M_{\phi_{k-}}^s])] M_{r,k^2}, \quad s = 0, 1, \dots, p-1. \end{aligned} \quad (29)$$

Furthermore,

$$\begin{aligned} x_i x_j &= E_{d,k}^{n-2} x_{j+1} \cdots x_n x_{i+1} \cdots x_{j-1} x_1 \cdots x_{i-1} x_i x_j \\ &= E_{d,k}^{n-j} W_{[k^j, k^{n-j}]} x_{i+1} \cdots x_{j-1} x_1 \cdots x_{i-1} x_i x_j \cdots x_n \\ &= E_{d,k}^{n-j} W_{[k^j, k^{n-j}]} W_{[k^i, k^{j-1-i}]} x_1 \cdots x_{i-1} x_i x_{i+1} \cdots x_{j-1} x_j \cdots x_n \\ &\triangleq M_{ij} x_1 x_2 \cdots x_n (i < j), \\ \bigvee_{s=0}^{p-1} \{[y_i \nabla \phi_{k+}^s(y_j)] \vee [y_i \nabla \phi_{k-}^s(y_j)]\} &= M_p M_{ij} x = M_{ij,p} x, \end{aligned} \quad (30)$$

where $x = x_1 x_2 \cdots x_n$, $M_{ij,p} = M_p M_{ij}$ ($i, j = 1, 2, \dots, n$),

$$M_{ij} = E_{d,k}^{n-j} W_{k^2, k^{n-j}} E_{d,k}^{j-1-i} W_{k, k^{j-1-i}} E_{d,k}^{i-1} (i < j), \quad (31)$$

$$\sum_{i=1}^n \sum_{j>i} \left\{ a_{ij} \bigvee_{s=0}^{p-1} \left[y_i \bar{\bigvee}_{\phi_{k+}^s} (y_j) \bigvee \left[y_i \bar{\bigvee}_{\phi_{k-}^s} (y_j) \right] \right] \right\} + b_{ij} \bigvee_{s=0}^{q-1} \left[y_i \bar{\bigvee}_{\phi_{k+}^s} (y_j) \bigvee \left[y_i \bar{\bigvee}_{\phi_{k-}^s} (y_j) \right] \right] \right\}, = \sum_{i=1}^n \sum_{j>i} (a_{ij} M_{ij,p} + b_{ij} M_{ij,q}) x. \quad (32)$$

Thus, logical equations (19) are solved if and only if there exists $x = \delta_{k^n}^v$ such that

$$\sum_{i=1}^n \sum_{j>i} (a_{ij} M_{ij,p} + b_{ij} M_{ij,q}) x = Mx = 0, \quad (33)$$

that is, the v -th column of M is zero. Then, the proof is complete. $\square$

Based on Theorem 2, we give the following algorithm to find all for the $L(p, q)$ coloring solutions of the given graph.

$$\begin{cases} S_{1,k}^n = \delta_k \left[\underbrace{1 \cdots 1}_{k^{n-1}} \underbrace{2 \cdots 2}_{k^{n-1}} \cdots \underbrace{k \cdots k}_{k^{n-1}} \right], \\ S_{2,k}^n = \delta_k \left[\underbrace{1 \cdots 1}_{k^{n-2}} \underbrace{2 \cdots 2}_{k^{n-2}} \cdots \underbrace{k \cdots k}_{k^{n-2}} \underbrace{1 \cdots 1}_{k^{n-2}} \underbrace{2 \cdots 2}_{k^{n-2}} \cdots \underbrace{k \cdots k}_{k^{n-2}} \right], \\ \vdots \\ S_{n,k}^n = \delta_k \left[\underbrace{12 \cdots k}_{k} \cdots \underbrace{12 \cdots k}_{k} \right]. \end{cases} \quad (34)$$

Using formula (24), compute $x_i = S_{i,k}^n \delta_{k^n}^{S_i}$ and $y_i = [k-1 \cdots 10] x_i$, $i = 1, 2, \dots, n$. Set

$$S(s_t) = \{f \mid f(v_i) = y_i, \quad i = 1, 2, \dots, n\}. \quad (35)$$

Then, all $L(p, q)$ -label coloring plans of $\mathcal{G}$ are $\{S(s_t) \mid t = 1, 2, \dots, m\}$. $\square$

4. Illustrative Example

In this section, we give an example to illustrate the effectiveness of the results/algorithms obtained in this paper.

In order to avoid interference with each other, different channels need to be assigned to different base stations in the wireless network. Moreover, the main object of the channel allocation problem is to search an allocation scheme which has the channel as least as possible. Some mathematical models can be used to study the channel allocation problem of the wireless network, including T -coloring, list coloring, set coloring, and $L(p_1, p_2, \dots, p_n)$ -coloring. The most commonly used model is $L(2, 1)$ -coloring. Denote by $\mathcal{G}(V, E)$ the topological graph of the wireless network, where V is a node set denoting base stations or their users and E denotes an edge set. Now, we will use the

and the product is " $\propto$ ".

Therefore,

$L(2, 1)$ -coloring model to analyze the channel allocation of the wireless network.

Example 1. Construct the telecommunication base stations among four cities denoted by $A \sim D$, respectively. Denote a city by one vertex of the graph. If the base station constructed in city X can cover city Y , then there is an edge between X and Y . Now, the covering graph of base stations $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is established as shown in Figure 1. Our target is to search all schemes for channel allocation of the wireless network. In the wireless network, the channel interval is greater than or equal to 2 for two adjacent base stations and greater than or equal to 1 for two base stations with distance 2.

From Figure 1, we have its adjacency matrix as the following:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \quad (36)$$

According to A , we have by the above definition

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \quad (37)$$

Using Algorithm 1, we can compute the matrix M defined in (26) by the semitensor product. Then, by Matlab, $k = 5$ different channels which are denoted by 0, 1, 2, 3, and 4 are needed to satisfy the requirement in the channel allocation of the wireless network. Moreover, all detailed schemes are corresponding to

$$\text{Col}_i(M) = 0, i = 97, 117, 221, 241, 385, 405, 509, 529. \quad (38)$$

The corresponding channel schemes are shown in Table 1.

For example, for $i = 97$, $y_1 = 4, y_2 = 1, y_3 = 0, y_4 = 3$. That is, channels 4, 1, 0, and 3 are, respectively, assigned to stations A, B, C , and D which satisfies the requirement in the channel allocation of the wireless network. Moreover, from the table, there are at least 4 channels to satisfy the graph.

FIGURE 1: A topological graph of the wireless network.

- (1) Compute matrix M by (20).
- (2) If there is no common k such that $Col_k(M) = 0$, then $\mathcal{G}$ has no $L(p, q)$ -label coloring solution, and stop the calculation. Otherwise, find out the number s such that $Col_s(M) = 0$ and denote these by $s_1, s_2, \dots, s_m$.
- (3) For each index $s_1, s_2, \dots, s_m$, let $\alpha_{i=1}^n x_i = \delta_{kn}^{s_i}$, $t = 1, 2, \dots, m$. Let [11]

ALGORITHM 1: Given a graph $\mathcal{G}$ with nodes $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$, establish a logical function $f(v_i) = y_i \in \{0, 1, \dots, k-1\}$ and let $x_i = \delta_k^{k-y_i}$, $i = 1, 2, \dots, n$. To determine all $L(p, q)$ -label coloring plans of $\mathcal{G}$, we can do by the following steps:

TABLE 1: Corresponding channel schemes.

i	(y_1, y_2, y_3, y_4)	(A, B, C, D)
97	(4, 1, 0, 3)	(4, 1, 0, 3)
117	(4, 0, 1, 3)	(4, 0, 1, 3)
221	(3, 1, 0, 4)	(3, 1, 0, 4)
241	(3, 0, 1, 4)	(3, 0, 1, 4)
385	(1, 4, 3, 0)	(1, 4, 3, 0)
405	(1, 3, 4, 0)	(1, 3, 4, 0)
509	(0, 4, 3, 1)	(0, 4, 3, 1)
529	(0, 3, 4, 1)	(0, 3, 4, 1)

5. Conclusion

The $L(p, q)$ -coloring problem of the graph is studied with application to channel allocation of the wireless network in this paper. The necessary and sufficient conditions for solving the $L(p, q)$ -coloring problem are given by introducing two new logical operators. Moreover, it is noted that all solutions of the obtained logical equations are corresponding to each coloring scheme. By using the semitensor product, the necessary and sufficient conditions are converted to an algebraic form. Based on this, all coloring schemes can be obtained through searching all column indices of the zero columns. Furthermore, an illustration example on channel allocation of the wireless network is given to show the effectiveness of the obtained results in this paper. In future, we plan to study other coloring problems by using the semitensor product method, i.e., T -coloring, list coloring, and set coloring.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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