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Research Article

Serre's Reduction and the Smith Forms of Multivariate Polynomial Matrices

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The equivalence of systems plays a critical role in multidimensional systems, which are usually represented by the multivariate polynomial matrices. The Smith form of a matrix is one of the important research contents in polynomial matrices. This paper mainly investigates the Smith forms of some multivariate polynomial matrices. We have obtained several new results and criteria on the reduction of a given multivariate polynomial matrix to its Smith form. These criteria are easily checked by computing the minors of lower order of the given matrix.

1. Introduction

The subject of multidimensional (nD) systems is concerned with a mathematical framework for tackling a broad range of paradigms whose analysis or synthesis requires the use of functions and polynomials in several complex variables. Many physical systems, multiple-input multiple-output systems, data analysis procedures, and learning algorithms have a natural nD structure due to the presence of one spatial variable. So the theory of nD systems is widely applied in areas of image processing, linear multipass processes, geophysical exploration, iterative learning control systems, etc. [1–9]. The equivalence of systems is one important research problem in the nD system theory. It is often required to transform a given system into a simpler but equivalent form. As we all know, a multivariate (nD) polynomial matrix is often used to represent an nD system. So the equivalence problem of nD system is often transformed into the equivalence problem of nD polynomial matrices. For 1D systems, the equivalence problem has been solved [5, 7] by the quite mature theory of 1D polynomial matrix. For nD ($n \geq 2$) case, since the equivalence problem is equivalent to a highly difficult problem, the isomorphism problem of two finitely presented modules, there is no hope that the equivalence problem can be solved completely. There exist two primary problems on equivalence of multivariate (nD) polynomial matrices: one is to reduce an nD polynomial matrix to its Smith form. Kung et al. have obtained some interesting results about the equivalence of nD polynomial matrices to their Smith forms [5, 6, 10–13]. Furthermore, the Smith forms of some nD polynomial matrices can be computed by Maple [14]. The other is called Serre's reduction problem. One of the motivations for doing Serre's reduction for an nD polynomial matrix is to reduce an nD system to an equivalent system containing fewer equations and unknowns. Cluzeau et al. have studied Serre's reduction and presented some new interesting results in [15, 16].

The following problem, proposed by Serre in 1960s, plays an important role in the research problems of nD systems. It is not only the problem of reducing a matrix to its Smith form, but also Serre's reduction problem.

Problem 1. When is an nD ($n \geq 2$) polynomial matrix $F(z)$ equivalent to the matrix

$$S(z) = \begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & d \end{pmatrix}, \qquad (1)$$

where $d = \det F(z)$? I_{l-1} is the $(l-1) \times (l-1)$ identity matrix and $0_{l,m}$ is the $l \times m$ zero matrix.

For the real number field $\mathbb{R}$, Lin et al. [12] have investigated Problem 1 for $F(z) \in \mathbb{R}^{l\times l}[z]$ with $\det F(z) = z_1 - f(z_2, \ldots, z_n)$ and proved that $F(z)$ is equivalent to its Smith form. Li et al. [17] have investigated Problem 1 for $F(z) \in \mathbb{R}^{l\times l}[z]$ with $\det F(z) = (z_1 - f(z_2, \ldots, z_n))^q$ and obtained that $F(z)$ is equivalent to its Smith form with $F(z)$ satisfying some criteria. Cluzeau et al. [13–16] also studied it and gave some new interesting results.

In this paper, we will investigate Problem 1 for the case of $F(z) \in K^{l\times l}[z]$ and $\det F(z) = (z_1 - f_1(z_2, \ldots, z_n))^{q_1}(z_2 - f_2(z_3, \ldots, z_n))^{q_2}$, where q_1, q_2 are nonnegative integers, K is an arbitrary field (even a function field or a finite field). Then, we investigate the Smith forms of some rectangular polynomial matrices and consider the following problem.

Problem 2. Let $F(z) \in K^{l\times m}[z]$ $(l \le m)$ with $d(F(z)) = (z_1 - f(z_2, \ldots, z_n))^{q_1} \cdot (z_2 - f_2(z_3, \ldots, z_n))^{q_2}$, where $d(F(z))$ is the greatest common divisor of the $l \times l$ minors of $F(z)$, q_1, q_2 are nonnegative integers. When is $F(z)$ equivalent to its Smith form?

The paper is organized as follows. In Section 2, we give some basic concepts on the equivalence of nD polynomial matrices. In Section 3, main results and some tractable criteria on equivalence of several kinds of polynomial matrices are proposed. In Section 4, an example is provided to illustrate the effectiveness of our constructive method.

2. Preliminaries

In the following, $K[z] = K[z_1, z_2, \ldots, z_n]$ will denote the polynomial ring in n variables $z_1, z_2, \ldots, z_n$ with coefficients in arbitrary field K, $\overline{K}$ will be an algebraic closed field of K, $\overline{K}^n$ will be the n dimensional vector space over $\overline{K}$, $K^{l\times m}[z]$ will denote the set of $l \times m$ matrices with their entries in $K[z]$, $K^{l\times m}[z_2, \ldots, z_n]$ will denote the set of $l \times m$ matrices with their entries in $K[z_2, \ldots, z_n]$, the $r \times r$ identity matrix will be denoted by I_r and the $r \times t$ zero matrix will be denoted by $0_{r,t}$. A matrix over $K[z]$ with its determinant in K is said to be unimodular. Throughout the paper, the argument (z) is omitted whenever its omission does not cause confusion.

Definition 1 (see [18]).Let $F(z) \in K^{l\times m}[z]$ be of full row (column) rank. Then $F(z)$ is said to be zero left prime (zero right prime) if the $l \times l\,(m \times m)$ minors of $F(z)$ generate the unit ideal $K[z]$.

If $F(z) \in K^{l\times m}[z]$ is zero left prime (zero right prime), then $F(z)$ is called simply to be ZLP (ZRP). According to the Quillen-Suslin theorem [19], we have that $F(z)$ is ZLP if and only if there is a matrix $N(z) \in K^{m\times m}[z]$ such that $F(z) \cdot N(z) = (I_l\ 0_{l\times(m-l)})$. It is also equivalent to say that any ZLP (ZRP) matrix over $K[z]$ can be completed to an unimodular (invertible) matrix.

Definition 2 Let $F(z) \in K^{l\times m}[z]$, $l \le m$, and Φ_i be polynomially defined as follows:

$$\Phi_i = \begin{cases} d_i(F)/d_{i-1}(F), & 1 \le i \le r, \\ 0, & r < i \le l, \end{cases} \tag{2}$$

where r is the rank of $F(z)$, $d_0(F) \equiv 1$, $d_i(F)$ is the greatest common divisor of the $i \times i$ minors of $F(z)$, and Φ_i satisfies $\Phi_i \mid \Phi_{i+1}$, $i = 1, \ldots, l$. Then, we define the Smith form of $F(z)$ as

$$S = \begin{pmatrix} \operatorname{diag}\{\Phi_i\} & 0_{l\times(m-l)} \end{pmatrix}. \tag{3}$$

Definition 3 (see [17]). Let $T_1(z)$ and $T_2(z)$ denote two matrices in $K^{l\times m}[z]$, $T_1(z)$, and $T_2(z)$ are said to be equivalent if there exist two invertible matrices $M(z) \in K^{l\times l}[z]$ and $N(z) \in K^{m\times m}[z]$ such that $T_2(z) = M(z)T_1(z)N(z)$.

3. Main Results

In this section, the main results are presented. First, we give some well-known results and provide an answer to Problem 1 for case of $\det F(z) = (z_1 - f_1(z_2, \ldots, z_n))(z_2 - f_2(z_3, \ldots, z_n))$ in Subsection 3.1. Then we extend this result to more general case of $\det F(z) = (z_1 - f_1(z_2, \ldots, z_n))(z_2 - f_2(z_3, \ldots, z_n))^{q_2}$ and present a complete answer to Problem 2 in Subsection 3.2.

3.1. Equivalent Theorem.

In order to prove our main results, we first give several useful lemmas.

Lemma 1 (see [20]). *Let $f_1(z), \ldots, f_s(z) \in K[z]$, then $f_1(z), \ldots, f_s(z)$ have no common zeros in $\overline{K}^n$ (are zero coprime) if and only if there exist $u_1(z), \ldots, u_s(z) \in K[z]$ such that*

$$u_1(z)f_1(z) + \cdots + u_s(z)f_s(z) = 1, \tag{4}$$

i.e., $f_1(z), \ldots, f_s(z)$ is a ZLP row vector in $K^{1\times s}[z]$, or $f_1(z), \ldots, f_s(z)$ generate the unit ideal $K[z]$.

Lemma 2 (see [20]). *Let $g(z) \in K[z_1, z_2, \ldots, z_n]$ and $f(z_2, \ldots, z_n) \in K[z_2, \ldots, z_n]$. Suppose that $g(f(z_2, \ldots, z_n), z_2, \ldots, z_n) = 0$, then $z_1 - f(z_2, \ldots, z_n)$ is a divisor of $g(z)$.*

Lemma 3 (see [17]). *Let $F(z), F_1(z), F_2(z) \in K^{l\times l}[z]$, $F(z) = F_1(z) \cdot F_2(z)$. If the $(l-1) \times (l-1)$ minors of $F(z)$ have no common zeros in $\overline{K}^n$ (generate $K[z]$), then the $(l-1) \times (l-1)$ minors of $F_i(z)$ have no common zeros in $\overline{K}^n$ (generate $K[z]$) for $i = 1, 2$.*

Proof. The proof is similar to that of Lemma 2.2 in [17], so it is omitted here. □

Lemma 4 (see [18]). *Let $Q \in K^{(l-1)\times l}[z]$ be of normal full rank, if the reduced minors of Q generate $K[z]$, then there exists a ZLP matrix $w \in K^{l\times 1}[z]$ such that $Q \cdot w = 0_{(l-1),1}$.*

The following result is presented in [20], for the convenience of the reader, we record it here.

Lemma 5 (see [20]). *Let $F(z) = F(z_1, z_2, \ldots, z_n) \in K^{l\times l}[z]$, $f(z_2, \ldots, z_n) \in K[z_2, \ldots, z_n]$, if the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$, then the $(l-1)\times(l-1)$ minors of $F(f(z_2, \ldots, z_n), z_2, \ldots, z_n)$ also generate $K[z]$.*

Proof. Assume that the $(l-1)\times(l-1)$ minors of $F(z)$ are $c_1(z_1, \ldots, z_n), \ldots, c_r(z_1, \ldots, z_n)$. If the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$, then there exist $u_1(z_1, z_2, \ldots, z_n), \ldots, u_r(z_1, z_2, \ldots, z_n) \in K[z]$ such that

$$u_1(z_1, z_2, \ldots, z_n)c_1(z_1, z_2, \ldots, z_n) + \cdots + u_r(z_1, z_2, \ldots, z_n)c_r(z_1, \ldots, z_n) = 1. \quad (5)$$

Substitute $z_1 = f(z_2, \ldots, z_n)$, then

$$u_1(f(z_2, \ldots, z_n), z_2, \ldots, z_n)c_1(f(z_2, \ldots, z_n), z_2, \ldots, z_n) + \cdots + u_r(f(z_2, \ldots, z_n), z_2, \ldots, z_n)c_r(f(z_2, \ldots, z_n), z_2, \ldots, z_n) = 1. \quad (6)$$

It is straightforward that $c_1(f(z_2, \ldots, z_n), z_2, \ldots, z_n)c_r(f(z_2, \ldots, z_n), z_2, \ldots, z_n)$ are the $(l-1)\times(l-1)$ minors of $F(f(z_2, \ldots, z_n), z_2, \ldots, z_n)$. Thus the $(l-1)\times(l-1)$ minors of $F(f(z_2, \ldots, z_n), z_2, \ldots, z_n)$ also generate $K[z]$. □

Lemma 6. *Let $F(z) \in K^{l\times l}[z]$, then $\det F(z)$ and the $(l-1)\times(l-1)$ minors of $F(z)$ have no common zeros if and only if the $(l-1)\times(l-1)$ minors of $F(z)$ have no common zeros.*

Proof. Necessity. Assuming that the $(l-1)\times(l-1)$ minors of $F(z)$ have a common zero point a_0, by Laplace expansion, we can easily know that a_0 is also a zero point of $\det F(z)$, contradicting the fact that $\det F(z)$ and the $(l-1)\times(l-1)$ minors of $F(z)$ have no common zeros. Thus, the $(l-1)\times(l-1)$ minors of $F(z)$ have no common zeros.

Sufficiency. It is straightforward.

For the convenience, we first define $P_1(z), P_2(z), P(z)$ as follows:

$$P_1(z) = \begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & z_1 - f_1(z_2, \ldots, z_n) \end{pmatrix},$$
$$P_2(z) = \begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & z_2 - f_2(z_3, \ldots, z_n) \end{pmatrix},$$
$$P(z) = \begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & (z_1 - f_1(z_2, \ldots, z_n))(z_2 - f_2(z_3, \ldots, z_n)) \end{pmatrix}. \quad (7)$$

We have the following key conclusion which is very important to derive our main results. □

Lemma 7. *Let $F(z) \in K^{l\times l}[z]$ and $F(z) = P_1(z)V(z)P_2(z)$, where $V(z) \in K^{l\times l}[z_2, \ldots, z_n]$ is unimodular. If all the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$, then $F(z)$ is equivalent to its Smith form $P(z)$.*

Proof. We prove this by induction on l. When $l = 2$, let

$$V(z) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}, \quad (8)$$

then

$$F(z) = \begin{pmatrix} v_{11} & v_{12}\cdot(z_2 - f_2(z_3, \ldots, z_n)) \\ v_{21}\cdot(z_1 - f_1(z_2, \ldots, z_n)) & v_{22}\cdot(z_1 - f_1(z_2, \ldots, z_n))(z_2 - f_2(z_3, \ldots, z_n)) \end{pmatrix}. \quad (9)$$

Obviously, $\det F(z) = c\cdot(z_1 - f_1(z_2, \ldots, z_n))(z_2 - f_2(z_3, \ldots, z_n))$, $0 \neq c \in K$. Note that $V(z)$ is unimodular, then v_{11}, v_{12} have no common zeros. Assume that $v_{11}, v_{12}(z_2 - f_2(z_3, \ldots, z_n))$ have a zero $\alpha_0 = (z_{20}, z_{30}, \ldots, z_{n0})$, where $z_{20} = f_2(z_{30}, \ldots, z_{n0})$, then $v_{11}, v_{12}\cdot(z_2 - f_2(z_3, \ldots, z_n))$, $v_{21}\cdot(z_1 - f_1(z_2, \ldots, z_n)), v_{22}\cdot(z_1 - f_1(z_2, \ldots, z_n))(z_2 - f_2(z_3, \ldots, z_n))$ has a common zero $(f_1(\alpha_0), z_{20}, z_{30}, \ldots, z_{no})$. This is a contradiction. Thus, $v_{11}, v_{12}\cdot(z_2 - f_2(z_3, \ldots, z_n))$ have no common zeros, and $(v_{11}, v_{12}\cdot(z_2 - f_2(z_3, \ldots, z_n)))$ is a unimodular row. According to Quillen-Suslin Theorem, there exists a 2×2 unimodular matrix $M_1(z) \in K^{2\times 2}[z]$ such that

$$(v_{11}, v_{12}(z_2 - f_2(z_3, \ldots, z_n)))M_1(z) = (1, 0). \quad (10)$$

So

$$F(z)M_1(z) = \begin{pmatrix} 1 & 0 \\ p_1(z) & p_2(z) \end{pmatrix}, \quad (11)$$

where $p_1(z), p_2(z) \in K[z]$. Then there exists the unimodular matrix $M_2(z) = \begin{pmatrix} 1 & 0 \\ -p_1(z) & 1 \end{pmatrix}$ such that

$$M_2(z)F(z)M_1(z) = \begin{pmatrix} 1 & 0 \\ 0 & p_2(z) \end{pmatrix}. \quad (12)$$

Note that $p_2(z) = \det(M_2(z)F(z)M_1(z)) = u\cdot\det F(z) = u'\cdot(z_1 - f_1(z_2, \ldots, z_n))(z_2 - f_2(z_3, \ldots, z_n))$, $0 \neq u, u' \in K$. Thus, $F(z)$ is equivalent to its Smith form

$$P(z) = \begin{pmatrix} 1 & 0 \\ 0 & (z_1 - f_1(z_2, \ldots, z_n))(z_2 - f_2(z_3, \ldots, z_n)) \end{pmatrix}. \quad (13)$$

So the conclusion is true for $l = 2$.

Assume that the conclusion is true for $l-1$, we investigate the case of l. Let $V(z) = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$, where $V_{11} \in K^{(l-1)\times(l-1)}[z_2, \ldots, z_n]$, $V_{12} \in K^{(l-1)\times 1}[z_2, \ldots, z_n]$, $V_{21} \in K^{1\times(l-1)}[z_2, \ldots, z_n]$, $V_{22} \in K^{1\times 1}[z_2, \ldots, z_n]$. Then

$$F(z)=\begin{pmatrix} V_{11} & V_{12}\cdot(z_2-f_2(z_3,\ldots,z_n)) \\ V_{21}\cdot(z_1-f_1(z_2,\ldots,z_n)) & V_{22}\cdot(z_1-f_1(z_2,\ldots,z_n))(z_2-f_2(z_3,\ldots,z_n)) \end{pmatrix}. \quad (14)$$

Let $(v_{11},\ldots,v_{1,l-1},v_{1l})$ be the first row of $V(z)$. Since $V(z)$ is unimodular, using Laplace Theorem and expanding its first row, we obtain that $v_{11}a_1(z)+\cdots+v_{1,l-1}a_{l-1}(z)+v_{1l}a_l(z)=k$, where $a_1(z),\ldots,a_{l-1}(z),a(z)$ are all the $(l-1)\times(l-1)$ minors of $V(z)$ and $0\neq k\in K$. Combined with Lemma 1, we have that $v_{11},\ldots,v_{1,l-1},v_{1l}$ have no common zeros. Assume that $v_{11},\ldots,v_{1,l-1},v_{1l}(z_2-f_2(z_3,\ldots,z_n))$ have a common zero $\alpha_0=(z_{20},z_{30},\ldots,z_{n0})$, then all the $(l-1)\times(l-1)$ minors of $F(z)$ have a common zero $(f_1(\alpha_0), z_{20},z_{30},\ldots,z_{n0})$. This is a contradiction. Thus, $(v_{11},\ldots, v_{1,l-1},v_{1l}(z_2-f_2(z_3,\ldots,z_n)))$ is a unimodular row in $K[z_2,\ldots,z_n]$, by the Quillen-Suslin Theorem, there exists a unimodular matrix $M_1(z)$ such that

$$\left(v_{11},\ldots,v_{1,l-1},v_{1l}(z_2-f_2(z_3,\ldots,z_n))\right)M_1(z)=(1,0,\ldots,0). \quad (15)$$

So

$$F(z)M_1(z)=\begin{pmatrix} 1 & 0_{1,l-1} \\ Q_1(z) & Q_2(z)\end{pmatrix}, \quad (16)$$

where $Q_1(z)\in K^{(l-1)\times 1}[z]$, $Q_2(z)\in K^{(l-1)\times(l-1)}[z]$. Then there exists the matrix

$$M_2(z)=\begin{pmatrix} 1 & 0_{1,l-1} \\ -Q_1(z) & I_{l-1}\end{pmatrix}, \quad (17)$$

such that

$$M_2(z)F(z)M_1(z)=\begin{pmatrix} 1 & 0_{1,l-1} \\ 0_{l-1,1} & Q_2(z)\end{pmatrix}. \quad (18)$$

Setting $N(z)=M_2(z)F(z)M_1(z)$, since all the $(l-1)\times(l-1)$ minors of $F(z)$ have no common zeros, combined with Lemma 3, we have that all the $(l-1)\times(l-1)$ minors of $N(z)$ have no common zeros. Note that an $(l-1)\times(l-1)$ minor of $N(z)$ is just an $(l-2)\times(l-2)$ minors of $Q_2(z)$ or $\det Q_2(z)$ or 0, then $\det Q_2(z)$ and the $(l-2)\times(l-2)$ minors of $Q_2(z)$ have no common zeros. By Lemma 6, the $(l-2)\times(l-2)$ minors of $Q_2(z)$ have no common zeros. By the inductive hypothesis, there exist two $(l-1)\times(l-1)$ unimodular matrices $N_1(z),N_2(z)$ such that

$$N_2(z)Q_2(z)N_1(z) =\begin{pmatrix} I_{l-2} & 0_{l-2,1} \\ 0_{1,l-2} & (z_1-f_1(z_2,\ldots,z_n))(z_2-f_2(z_3,\ldots,z_n))\end{pmatrix}, \quad (19)$$

then

$$\begin{aligned}&\begin{pmatrix} 1 & 0_{1,l-1} \\ 0_{l-1,1} & N_2(z)\end{pmatrix}N(z,w)\begin{pmatrix} 1 & 0_{1,l-1} \\ 0_{1,l-1} & N_1(z)\end{pmatrix} \\ &=\begin{pmatrix} 1 & 0_{1,l-1} \\ 0_{l-1,1} & N_2(z)Q_2(z)N_1(z)\end{pmatrix} \\ &=\begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & (z_1-f_1(z_2,\ldots,z_n))(z_2-f_2(z_3,\ldots,z_n))\end{pmatrix}.\end{aligned} \quad (20)$$

Thus, $N(z)$ is equivalent to the matrix $P(z)$, combined with $F(z)$ is equivalent to the matrix $N(z)$; then we obtain that $F(z)$ is equivalent to the matrix $P(z)$, and $P(z)$ is the Smith form of $F(z)$.

Now we are going to state one of our main results, which give partial answer to Problem 1. We recall the notation

$$P(z)=\begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & (z_1-f_1(z_2,\ldots,z_n))(z_2-f_2(z_3,\ldots,z_n))\end{pmatrix}. \quad (21)$$

□

Theorem 1. *Let $F(z)\in K^{l\times l}[z]$ with $\det F(z)=(z_1-f_1(z_2,\ldots,z_n))(z_2-f_2(z_3,\ldots,z_n))$. $F(z)$ is equivalent to its Smith form $P(z)$ if and only if all the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$.*

Proof. Sufficiency. Because $\det F(z_1,f_2(z_3,\ldots,z_n),z_3,\ldots,z_n)=0$, then the rank of $F(z_1,f_2(z_3,\ldots,z_n),z_3,\ldots,z_n)\leq l-1$. Since all the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$, by Lemma 5, the $(l-1)\times(l-1)$ minors of $F(z_1,f_2(z_3,\ldots,z_n),z_3,\ldots,z_n)$ also generate $K[z]$, then $\operatorname{rank} F(z_1,f_2(z_3,\ldots,z_n),z_3,\ldots,z_n)=l-1$ for every $(z_1,z_3,\ldots,z_n)\in\overline{K}$. By Lemma 4, there exists a ZRP column vector $Y_1(z)\in K^{l\times 1}[z]$ such that

$$F(z_1,f_2(z_3,\ldots,z_n),z_3,\ldots,z_n)Y_1(z)\equiv(0,\ldots,0)^T. \quad (22)$$

By the Quillen-Suslin theorem, an $l\times l$ unimodular matrix $U_{11}(z)$ can be constructed such that $Y_1(z)$ is its last column. Then the elements of the last column of $F(z_1,f_2(z_3,\ldots,z_n),z_3,\ldots,z_n)\cdot U_{11}(z)$ are zero polynomials. By Lemma 2, the last column of $F(z)\cdot U_{11}(z)$ have the common divisor $z_2-f_2(z_3,\ldots,z_n)$, i.e.,

$$F(z)\cdot U_{11}(z)=F_1(z)\cdot\begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & z_2-f_2(z_3,\ldots,z_n)\end{pmatrix}, \quad (23)$$

for some $F_1(z)\in K^{l\times l}[z]$. Let $U_1(z)=U_{11}^{-1}(z)$, then we have

$$F(z)=F_1(z)P_2(z)U_1(z). \quad (24)$$

Note that $U_1(z), U_{11}(z)$ are unimodular, and $\det F(z) = (z_1 - f_1(z_2, \ldots, z_n))(z_2 - f_2(z_3, \ldots, z_n))$, then

$$\det F_1(z) = c \cdot (z_1 - f_1(z_2, \ldots, z_n)), \quad 0 \neq c \in K. \tag{25}$$

From Lemma 3, we have that the $(l-1)\times(l-1)$ minors of $F_1(z)$ also generate $K[z]$. Note that $\det F_1(f_1(z_2, \ldots, z_n), z_2, z_3, \ldots, z_n) = 0$, combined with Lemma 5, we obtain that the $(l-1)\times(l-1)$ minors of $F_1(f_1(z_2, \ldots, z_n), z_2, z_3, \ldots, z_n)$ also generate $K[z]$ for $(z_2, z_3, \ldots, z_n) \in \overline{K}$. Similarly, a unimodular matrix $V_{11}(z) \in K^{l\times l}[z_2, \ldots, z_n]$ can be constructed such that

$$F_1(z)V_{11}(z) = F_2(z) \cdot \begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & z_1 - f_1(z_2, \ldots, z_n) \end{pmatrix}, \tag{26}$$

where $F_2(z) \in K^{l\times l}[z]$. Let $V_1(z) = V_{11}^{-1}(z)$, then we have

$$F_1(z) = F_2(z)P_1(z)V_1(z). \tag{27}$$

Note that $V_1(z) \in K^{l\times l}[z_2, \ldots, z_n]$ is unimodular, and $\det F_1(z) = c_1 \cdot (z_1 - f_1(z_2, \ldots, z_n))$, then $\det F_2(z) = c_2$, $0 \neq c_2 \in K$, $F_2(z)$ is unimodular, and

$$F(z) = F_2(z)P_1(z)V_1(z)P_2(z)U_1(z). \tag{28}$$

By Lemma 7, the matrix $P_1(z)V_1(z)P_2(z)$ is equivalent to

$$P(z) = \begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & (z_1 - f_1(z_2, \ldots, z_n))(z_2 - f_2(z_3, \ldots, z_n)) \end{pmatrix}. \tag{29}$$

Combined with $F_2(z), U_1(z)$ are unimodular, we obtain that $F(z)$ is equivalent to its Smith form $P(z)$.

Necessity. If $F(z)$ is equivalent to $P(z)$, there exist two unimodular matrices $U(z), V(z)$ such that $F(z) = V(z)P(z)U(z)$. Note that all the $(l-1)\times(l-1)$ minors of $P(z)$ have no common zeros; combined with Lemma 3, we have that all the $(l-1)\times(l-1)$ minors of $F(z)$ have no common zeros, i.e., all the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$.

In the following, we will extend the above result to more general case. □

3.2. Generalization

Lemma 8. *Let $D(z) \in K^{l\times l}[z]$ with $D(z) = \begin{pmatrix} I_{l-1} & 0 \\ 0 & h(z) \end{pmatrix}$, $h(z) \in K[z]$, and $V(z) \in K^{l\times l}[z]$ be a unimodular matrix. If all the $(l-1)\times(l-1)$ minors of $D^s(z)V(z)D^t(z)$ generate $K[z]$ and s, t are positive integers, then $D^s(z)V(z)D^t(z)$ is equivalent to $D^{s+t}(z)$.*

Proof. Let

$$V(z) = \begin{pmatrix} V_{11}(z) & V_{12}(z) \\ V_{21}(z) & V_{22}(z) \end{pmatrix}, \tag{30}$$

where $V_{11}(z) \in K^{(l-1)\times(l-1)}[z]$, $V_{12}(z) \in K^{(l-1)\times 1}[z]$, $V_{21}(z) \in K^{1\times(l-1)}[z]$, $V_{22}(z) \in K^{1\times 1}[z]$.

Then

$$F(z) = D^s(z)V(z)D^t(z) = \begin{pmatrix} V_{11}(z) & V_{12}(z)h^t(z) \\ V_{21}(z)h^s(z) & V_{22}(z)h^{s+t}(z) \end{pmatrix}, \tag{31}$$

and $\det F(z) = c \cdot h^{s+t}(z)$, $0 \neq c \in K$.

Next, we will proof that $(V_{11}(z) \;\; V_{12}(z)h^t(z))$ is a ZLP matrix. Note that $V(z)$ is unimodular, then $(V_{11}(z) \;\; V_{12}(z))$ is a ZLP matrix. Let $c_1(z), c_2(z), \ldots, c_l(z)$ denote all the $(l-1)\times(l-1)$ minors of $(V_{11}(z) \;\; V_{12}(z))$. Then all the $(l-1)\times(l-1)$ minors of $(V_{11}(z) \;\; V_{12}(z)h^t(z))$ are $c_1(z), c_2(z)h^t(z), \ldots, c_l(z)h^t(z)$. We can prove that $c_1(z), c_2(z)h^t(z), \ldots, c_l(z)h^t(z)$ have no common zeros. Suppose that $c_1(z), c_2(z)h^t(z), \ldots, c_l(z)h^t(z)$ have a common zero α_0 and combined with $c_1(z), c_2(z), \ldots, c_l(z)$ have no common zeros, so α_0 is a zero of $c_1(z)$ and $h(z)$. Note that the elements of the last row of matrix $F(z)$ all have the factor $h(z)$ and that an $(l-1)\times(l-1)$ minor of $F(z)$ is just $c_1(z)$ or includes the factor $h(z)$, so the $(l-1)\times(l-1)$ minors of $F(z)$ have the common zero α_0, this is a contradiction. Thus, all the $(l-1)\times(l-1)$ minors of $(V_{11}(z) \;\; V_{12}(z)h^t(z))$ have no common zeros, and it is a ZLP matrix. According to Quillen-Suslin theorem, there exists an $m \times m$ unimodular matrix $N_1(z)$ such that

$$(V_{11}(z) \;\; V_{12}(z)h^t(z))N_1(z) = (I_{l-1}(z) \;\; 0_{l-1,1}). \tag{32}$$

Then,

$$F(z)N_1(z) = \begin{pmatrix} I_{l-1}(z) & 0_{l-1,1} \\ H_1(z) & H_2(z) \end{pmatrix}, \tag{33}$$

for some $H_1(z), H_2(z) \in K[z]$. Let

$$N_2(z) = \begin{pmatrix} I_{l-1}(z) & 0_{l-1,1} \\ -H_1(z) & 1 \end{pmatrix}. \tag{34}$$

Then,

$$N_2(z)F(z)N_1(z) = \begin{pmatrix} I_{l-1}(z) & 0_{l-1,1} \\ 0_{1,l-1} & H_2(z) \end{pmatrix}. \tag{35}$$

Note that $N_1(z), N_2(z)$ are unimodular, then $H_2(z) = u \cdot \det F(z) = u' \cdot h^{s+t}(z)$, $0 \neq u \in K$, $u' = u \cdot c$, thus $F(z)$ is equivalent to $D^{s+t}(z)$.

Now we investigate Problem 1 for the case of $q_1 = q_2$. Denote

$$P^q(z) = \begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & (z_1 - f_1(z_2, \ldots, z_n))^q (z_2 - f_2(z_3, \ldots, z_n))^q \end{pmatrix}. \tag{36}$$

□

Theorem 2. *Let $F(z) \in K^{l\times l}[z]$ with $\det F(z) = (z_1 - f_1(z_2, .., z_n))^q (z_2 - f_2(z_3, \ldots, z_n))^q$, where q is a positive integer. Then $F(z)$ is equivalent to its Smith form $P^q(z)$ if and only if all the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$.*

Proof. Sufficiency. Notice that $\det F(z_1, f_2(z_3, \ldots, z_n), z_3, \ldots, z_n) = 0$, we have that rank $F(z_1, f_2(z_3, \ldots, z_n)$

$, z_3, \ldots, z_n) \leq l-1$. Since all the $(l-1)\times(l-1)$ minors of $F(z)$ generate the unit ideal $K[z]$, according to Lemma 5, then the $(l-1)\times(l-1)$ minors of $F(z_1, f_2(z_3,\ldots,z_n), z_3,\ldots,z_n)$ also generate the unit ideal $K[z]$. Hence, $\operatorname{rank} F(z_1, f_2(z_3,\ldots,z_n), z_3,\ldots,z_n) = l-1$ for every $(z_1, z_3, \ldots, z_n) \in \overline{K}$, repeating the proof of Theorem 1, we have that

$$F(z) = F_1(z)P_1(z)V_1(z)P_2(z)U_1(z), \tag{37}$$

where $V_1(z) \in K^{l\times l}[z_2,\ldots,z_n]$, $U_1(z) \in K^{l\times l}[z]$ are unimodular, $F_1(z) \in K^{l\times l}[z]$ with $\det F_1(z) = (z_1 - f_1(z_2,\ldots,z_n))^{q-1}(z_2 - f_2(z_3,\ldots,z_n))^{q-1}$. By Lemma 3, all the $(l-1)\times(l-1)$ minors of $P_1(z)V_1(z)P_2(z)$ also generate $K[z]$. According to Lemma 7, there exist unimodular matrices $M_1(z), N_1(z) \in K^{l\times l}[z]$ such that $P_1(z)V_1(z)P_2(z) = M_1(z)P_1(z)P_2(z)N_1(z) = M_1(z)P(z)N_1(z)$. Thus

$$F(z) = F_1(z)M_1(z)P(z)N_1(z)U_1(z). \tag{38}$$

Again by Lemma 3, we obtain that all the $(l-1)\times(l-1)$ minors of $F_1(z)$ generate $K[z]$. If $q \geq 2$, iterating the preceding process, we obtain that

$$F_1(z) = F_2(z)M_2(z)P(z)N_2(z), \tag{39}$$

where $F_2(z) \in K^{l\times l}[z]$ with $\det F_2(z) = (z_1 - f_1(z_2,\ldots,z_n))^{q-2}(z_2 - f_2(z_3,\ldots,z_n))^{q-2}$. Then

$$F(z) = F_2(z)M_2(z)P(z)N_2(z)M_1(z)P(z)N_1(z)U_1(z). \tag{40}$$

Let $S_2(z) = N_2(z)M_1(z)$, we have that $S_2(z)$ is unimodular and all the $(l-1)\times(l-1)$ minors of $P(z)S_2(z)P(z)$ generate $K[z]$. By Lemma 8, $P(z)S_2(z)P(z)$ is equivalent to $P^2(z)$, that is, there exist unimodular matrices $L_1(z), T_1(z) \in K^{l\times l}[z]$ such that $P(z)S_2(z)P(z) = L_1(z)P^2(z)T_1(z)$. Furthermore,

$$F(z) = F_2(z)M_2(z)L_1(z)P^2(z)T_1(z)N_1(z)U_1(z). \tag{41}$$

Let $V_2(z) = M_2(z)L_1(z)$, $U_2(z) = T_1(z)N_1(z)U_1(z)$, then $V_2(z), U_2(z)$ are unimodular, and

$$F(z) = F_2(z)V_2(z)P^2(z)U_2(z). \tag{42}$$

If $q \geq 3$, iterating the same procedure successively, we obtain that

$$F(z) = F_q(z)V_q(z)P^q(z)U_q(z), \tag{43}$$

where $V_q(z), U_q(z)$ are unimodular. Note that $\det F(z) = (z_1 - f_1(z_2,\ldots,z_n))^q(z_2 - f_2(z_3,\ldots,z_n))^q$, then $\det F_q(z) = c$, $c \in K$; that is, $F_q(z)$ is a unimodular matrix. So $F(z)$ is equivalent to its Smith form $P^q(z)$.

Necessity. It is straightforward that all the $(l-1)\times(l-1)$ minors of $P^q(z)$ generate $K[z]$. According to Lemma 3, all the $(l-1)\times(l-1)$ minors of $F(z)$ also generate $K[z]$.

Next we investigate Problem 1 for the case of $F(z) \in K^{l\times l}[z]$ and $\det F(z) = (z_1 - f_1(z_2,\ldots,z_n))^{q_1}(z_2 - f_2(z_3,\ldots,z_n))^{q_2}$, where q_1, q_2 are nonnegative integers. □

Theorem 3. *Let $F(z) \in K^{l\times l}[z]$ with $\det F(z) = (z_1 - f_1(z_2,\ldots,z_n))^{q_1}(z_2 - f_2(z_3,\ldots,z_n))^{q_2}$, where q_1, q_2 are nonnegative integers. If all the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$, then $F(z)$ is equivalent to its Smith form*

$$Q(z) = \begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & (z_1 - f_1(z_2,\ldots,z_n))^{q_1}(z_2 - f_2(z_3,\ldots,z_n))^{q_2} \end{pmatrix}. \tag{44}$$

Proof. (1) If $q_1 = q_2 = 0$, then $\det F(z) = 1$, i.e., $F(z)$ is unimodular, so $F(z)$ is equivalent to I_l.

(2) If $q_1 = 0$ or $q_2 = 0$, we have that $\det F(z) = (z_1 - f_1(z_2,\ldots,z_n))^{q_1}$ or $(z_2 - f_2(z_3,\ldots,z_n))^{q_2}$. Without loss of generality, we assume that $\det F(z) = (z_1 - f_1(z_2,\ldots,z_n))^{q_1}$. Because $\det F(f_1(z_2,\ldots,z_n), z_2,\ldots,z_n) = 0$, then the rank of $F(f_1(z_2,\ldots,z_n), z_2,\ldots,z_n) \leq l-1$. Since all the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$, by Lemma 5, the $(l-1)\times(l-1)$ minors of $F(f_1(z_2,\ldots,z_n), z_2,\ldots,z_n)$ also generate $K[z]$; hence, $\operatorname{rank} F(f_1(z_2,\ldots,z_n), z_2,\ldots,z_n) = l-1$ for every $(z_2, z_3,\ldots,z_n) \in \overline{K}^{n-1}$. By Lemma 4, we obtain a ZRP column vector $Y_1(z) \in K^{l\times 1}[z]$ such that

$$F(f_1(z_1,\ldots,z_n), z_2,\ldots,z_n)Y_1(z) \equiv (0,\ldots,0)^T. \tag{45}$$

According to the Quillen-Suslin Theorem, an $l\times l$ unimodular matrix $U_{11}(z)$ $(\det U_{11}(z) = 1)$ can be constructed such that $Y_1(z)$ is its last column. Then the last column of $F(f_1(z_1,\ldots,z_n), z_2,\ldots,z_n)\cdot U_{11}(z)$ are zero polynomials. By Lemma 2, the last column of $F(z)\cdot U_{11}(z)$ have the common divisor $z_1 - f_1(z_2,\ldots,z_n)$, i.e.,

$$F(z)\cdot U_{11}(z) = F_1(z)\cdot\begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & z_1 - f_1(z_2,\ldots,z_n) \end{pmatrix}, \tag{46}$$

for some $F_1(z) \in K^{l\times l}[z]$. Let $U_1(z) = U_{11}^{-1}(z)$, then we obtain that

$$F(z) = F_1(z)P_1(z)U_1(z), \qquad \det F_1(z) = (z_1 - f_1(z_2,\ldots,z_n))^{q_1-1}. \tag{47}$$

By Lemma 3, all the $(l-1)\times(l-1)$ minors of $F_1(z)$ also generate the unit ideal $K[z]$. If $q_1 \geq 2$, iterating the preceding process, we obtain that

$$F_1(z) = F_2(z)P_1(z)U_2(z). \tag{48}$$

Furthermore,

$$F(z) = F_2(z)P_1(z)U_2(z)P_1(z)U_1(z). \tag{49}$$

By Lemma 3, the $(l-1)\times(l-1)$ minors of $P_1(z)U_2(z)P_1(z)$ generate the ideal $K[z]$, according to Lemma 8, there exist unimodular matrices $N_1(z), V_1(z)$ such that $P_1(z)U_2(z)P_1(z) = N_1(z)P_1^2(z)V_1(z)$. Then

$$F(z) = F_2(z)N_1(z)P_1^2(z)V_1(z)U_1(z). \tag{50}$$

Furthermore, $\det F_2(z)N_1(z) = (z_1 - f_1(z_2,\ldots,z_n))^{q_1-2}$, combined with Lemma 3, the $(l-1)\times(l-1)$ minors of $F_2(z)N_1(z)$ also generate $K[z]$.

If $q_1 \geq 3$, iterating the same procedure successively, we obtain that

$$F(z) = F_{q_1}(z) N_{q_1-1}(z) P_1^{q_1}(z) V_{q_1-1}(z), \ldots, V_1(z) U_1(z), \tag{51}$$

with $\det F_{q_1}(z) N_{q_1-1}(z) = 1$. Thus, $F(z)$ is equivalent to its Smith form $P_1^{q_1}(z) = Q(z)$.

(3) If q_1, q_2 are positive integers, then $q_1 \geq q_2$ or $q_2 \geq q_1$. Without loss of generality, we assume that $q_1 \geq q_2$. By Theorem 2, combined with the conclusion of the case (2) above, we have that $F(z)$ is equivalent to the matrix $P^{q_2}(z) V(z) P_1^{q_1-q_2}(z)$. In the following, we prove that $P^{q_2}(z) V(z) P_1^{q_1-q_2}(z)$ is equivalent to the matrix $Q(z)$

Let

$$V(z) = \begin{pmatrix} V_{11}(z) & V_{12}(z) \\ V_{21}(z) & V_{22}(z) \end{pmatrix}, \tag{52}$$

where $V_{11}(z) \in K^{(l-1)\times(l-1)}[z]$, $V_{12}(z) \in K^{(l-1)\times 1}[z]$, $V_{21}(z) \in K^{1\times(l-1)}[z]$, and $V_{22}(z) \in K^{1\times 1}[z]$.

Then

$$P^{q_2}(z) V(z) P_1^{q_1-q_2}(z) = \begin{pmatrix} V_{11}(z) & V_{12}(z) p_1^{q_1-q_2}(z) \\ V_{21}(z) p_1^{q_2}(z) p_2^{q_2}(z) & V_{22}(z) p_1^{q_1}(z) p_2^{q_2}(z) \end{pmatrix}, \tag{53}$$

where $p_1(z) = z_1 - f_1(z_2, \ldots, z_n)$, $p_2(z) = z_2 - f_2(z_3, \ldots, z_n)$.

In fact, we can prove that $\begin{pmatrix} V_{11}(z) & V_{12}(z) p_1^{q_1-q_2}(z) \end{pmatrix}$ is a ZLP matrix. Note that $V(z)$ is unimodular, then $\begin{pmatrix} V_{11}(z) & V_{12}(z) \end{pmatrix}$ is a ZLP matrix. Let $c_1(z), c_2(z), \ldots, c_l(z)$ denote all the $(l-1)\times(l-1)$ minors of $\begin{pmatrix} V_{11}(z) & V_{12}(z) \end{pmatrix}$. Then all the $(l-1)\times(l-1)$ minors of $\begin{pmatrix} V_{11}(z) & V_{12}(z) p_1^{q_1-q_2}(z) \end{pmatrix}$ are $c_1(z), c_2(z) p_1^{q_1-q_2}(z), \ldots, c_l(z) p_1^{q_1-q_2}(z)$. We can prove that $c_1(z), c_2(z) p_1^{q_1-q_2}(z), \ldots, c_l(z) p_1^{q_1-q_2}(z)$ have no common zeros. Suppose that $c_1(z), c_2(z) p_1^{q_1-q_2}(z), \ldots, c_l(z) p_1^{q_1-q_2}(z)$ have a common zero α_0, note that $c_1(z), c_2(z), \ldots, c_l(z)$ have no common zeros, so $c_1(z)$ and $p_1(z)$ have the common zero α_0. Moreover, the elements of the last row of matrix $P^{q_2}(z) V(z) P_1^{q_1-q_2}(z)$ have the common factor $p_1(z)$, since an $(l-1)\times(l-1)$ minors of $P^{q_2}(z) V(z) P_1^{q_1-q_2}(z)$ is just $c_1(z)$ or includes the factor $p_1(z)$, so the $(l-1)\times(l-1)$ minors of $P^{q_2}(z) V(z) P_1^{q_1-q_2}(z)$ have the common zero α_0, this is a contradiction. Thus, all the $(l-1)\times(l-1)$ minors of $\begin{pmatrix} V_{11}(z) & V_{12}(z) p_1^{q_1-q_2}(z) \end{pmatrix}$ have no common zeros, and it is a ZLP matrix. By the Quillen-Suslin theorem, there exists an $l \times l$ unimodular matrix $N(z)$ such that

$$\begin{pmatrix} V_{11}(z) & V_{12}(z) p_1^{q_1-q_2}(z) \end{pmatrix} N(z) = \begin{pmatrix} I_{l-1}(z) & 0_{l-1,1} \end{pmatrix}. \tag{54}$$

Then,

$$P^{q_2}(z) V(z) P_1^{q_1-q_2}(z) N(z) = \begin{pmatrix} I_{l-1}(z) & 0_{l-1,1} \\ H_1(z) & H_2(z) \end{pmatrix}, \tag{55}$$

for some $H_1(z), H_2(z) \in K[z]$. There exists a unimodular matrix $N_1(z)$ such that

$$N_1(z) P^{q_2}(z) V(z) P_1^{q_1-q_2}(z) N(z) = \begin{pmatrix} I_{l-1}(z) & 0_{l-1,1} \\ 0_{1,l-1} & H_2(z) \end{pmatrix}. \tag{56}$$

Note that $N(z), N_1(z)$ are unimodular, then $H_2(z) = u \cdot \det P^{q_2}(z) V(z) P_1^{q_1-q_2}(z) = u\prime \cdot p_1^{q_1}(z) p_2^{q_2}(z)$, $0 \neq u \in K$, $u\prime = u \cdot c$; thus, $P^{q_2}(z) V(z) P_1^{q_1-q_2}(z)$ is equivalent to $Q(z)$. Note that $F(z)$ is equivalent to the matrix $P^{q_2}(z) V(z) P_1^{q_1-q_2}(z)$, combined with the definition of the Smith form of a matrix, we obtain that $F(z)$ is equivalent to its Smith form $Q(z)$.

Theorem 3 provides a positive answer to Problem 1 for the case of $F(z) \in K^{l\times l}[z]$ and $\det F(z) = (z_1 - f_1(z_2, \ldots, z_n))^{q_1} (z_2 - f_2(z_3, \ldots, z_n))^{q_2}$. It also gives a sufficient condition to check this kind of matrices are equivalent to their Smith forms; in fact, this condition is also a necessary condition. □

Theorem 4. *Let $F(z) \in K^{l\times l}[z]$ with $\det F(z) = (z_1 - f_1(z_2, \ldots, z_n))^{q_1} (z_2 - f_2(z_3, \ldots, z_n))^{q_2}$, where q_1, q_2 are non-negative integers, then $F(z)$ is equivalent to its Smith form*

$$Q(z) = \begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & (z_1 - f_1(z_2, \ldots, z_n))^{q_1} (z_2 - f_2(z_3, \ldots, z_n))^{q_2} \end{pmatrix}, \tag{57}$$

if and only if all the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$.

Proof. Sufficiency. From Theorem 3, it is straightforward.

Necessity. By computing, we can easily obtain that all the $(l-1)\times(l-1)$ minors of $Q(z)$ generate $K[z]$. By Lemma 3, all the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$. □

Remark 1. In the theorem above, K is an arbitrary field. When $K = \mathbb{R}$ is the real field and $q_2 = 0$, Theorem 4 is same as Theorem 2.5 in Li et al. [17]. Furthermore, if $q_1 = 1$ and $q_2 = 0$, Theorem 4 is just Proposition 4 in Lin et al. [12]. So Theorem 4 extends the above two results.

In the following, we will investigate the Smith forms of some rectangular polynomial matrices and consider Problem 2. Let $d(z)$ denote the greatest common divisor (g.c.d) of the $l \times l$ minors of the matrix $F(z)$, $d_{l-1}(F(z))$ denote the g.c.d of all the $(l-1)\times(l-1)$ minors of $F(z)$ and denote

$$B(z) = \begin{pmatrix} I_{l-1} & 0_{l-1,1} \\ 0_{1,l-1} & d(z) \end{pmatrix}. \tag{58}$$

Theorem 5. *Let $F(z) \in K^{l\times m}[z]$ $(l \le m)$ be of full row rank and suppose $d(z) = (z_1 - f_1(z_2,\ldots,z_n))^{q_1}(z_2 - f_2(z_3,\ldots,z_n))^{q_2}$, where q_1, q_2 are nonnegative integers. Then $F(z)$ is equivalent to its Smith form*

$$Q(z) = \begin{pmatrix} B(z) & 0_{l\times(m-l)} \end{pmatrix}, \tag{59}$$

if and only if all the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$.

Proof. Sufficiency. According to Theorem 3.3 in [21], there exists $G(z) \in K^{l\times l}[z]$, $F_0(z) \in K^{l\times m}[z]$ such that $F(z) = G(z)\cdot F_0(z)$, where $\det G(z) = d(z)$, and $F_0(z)$ is a ZLP matrix. Combined with Theorem 5, we can obtain two unimodular matrices $V_1(z), V_2(z) \in K^{l\times l}[z]$ such that $G(z) = V_1(z)\cdot B(z)\cdot V_2(z)$. Then

$$F(z) = V_1(z)\cdot B(z)\cdot V_2(z)\cdot F_0(z). \tag{60}$$

It is obviously that $V_2(z)\cdot F_0(z)$ is also ZLP. According to the Quillen-Suslin Theorem, we can construct an $m\times m$ unimodular matrix $U(z)$ such that $V_2(z)\cdot F_0(z)$ is its first l rows. Hence

$$F(z) = V_1(z)\cdot \begin{pmatrix} B(z) & 0_{l,m-l} \end{pmatrix}\cdot U(z) = V_1(z)\cdot Q(z)\cdot U(z), \tag{61}$$

and $d_{l-1}(Q(z)) = 1$. Since $Q(z) = V_1^{-1}(z)\cdot F(z)\cdot U^{-1}(z)$, combined with Lemma 3, we have that $d_{l-1}(F(z)) = 1$, then the Smith form of $F(z)$ is $Q(z)$.

Necessity. If $F(z)$ is equivalent to its Smith form $Q(z) = \begin{pmatrix} B(z) & 0_{l,(m-l)} \end{pmatrix}$, we see that all the $(l-1)\times(l-1)$ minors of $Q(z)$ generate $K[z]$. Using Lemma 3, we obtain that all the $(l-1)\times(l-1)$ minors of $F(z)$ generate $K[z]$. □

Remark 2. Let $f_1(z), f_2(z), \ldots, f_t(z) \in K[z]$ be nonzero polynomials, a necessary and sufficient condition for $G = \{f_1(z), f_2(z), \ldots, f_t(z)\}$ generates that $K[z]$ is the reduced Gröbner basis of the ideal generated by G includes a unit in the field K. According to Theorems 4 and 5, we can check whether an nD polynomial matrix $F(z)$ is equivalent to its Smith form by using the existing Gröbner basis algorithms to the ideal generated by the lower minors of $F(z)$. Thus, the conditions of Theorems 4 and 5 can be verified easily.

4. Example and Algorithm

In this section, we first present a 2D example to illustrate our result and explain how to obtain the unimodular matrices associated with the equivalence of system matrices in the method. Then we give an algorithm to deal with the equivalence of the kind of matrices we discussed to their Smith forms.

In many areas of engineering such as Circuits and Signals, the general 2D systems can be defined in terms of the generalized Rosenbrock system [8] as

$$\begin{aligned} F(z_1,z_2)x &= U(z_1,z_2)u, \\ y &= V(z_1,z_2)x + W(z_1,z_2)u, \end{aligned} \tag{62}$$

where $x \in K^n$ is the state vector, $u \in K^l$ is the input vector, $y \in K^m$ is the output vector, $F(z_1,z_2) \in K^{n\times n}[z_1,z_2]$, $U(z_1,z_2) \in K^{n\times l}[z_1,z_2]$, $V(z_1,z_2) \in K^{m\times n}[z_1,z_2]$, $W(z_1,z_2) \in K^{m\times l}[z_1,z_2]$ are polynomial matrices, K is a field. The operators z_1 and z_2 may have various meanings depending on the type of system. For example, in delay-differential systems, z_1 represents a differential operator and z_2 a delay-operator. For 2D discrete systems, z_1 and z_2 represent horizontal and vertical shift operators, respectively. This system gives rise to the system matrix in the general form:

$$T(z_1,z_2) = \begin{pmatrix} F(z_1,z_2) & U(z_1,z_2) \\ -V(z_1,z_2) & W(z_1,z_2) \end{pmatrix}. \tag{63}$$

Example 1. Consider a system matrix

$$\begin{aligned} T(z_1,z_2) &= \begin{pmatrix} F(z_1,z_2) & U(z_1,z_2) \\ -V(z_1,z_2) & W(z_1,z_2) \end{pmatrix} \\ &= \begin{pmatrix} z_2^3 & -6z_2^2z_1^2 - 8z_2^2z_1 - z_2^2 + 4z_2z_1^2 + 2z_2z_1 - z_1^2 & 0 & z_1 \\ z_1^2 & -4 - 8z_1 + z_2z_1 - 2z_2^2 + z_2^3 & z_1+1 & z_2 \\ z_1 - 1 & 2z_2^2 - z_1z_2 - 6z_2 + 4z_1 & 1 & 0 \\ z_1 & z_2 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{64}$$

where

$$
\begin{aligned}
F(z_1,z_2) &= \begin{pmatrix} z_2^3 & -6z_2^2z_1^2 - 8z_2^2z_1 - z_2^2 + 4z_2z_1^2 + 2z_2z_1 - z_1^2 & 0 \\ z_1^2 & -4 - 8z_1 + z_2z_1 - 2z_2^2 + z_2^3 & z_1 + 1 \\ z_1 - 1 & 2z_2^2 - z_1z_2 - 6z_2 + 4z_1 & 1 \end{pmatrix}, \\
U(z_1,z_2) &= (z_1\ z_2\ 0)^T, \\
V(z_1,z_2) &= (-z_1\ -z_2\ 0), \\
W(z_1,z_2) &= (0).
\end{aligned} \tag{65}
$$

By computing, $\det F(z) = (z_1 - z_2)^2(z_2 - 1)^4$, the 2×2 minors of $F(z)$ is as follows: $c_1(z) = z_2^6 - 2z_2^5 + z_2^4z_1 - 8z_2^3z_1 - 4z_2^3 + 6z_2^2z_1^4 + 8z_2^2z_1^3 + z_2^2z_1^2 - 4z_2z_1^4 - 2z_2z_1^3 + z_1^4$, $c_2(z) = z_2^3(z_1 + 1)$, $c_3(z) = -(z_1 + 1)(6z_2^2z_1^2 + 8z_2^2z_1 + z_2^2 - 4z_2z_1^2 - 2z_2z_1 + z_1^2)$, $c_4(z) = 2z_2^5 - z_2^4z_1 - 6z_2^4 + 4z_2^3z_1 + 6z_2^2z_1^3 + 2z_2^2z_1^2 - 7z_2^2z_1 - z_2^2 - 4z_2z_1^3 + 2z_2z_1^2 + 2z_2z_1 + z_1^3 - z_1^2$, $c_5(z) = z_1^3$, $c_6(z) = -6z_2^2z_1^2 - 8z_2^2z_1 - z_2^2 + 4z_2z_1^2 + 2z_2z_1 - z_1^2$, $c_7(z) = -z_2^3z_1 + z_2^3 + 2z_2^2z_1^2 + 2z_2^2z_1 - 2z_2^2 - z_2z_1^3 - 7z_2z_1^2 + z_2z_1 + 4z_1^3 + 8z_1^2 - 4z_1 - 4$, $c_8(z) = 1$, $c_9(z) = z_2^3 - 2z_2^2z_1 - 4z_2^2 + z_2z_1^2 + 8z_2z_1 + 6z_2 - 4z_1^2 - 12z_1 - 4$.

The reduced Gröbner basis of the ideal generated by $c_1(z), \ldots, c_9(z)$ is $\{1\}$. So the 2×2 minors of $F(z_1,z_2)$ generate unit ideal $K[z_1,z_2]$, by Theorem 4, $F(z_1,z_2)$ is equivalent to its Smith form. Consider

$$
F(z_1,1) = \begin{pmatrix} 1 & -3z_1^2 - 6z_1 - 1 & 0 \\ z_1^2 & -7z_1 - 5 & z_1 + 1 \\ z_1 - 1 & 2z_1 - 4 & 1 \end{pmatrix}. \tag{66}
$$

Solving the equations, $F(z_1,1)Y_1 = 0$, where $Y_1 \in K^{3\times1}[z_1]$, we obtain $Y_1^T = (3z_1^2 + 6z_1 + 1, 1, -3z_1^3 - 3z_1^2 + 2z_1 + 5)$, where Y_1^T is the transpose of Y_1. It then follows that

$$
F(z_1,1)\begin{pmatrix} 3z_1^2 + 6z_1 + 1 \\ 1 \\ -3z_1^3 - 3z_1^2 + 2z_1 + 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{67}
$$

We complete $(3z_1^2 + 6z_1 + 1, 1, -3z_1^3 - 3z_1^2 + 2z_1 + 5)^T$ into a unimodular matrix $U_{41} \in K^{3\times3}[z_1]$ such that $FU_{41} = F_1P_2$, where

$$
\begin{aligned}
F_1 &= \begin{pmatrix} z_2^3 & 0 & 3z_2^2z_1^2 + 6z_2^2z_1 + z_2^2 - 3z_2z_1^2 - 2z_2z_1 + z_1^2 \\ z_1^2 & -z_1 - 1 & z_2^2 - z_2 + z_1 - 1 \\ z_1 - 1 & -1 & 2z_2 - z_1 - 4 \end{pmatrix}, \\
P_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_2 - 1 \end{pmatrix}, \\
U_{41} &= \begin{pmatrix} 1 & 0 & 3z_1^2 + 6z_1 + 1 \\ 0 & 0 & 1 \\ 0 & -1 & -3z_1^3 - 3z_1^2 + 2z_1 + 5 \end{pmatrix}.
\end{aligned} \tag{68}
$$

We then have

$$
F = F_1P_2U_4, \tag{69}
$$

where

$$
U_4 = U_{41}^{-1} = \begin{pmatrix} 1 & -3z_1^2 - 6z_1 - 1 & 0 \\ 0 & -3z_1^3 - 3z_1^2 + 2z_1 + 5 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{70}
$$

We consider $F_1(z_1,z_2)$ again

$$
F_1(z_1,1) = \begin{pmatrix} 1 & 0 & z_1^2 + 4z_1 + 1 \\ z_1^2 & -z_1 - 1 & z_1 - 1 \\ z_1 - 1 & -1 & -2 - z_1 \end{pmatrix}. \tag{71}
$$

Repeating the procedure above for F_1, we have that

$$
F_1 = F_2P_2U_3, \tag{72}
$$

where

$$F_2 = \begin{pmatrix} z_2^3 & 0 & z_2^2 z_1^2 + 4z_2^2 z_1 + z_2^2 - 2z_2 z_1^2 - 2z_2 z_1 + z_1^2 \\ z_1^2 & z_1 + 1 & -z_2 \\ z_1 - 1 & 1 & -2 \end{pmatrix},$$
$$U_3 = \begin{pmatrix} 1 & 0 & z_1^2 + 4z_1 + 1 \\ 0 & -1 & -z_1^3 - 3z_1^2 + 2z_1 - 1 \\ 0 & 0 & -1 \end{pmatrix}. \quad (73)$$

Then,

$$F = F_1 P_2 U_4 = F_2 P_2 U_3 P_2 U_4. \quad (74)$$

By Theorem 3.5 in [17], we have that $P_2 U_3 P_2$ is equivalent to P_2^2, and

$$P_2 U_3 P_2 = P_2^2 M_2, \quad (75)$$

where

$$M_2 = \begin{pmatrix} 1 & 0 & (z_2 - 1)(z_1^2 + 4z_1 + 1) \\ 0 & -1 & -(z_2 - 1)(z_1^3 + 3z_1^2 - 2z_1 + 1) \\ 0 & 0 & -1 \end{pmatrix}. \quad (76)$$

So,

$$F = F_1 P_2 U_4 = F_2 P_2 U_3 P_2 U_4 = F_2 P_2^2 M_2 U_4. \quad (77)$$

Consider F_2, factor it like above, we obtain that

$$F_2 = F_3 P_2 U_2, \quad (78)$$

where

$$F_3 = \begin{pmatrix} z_2^3 & 0 & 2z_2^2 z_1 + z_2^2 - z_2 z_1^2 - 2z_2 z_1 + z_1^2 \\ z_1^2 & -z_1 - 1 & 1 \\ z_1 - 1 & -1 & 0 \end{pmatrix},$$
$$U_2 = \begin{pmatrix} 1 & 0 & 2z_1 + 1 \\ 0 & -1 & 2z_1^2 - z_1 + 1 \\ 0 & 0 & -1 \end{pmatrix}. \quad (79)$$

Then,

$$F = F_2 P_2^2 M_2 U_4 = F_3 P_2 U_2 P_2^2 M_2 U_4. \quad (80)$$

We consider F_3 again

$$F_3(z_2, z_2) = \begin{pmatrix} z_2^3 & 0 & z_2^3 \\ z_2^2 & -z_2 - 1 & 1 \\ z_2 - 1 & -1 & 0 \end{pmatrix}. \quad (81)$$

Solving the equations, $F_3(z_2, z_2) Y_2 = 0$, where $Y_2 \in K^{3\times 1}[z_2]$, we obtain $Y_2^T = (1, z_2 - 1, -1)$, where Y_2^T is the transpose of Y_2. It then follows that

$$F_3(z_2, z_2) \begin{pmatrix} 1 \\ z_2 - 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (82)$$

We complete $(1, z_2 - 1, -1)^T$ into a unimodular matrix $V_{21} \in K^{3\times 3}[z_2]$ such that $F_3 V_{21} = F_4 P_1$, where

$$F_4 = \begin{pmatrix} z_2^3 & 0 & (z_2 - 1)(z_1 - z_2) \\ z_1^2 & z_1 + 1 & z_1 + 1 \\ z_1 - 1 & 1 & 1 \end{pmatrix},$$
$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_1 - z_2 \end{pmatrix}, \quad (83)$$
$$V_{21} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & z_2 - 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

We obtain that

$$F_3 = F_4 P_1 V_2, \quad (84)$$

where

$$V_2 = V_{21}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 - z_2 \\ 0 & 0 & -1 \end{pmatrix}. \quad (85)$$

Thus,

$$F = F_3 P_2 U_2 P_2^2 M_2 U_4 = F_4 P_1 V_2 P_2 U_2 P_2^2 M_2 U_4. \quad (86)$$

Now we consider $P_1 V_2 P_2$. By Lemma 7, we have that $P_1 V_2 P_2$ is equivalent to P. And

$$P_1 V_2 P_2 = P M_1, \quad (87)$$

where

$$M_1 = \begin{pmatrix} 1 & 0 & z_2 - 1 \\ 0 & -1 & -(z_2 - 1)^2 \\ 0 & 0 & -1 \end{pmatrix}. \quad (88)$$

Thus,

$$F = F_4 P_1 V_2 P_2 U_2 P_2^2 M_2 U_4 = F_4 P M_1 U_2 P_2^2 M_2 U_4. \quad (89)$$

Repeating the procedure above for F_4, we obtain that $F_4 = F_5 P_2 U_1$. Furthermore, we consider F_5 and obtain $F_5 = F_6 P_1$. So

$$\begin{aligned} F &= F_4 P M_1 U_2 P_2^2 M_2 U_4 = F_5 P_2 U_1 P M_1 U_2 P_2^2 M_2 U_4 \\ &= F_6 P_1 P_2 U_1 P M_1 U_2 P_2^2 M_2 U_4, \end{aligned} \quad (90)$$

where

$$F_5 = \begin{pmatrix} z_2^3 & 0 & 1-z_2 \\ z_1^2 & -z_1-1 & 0 \\ z_1-1 & -1 & 0 \end{pmatrix},$$
$$F_6 = \begin{pmatrix} z_2^3 & 0 & -1 \\ z_1^2 & -z_1-1 & 0 \\ z_1-1 & -1 & 0 \end{pmatrix}, \quad (91)$$
$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Setting

$$P = P_1P_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (z_1-z_2)(z_2-1) \end{pmatrix}, \quad (92)$$

we obtain that

$$F = F_6P_1P_2U_1PM_1U_2P_2^2M_2U_4 = F_6PU_1PM_1U_2P_2^2M_2U_4. \quad (93)$$

Let $L_1 = M_1U_2$, $L_2 = M_2U_4$, we have

$$F = F_6PU_1PL_1P_2^2L_2. \quad (94)$$

Now we consider the matrix PU_1P. From Theorem 2, we know that PU_1P is equivalent to P^2. And

$$PU_1P = P^2M_3, \quad (95)$$

where

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & (z_2-z_1)(z_2-1) \\ 0 & 0 & -1 \end{pmatrix}. \quad (96)$$

Then we have

$$F = F_6P^2M_3L_1P_2^2L_2. \quad (97)$$

Let $L_3 = M_3L_1$, now consider the matrix $P^2L_3P_2^2$. By Theorem 3, we have that $P^2L_3P_2^2$ is equivalent to B, where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (z_1-z_2)^2(z_2-1)^4 \end{pmatrix}, \quad (98)$$
$$P^2L_3P_2^2 = BL_4,$$

where

$$L_4 = \begin{pmatrix} 1 & 0 & (z_2-1)^2(2z_1-z_2+2) \\ 0 & -1 & (z_2-1)^2(2z_1^2-z_2z_1+z_2) \\ 0 & 0 & -1 \end{pmatrix}. \quad (99)$$

Now

$$F = F_6BL_4L_2. \quad (100)$$

Let $L_5 = L_4L_2$, then

$$L_5 = \begin{pmatrix} 1 & z_2^3-2z_2^2z_1-4z_2^2+z_2z_1^2+8z_2z_1+6z_2-4z_1^2-12z_1-4 & 0 \\ 0 & z_2^3z_1-z_2^3-2z_2^2z_1^2-2z_2^2z_1+2z_2^2+z_2z_1^3+7z_2z_1^2-z_2z_1+4z_1^3-8z_1^2+4z_1+4 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (101)$$

we have

$$F = F_6BL_5. \quad (102)$$

By computing, $\det F_6 = \det F_5 = 1$.
Thus F is equivalent to the matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (z_1-z_2)^2(z_2-1)^4 \end{pmatrix}. \quad (103)$$

Now we can find $X = 0_{1\times 3}$, $Y = 0_{3\times 1}$, such that

$$\begin{pmatrix} F_6^{-1} & 0_{3\times 1} \\ X & 1 \end{pmatrix}\begin{pmatrix} F & U \\ -V & W \end{pmatrix} = \begin{pmatrix} B & U_1 \\ -V_1 & W_1 \end{pmatrix}\begin{pmatrix} L_5 & Y \\ 0_{1\times 3} & 1 \end{pmatrix}, \quad (104)$$

where $U_1 = F_6U = (z_2\ z_2z_1 - z_2\ z_2^4 - z_1)^T$, $V_1 = VL_5^{-1} = (-z_1\ 0\ -a_1)$, $a_1 = -z_1z_2^3 + 2z_1^2z_2^2 + 4z_1z_2^2 - z_2z_1^3 - 8z_2z_1^2 - 6z_2z_1 + z_2 + 4z_1^3 + 12z_1^2 + 4z_1$, $W_1 = 0$.

Thus, the system

$$\begin{pmatrix} F(z_1,z_2) & U(z_1,z_2) \\ -V(z_1,z_2) & W(z_1,z_2) \end{pmatrix}, \quad (105)$$

is equivalent to the system

$$\begin{pmatrix} B(z_1,z_2) & U_1(z_1,z_2) \\ -V_1(z_1,z_2) & W_1(z_1,z_2) \end{pmatrix}, \quad (106)$$

which is simpler.

With the help of Example 1 and Theorem 3, we now can get the Algorithm 1.

The program of the function SyzygyModule which we use in algorithm can be found in https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2-1.15/share/doc/Macaulay2/MCMApproximations/html/_syzygy__Module.html.

Another function CompleteMatrix is available in http://wwwb.math.rwth-aachen.de/QuillenSuslin. Moreover, for a new algorithm for CompleteMatrix, see the package MatrixFactorization in http://www.mmrc.iss.ac.cn/dwang/software.html, which contained a ZLP algorithm. This

(i) Step 1. Declare the ring $K[z] = K[z_1, \ldots, z_n]$ over which the matrix is defined by declaring the indeterminates and the field of coefficients. Factor the determinant of F. Check that the determinant of F is the form $(z_j - f_j(z_2, \ldots, z_n))^{q_j}(z_k - f_k(z_3, \ldots, z_n))^{q_k}$. If yes, set $q_1 = \max\{q_j, q_k\}$, the polynomial it corresponds is $z_1 - f_1$, $q_2 = \min\{q_j, q_k\}$, the polynomial it corresponds is $z_2 - f_2$, go to Step 2. Otherwise, return this method is not fit for F.
(ii) Step 2. Compute the reduced Gröbner basis G of ideal generated by the lower minors of F. If $G = \{1\}$, go to Step 3; otherwise, return this method is not fit for F.
(iii) Step 3. Set $F_0 = F$, $P_1 = \mathrm{diag}(1, \ldots, 1, z_1 - f_1)$, $P_2 = \mathrm{diag}(1, \ldots, 1, z_2 - f_2)$, $P = P_1 P_2$, $i = 0$ and $t = q_1 - q_2$.
(iv) Step 4. Substitute $z_1 = f_1$ in F_i to obtain $\overline{F_i}$. Compute a ZRP vector $Y_i \in K[z_2, \ldots, z_n]$ such that $\overline{F_i} Y_i = 0$ by using the function SyzygyModule. Then compute a unimodular matrix U with Y_i is its last column by using the function CompleteMatrix. Compute F_{i+1} such that $F_i U = F_{i+1} P_1$. Compute U^{-1} and set $U_{i+1} = U^{-1}$. Compute $P_1^i U_{i+1} P_1 = N_i P_1^{i+1} V_i$ and obtain $F = F_{i+1} N_i P_1^{i+1} V_i$, store F_{i+1}. $L_3 = F_{q_1} L_1$, $i = i + 1$.
(v) Step 5. When $i \leq t$, go to Step 4. When $t < i < q_1$, do Step 6; otherwise, compute $P^{q_2} N_{q_1} P_1^t = L_1 P^{q_2} P_1^t L_2$, let return $F = L_3 P^{q_2} P_1^t L_2$.
(vi) Step 6. Substitute $z_1 = f_1$ in F_i to obtain $\overline{F_i}$, do procedure similar to the step 4. And obtain a ZRP vector Y_i such that $\overline{F_i} Y_i = 0$ and a unimodular matrix U with Y_i is its last column. Then compute F_{i+1} such that $F_i U = F_{i+1} P_1$, and compute U^{-1} and set $U_{i+1} = U^{-1}$. Set $F_i = F_{i+1}$, substitute $z_2 = f_2$ in F_i to obtain $\overline{F_i}$. Compute a ZRP vector $X \in K[z_1, z_3, \ldots, z_n]$ such that $\overline{F_i} X = 0$ by using the function SyzygyModule. Then compute a unimodular matrix V with X is its last column by using the function CompleteMatrix. Compute $V_{i-t} = V^{-1}$ and F_{i+1} such that $F_i V_{i-t} = F_{i+1} P_2$. Compute $P_1 U_{i+1} P_2 = N_i P M_i$ and obtain $F = F_{i+1} P^{i-t} N_i P_1^t V_i$, where $V_i = M_i V_{i-t}$. Store F_{i+1}, $i = i + 1$. Go to Step 5.

Algorithm 1: A matrix equivalence (ME) algorithm.

algorithm can obtain a unimodular matrix whose inverse is the complete matrix of a given ZLP matrix.

5. Conclusions

In this paper, we have investigated the equivalence problem of several kinds of nD polynomials matrices over an arbitrary field, and have presented some interesting results. We have obtained some criteria for these matrices to equivalent to their Smith forms respectively. These criteria are easily checked by the existing Gröbner basis algorithm for the ideal generated by the minors of lower order of a given matrix. We also give an example to illustrate our method. All of these could provide useful information for engineers to reduce nD systems.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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