Research Article

Finite-Time Attitude Control for Quadrotor with Input Constraints and Disturbances

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This paper investigates the adaptive output feedback attitude control of a quadrotor. First, a nonsingular terminal sliding-mode variable and auxiliary variable are introduced into a closed-loop structure. Meanwhile, a fuzzy logic system is incorporated into an adaptive algorithm to compensate for the adverse influence caused by lumped disturbances including system uncertainty and external disturbances on the attitude adjustment performance of a quadrotor. Then, a novel finite-time output feedback controller equipped with the saturation suppression algorithm is designed. Rigorous proof shows that the design control strategy ensures the closed-loop system stability and guarantees the attitude of the spacecraft to track desired command signals in finite time. Simulation results are presented to illustrate the performance of the proposed control scheme.

1. Introduction

The challenges faced in the control of a quadrotor with excellent capability have been concerned and developed widely in the last several decades based on the demand for a reliable one for both commercial and military applications. During the quadrotor entry process, the attitude adjustment of the quadrotor involves attitude maneuvering through a wide range of physical limitation, lumped external disturbances, and system parameter uncertainties [1–3]. Many advanced flight control methods have been proposed for the quadrotor's attitude control in order to improve the performance of the control system, such as adaptive backstepping method [4, 5], adaptive compensation control [6–8], sliding-mode control strategy [9, 10], and linearization control technique [11, 12].

Although the high accuracy of attitude adjustment can be accomplished by using the previously discussed control strategy, these works require infinite time to achieve attitude maneuver. It is obvious that the convergence rate with infinite settling time is not ideal during critical phases of some kind of specialized real-time tasks. There can be no doubt that the finite-time control technology [13, 14] can provide higher tracking accuracy, faster dynamic response,

and better robustness than the asymptotic control approach. Therefore, the finite-time control technique is one of the most effective control strategies that can handle the tracking control task and the attitude stabilization problem. In [15], a time-specified control strategy based on the nonsingular terminal sliding-mode technique was designed to solve the problem of trajectory tracking for robotic airships. In [16], a finite-time control method with time-delay estimation is proposed for shape memory alloy actuators. In [17], an adaptation supertwisting algorithm control scheme using the finite-time technique was designed for attitude tracking of the quadrotor. In [18], a smooth variable structure control scheme with multivariable form and disturbance observer was designed for quadrotor attitude adjustment. A nonsingular terminal sliding-mode control (NTSMC) method using the finite-time technique was thoroughly examined in [19] to achieve attitude tracking maneuver for the quadrotor with atmospheric disturbances. In a more recent work [20], an output feedback attitude controller based on the finite-time strategy was developed for the quadrotor with both matched and mismatched disturbances. Although many finite-time control strategies are focused on how to deal with attitude maneuver of the quadrotor, there is still a challenging problem for improving both the fast response ability

and robustness of the closed-loop system with the factor of numerous disadvantages in different working environments, which is one motivation of this paper.

Another problem to be noted is actuator saturation, which is generally encountered for nonsmooth nonlinear constraints of actuators in quadrotor applications. In fact, if an actuator falls into its physical saturation limitation boundary, any operation to enlarge the output of the actuator would not make any variation in the system output. It is likely to cause control performance degradation and even leading to task failure due to actuator saturation when the designed controller is not equipped with an effectiveness adjustment strategy to dump the saturated actuators [21–26]. Besides, taking quadrotor parameter uncertainties and large external disturbances into consideration makes the attitude tracking control problem of the quadrotor more difficult. How to cope with the simultaneous action of the abovementioned adverse factors has been turned into a challenging subject, and extensive attentions have been paid to solve the issue of saturation nonlinearity. To the best knowledge of the authors because of the complex characteristics of the quadrotor system, there are few results for the case where a finite-time control method is designed to achieve attitude adjustment of the quadrotor with control torque limitation and lumped disturbances.

In order to respond to the above discussions, we will investigate the adaptive output feedback attitude control based on the finite-time control technique for quadrotor systems with lumped disturbance and input saturation in this paper. In particular, we are interested in developing an effective adaptation compensation algorithm capable of accommodating the lumped disturbances as well as input saturation suppression. The main contributions are summarized as follows:

- ① A control input auxiliary variable is designed into the design of the closed-loop system, and the design parameter update laws are able to make the control input boundedness by the saturation suppression algorithm only.
- ② A fuzzy logic system is incorporated into the parameter updating algorithm to estimate the complex nonlinear function including system uncertainty and external disturbances. Under the design NTSMC structure, a novel compensation control law equipped with the adaptive algorithm is designed to achieve both terminal surface variable stabilization and attitude tracking.
- ③ A sliding-mode differentiator is utilized to relax the differential operation requirement of the virtual control law, which avoids the related singular problems. Moreover, the introduced differentiator can achieve virtual law approximation with lower computational complexity, facilitating the control law design.

The paper is organized as follows. In Section 2, the system model with adverse factors is formulated. The closed-loop system structure is established by the nonsingular

terminal surface and the control input auxiliary variable, and the finite-time stability of the design control strategy with the parameters update law is analyzed in Section 3. Illustrative simulation of the design control strategy for the quadrotor and conclusion is given in Sections 4 and 5, respectively.

2. Problem Formulation

2.1. System Model. The quadrotor system model contains the rotational equations and the translational equations of motion. This paper will focus on the attitude controller design for the quadrotor, and therefore a control-oriented model can be expressed as [5–8]

$$\mathbf{J}\ddot{\Theta} + \mathbf{N}\dot{\Theta} = \mathbf{M} + \mathbf{d}, \quad (1)$$

where $\Theta = [\alpha, \beta, \sigma]^T \in \mathbb{R}^3$ is the attitude angle vector including the yaw angle α , pitch angle β , and roll angle σ ; $\dot{\Theta}$ and $\ddot{\Theta}$ are the attitude angular rate vector and the angular acceleration vector, respectively; $\mathbf{d} = [d_1, d_2, d_3]^T$ is the external disturbance; $\mathbf{M} = [M_1, M_2, M_3] \in \mathbb{R}^3$ is control torque vector subject to nonlinear saturation described as

$$M_i = \begin{cases} \text{sign}(u_i) \cdot u_M, & \text{if } |u_i| \geq u_M, \\ u_i, & \text{if } |u_i| < u_M, \end{cases} \quad (i = 1, 2, 3), \quad (2)$$

where u_M is a known bound of M_i and u_i is the design control command. In addition, $\mathbf{J} \in \mathbb{R}^{3 \times 3}$ is a symmetric and positive definite matrix, and the matrix $\mathbf{N} \in \mathbb{R}^{3 \times 3}$ is the Coriolis term of the quadrotor containing centrifugal and gyroscope terms ($\mathbf{J}$ and $\mathbf{N}$ will be given later).

It is worth noting that the expression of M_i in (2) is a discontinuous function which will lead to the sharp corner. In order to improve the smoothness of the control input, the saturation nonlinear function M_i is approximated by a smooth function defined as

$$\begin{cases} \Gamma_u(u_i) = u_M \cdot \tanh\left(\frac{u_i}{u_M}\right), \\ \Gamma_u = [\Gamma_u(u_1), \Gamma_u(u_2), \Gamma_u(u_3)]^T. \end{cases} \quad (3)$$

In light of (3), (2) can be expressed as

$$M_i = \Gamma_u(u_i) + \Delta M_i, \quad i = 1, 2, 3, \quad (4)$$

where $|\Delta M_i| = |M_i - \Gamma_u(u_i)| \leq u_M(1 - \tanh(1))$ is a bounded approximation error. Furthermore, we have

$$\mathbf{M} = \Gamma_u + \Delta \mathbf{M}, \quad (5)$$

where $\Delta \mathbf{M} = [\Delta M_1, \Delta M_2, \Delta M_3]^T$, satisfying $\|\Delta \mathbf{M}\| = \sum_{i=1}^3 |\Delta M_i| \leq 3 \cdot u_M(1 - \tanh(1))$. Let $\dot{\Theta} = \omega$, then (1) can be rewritten as

$$\begin{cases} \dot{\Theta} = \omega, \\ \mathbf{J}\dot{\omega} = -\mathbf{N}\omega + \Gamma_u + \Delta \mathbf{M} + \mathbf{d}. \end{cases} \quad (6)$$

In order to further develop our control scheme, the following lemmas and assumption are given.

Assumption 1. It is supposed that lumped disturbances d_i ($i = 1, 2, 3$) is continuous, which satisfy $\max\{\|\mathbf{d}\|, \|\dot{\mathbf{d}}\|\} \leq \rho$ where ρ is an unknown positive constant.

Lemma 1 (see [27]). *A first-order sliding-mode differentiator is designed as*

$$\begin{cases} \dot{\mu}_1 = \varsigma, \\ \varsigma = \mu_2 - b_1^* \cdot |\mu_1 - F(t)|^{1/2} \cdot \text{sign}(\mu_1 - F(t)), \\ \dot{\mu}_2 = -b_2^* \cdot \text{sign}(\mu_2 - \varsigma), \end{cases} \quad (7)$$

where μ_1 and μ_2 are the states of the differentiator (7), ς is the output of the differentiator, b_1^* and b_2^* are the positive constants, and $F(t)$ is a known function. Then, ς can approximate the differential term $\dot{F}(t)$ to any arbitrary accuracy if the initial deviations $\mu_1 - F(t_0)$ and $\varsigma - \dot{F}(t_0)$ are bounded.

Lemma 2 (see [27]). *Consider the system $\dot{x} = f(x, u)$. Suppose that there exist a continuous function $V(x)$ and scalars $\psi_1 > 0$, $0 < \psi_2 < 1$, and $0 < \psi_3 < \infty$, such that*

$$\dot{V}(x) \leq -\psi_1 \cdot V^{\psi_2}(x) + \psi_3. \quad (8)$$

Then, this system's trajectory is practical finite-time stable.

2.2. Description of a Fuzzy Logic System. From [28], any continuous functions on a compact set can be approximated by the fuzzy logic system, with arbitrary accuracy. The fuzzy logic system is composed of four principal sections: an inference engine, a fuzzy rule, a fuzzifier, and a defuzzifier. Indeed, a set of 'If-Then' linguistic rules can describe the fuzzy logic system, i.e.,

$$R^{(j)}: \text{If } x_1 \text{ is } A_1^j \text{ and } \dots \text{ and } x_n \text{ is } A_n^j, \text{ then } z \text{ is } B^j, \quad (9)$$

where $j = 1, 2, \dots, N$; $x = (x_1, x_2, \dots, x_n)^T$ and $z \in \mathbb{R}$ are the inputs and outputs of the fuzzy logic system, respectively; in addition, A_i^j , $i = 1, 2, \dots, N$ and B^j are characterized by the fuzzy membership functions $\mu_{A_i^j}(x_i)$ and $\mu_{B^j}(z)$, respectively. Based on the singleton fuzzifier, product inference, center-average defuzzifier, and Gaussian membership function, the output of the fuzzy logic system can be written as

$$f(x) = \frac{\sum_{j=1}^M \bar{z}^j \left(\prod_{i=1}^n \mu_{A_i^j}(x_i) \right)}{\sum_{j=1}^M \left(\prod_{i=1}^n \mu_{A_i^j}(x_i) \right)} = \Phi \cdot \zeta(x), \quad (10)$$

where $\bar{z}^j \in \mathbb{R}$ is the point at which $\mu_{B^j}(z)$ achieves the maximum value, $\Phi = [\bar{z}^1, \bar{z}^2, \dots, \bar{z}^M]$ is an adjustable parameter vector, and $\zeta = [\zeta_1(x), \zeta_2(x), \dots, \zeta_n(x)]^T$ is the fuzzy basis function vector; each element of $\zeta(x)$ is

$$\zeta_i(x) = \frac{\sum_{j=1}^M \prod_{i=1}^n \mu_{A_i^j}(x_i)}{\sum_{j=1}^M \left(\prod_{i=1}^n \mu_{A_i^j}(x_i) \right)}. \quad (11)$$

It should be pointed out that fuzzy logic system is a powerful function approximation tool, and it can effectively

deal with uncertainty [29]. For a continuous nonlinear function $\mathbf{D}$, it can be approximated by

$$\mathbf{D} = f(x) = \Phi \cdot \zeta^T(x) + \mathbf{v}, \quad (12)$$

where $\Phi = [\Phi_1, \Phi_2, \dots, \Phi_n]^T$ is an optimal weight vector and $\mathbf{v} = [\nu_1, \nu_2, \dots, \nu_n]^T$ is the error vector of fuzzy approximation.

Assumption 2. The optimal weight and approximation error of FLS are bounded such that $\|\Phi\| \leq \Phi_M$ and $\|\mathbf{v}\| \leq \nu_M$, where Φ_M and ν_M are the positive constants.

The aim of this paper is to construct a closed-loop system with the self-adaption compensation algorithm which guarantees that the commanded signal Θ_d is tracked by the system output Θ in finite time in the presence of the lumped disturbance and input constraints.

3. Control Strategy Design

In this section, an output feedback attitude control strategy is proposed under the framework of the NTSMC method, which guarantees the boundedness of all the signals in the control loop. To achieve high-accuracy tracking control, a control input updating algorithm is further designed to avoid control input constraints and to handle lumped disturbances in the system. The stability of the closed-loop system can be proved at the same time. Now, the detailed contents are stated as follows.

First, we define the system tracking error $\mathbf{e}_1 \in \mathbb{R}^3$ and its derivative $\mathbf{e}_2 \in \mathbb{R}^3$ as

$$\mathbf{e}_1 = [e_{11}, e_{12}, e_{13}]^T = \Theta - \Theta_d, \quad (13)$$

$$\mathbf{e}_2 = \dot{\mathbf{e}}_1 = [e_{21}, e_{22}, e_{23}]^T = \dot{\Theta} - \dot{\Theta}_d, \quad (14)$$

where $\Theta_d = [\Theta_{d1}, \Theta_{d2}, \Theta_{d3}]^T$ represents the desired commanded signal. To ensure fast and accurate attitude tracking performance of system (1), an NTSM-type sliding surface [30] $\mathbf{S}_a \in \mathbb{R}^3$ and an auxiliary variable $\mathbf{z} \in \mathbb{R}^3$ are defined as

$$\mathbf{S}_a = [S_{a1}, S_{a2}, S_{a3}]^T = \mathbf{e}_1 + c \cdot \mathbf{e}_2^{\kappa} \cdot \text{sign}(\mathbf{e}_2), \quad 1 < \kappa < 2, c > 0.5, \quad (15)$$

$$\begin{aligned} \mathbf{z} = [z_1, z_2, z_3]^T &= \Gamma_{\mathbf{u}} - \Gamma_{\Phi} = \begin{bmatrix} \Gamma_{\mathbf{u}}(u_1) \\ \Gamma_{\mathbf{u}}(u_2) \\ \Gamma_{\mathbf{u}}(u_3) \end{bmatrix} - \begin{bmatrix} \Gamma_{\Phi}(\varphi_1) \\ \Gamma_{\Phi}(\varphi_2) \\ \Gamma_{\Phi}(\varphi_3) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} u_M \cdot \tanh\left(\frac{u_1}{u_M}\right) \\ u_M \cdot \tanh\left(\frac{u_2}{u_M}\right) \\ u_M \cdot \tanh\left(\frac{u_3}{u_M}\right) \end{bmatrix}}_{\Gamma_{\mathbf{u}}} - \underbrace{\begin{bmatrix} \varphi_M \cdot \tanh\left(\frac{\varphi_1}{\varphi_M}\right) \\ \varphi_M \cdot \tanh\left(\frac{\varphi_2}{\varphi_M}\right) \\ \varphi_M \cdot \tanh\left(\frac{\varphi_3}{\varphi_M}\right) \end{bmatrix}}_{\Gamma_{\Phi}}, \end{aligned} \quad (16)$$

where $\mathbf{e}_2^{\kappa} \text{sign}(\mathbf{e}_2) = [|e_{21}|^{\kappa} \text{sign}(e_{21}), |e_{22}|^{\kappa} \text{sign}(e_{22}), |e_{23}|^{\kappa} \text{sign}(e_{23})]^T$, $\kappa = (p/q)$ with odd integers p and q , and in addition, $\varphi_M = u_M \cdot \tanh(1) - b > 0$ (b is the design parameter satisfying $2b \ll 1$) and φ_i ($i = 1, 2, 3$) will be designed later. For analysis simplification, $\varphi \triangleq [\varphi_1, \varphi_2, \varphi_3]^T$ in which φ will be given later.

$$\begin{aligned} \dot{\mathbf{S}}_a &= [\dot{S}_{a1}, \dot{S}_{a2}, \dot{S}_{a3}]^T = \mathbf{e}_2 + \kappa c \cdot |\mathbf{e}_2|^{\kappa-1} \cdot \dot{\mathbf{e}}_2 \\ &= \mathbf{e}_2 + \kappa c \cdot |\mathbf{e}_2|^{\kappa-1} \cdot [\ddot{\Theta} - \ddot{\Theta}_d] = \mathbf{e}_2 + \kappa c \cdot |\mathbf{e}_2|^{\kappa-1} \cdot [-\mathbf{J}^{-1} \cdot \mathbf{N} \cdot \boldsymbol{\omega} + \mathbf{J}^{-1} \cdot \mathbf{z} + \mathbf{J}^{-1} \boldsymbol{\varphi} + \mathbf{J}^{-1} \mathbf{d} - \mathbf{J}^{-1}(\boldsymbol{\varphi} - \Gamma_{\varphi}) + \mathbf{J}^{-1} \Delta \mathbf{M} - \ddot{\Theta}_d] = \mathbf{e}_2 + \kappa c \cdot |\mathbf{e}_2|^{\kappa-1} \cdot (\boldsymbol{\varphi} + \mathbf{D}), \end{aligned} \quad (17)$$

with $|\mathbf{e}_2|^{\kappa-1} = \text{diag}\{|e_{21}|^{\kappa-1}, |e_{22}|^{\kappa-1}, |e_{23}|^{\kappa-1}\} \geq 0$ and $\mathbf{D} = -\mathbf{J}^{-1} \mathbf{N} \boldsymbol{\omega} + \mathbf{J}^{-1} \mathbf{d} + \mathbf{J}^{-1} \Delta \mathbf{M} - \ddot{\Theta}_d + \mathbf{J}^{-1} \mathbf{z} + \mathbf{J}^{-1} \Gamma_{\varphi} - \boldsymbol{\varphi}$. Note that the mixture term $\mathbf{D}$ is a nonlinear function including the external disturbances and parameter perturbations. To eliminate the adverse effect caused by the lumped disturbance $\mathbf{D}$ on the system, the fuzzy logic system represented by (10) is employed to learn the unknown nonlinear function $\mathbf{D}$ due to its universal approximation property, i.e.,

$$\mathbf{D} = \Phi \cdot \zeta^T + \mathbf{v} = \|\Phi\| \cdot \Phi^* \cdot \zeta^T + \mathbf{v}, \quad (18)$$

with $\Phi^* = (\Phi / \|\Phi\|)$. With the help of Young's inequality, the relationships (19) and (20) can be established:

$$\mathbf{S}_a^T \mathbf{e}_2 \leq \frac{\|\mathbf{S}_a^T \mathbf{e}_2\|_2^2}{2a_1^2} + \frac{a_1^2}{2}, \quad (19)$$

$$\begin{aligned} \kappa c \mathbf{S}_a^T |\mathbf{e}_2|^{\kappa-1} \mathbf{D} &= \kappa c \mathbf{S}_a^T |\mathbf{e}_2|^{\kappa-1} \cdot (\|\Phi\| \cdot \Phi^* \cdot \zeta + \mathbf{v}) \\ &\leq \frac{\kappa^2 c^2 \|\mathbf{S}_a^T \mathbf{e}_2^{\kappa-1}\|_2^2 \|\Phi\|_2^2 \|\zeta\|_2^2}{2a_2^2} \\ &\quad + \frac{a_2^2}{2} + \frac{\kappa^2 c^2 \|\mathbf{S}_a^T \mathbf{e}_2^{\kappa-1}\|_2^2 \|\mathbf{v}\|_2^2}{2a_2^2} + \frac{a_2^2}{2}, \end{aligned} \quad (20)$$

where $\|\mathbf{S}_a^T \mathbf{e}_2^{\kappa-1}\|_2^2 = (S_{a1} e_{21}^{\kappa-1})^2 + (S_{a2} e_{22}^{\kappa-1})^2 + (S_{a3} e_{23}^{\kappa-1})^2$ and $\|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 = (S_{a1} e_{21})^2 + (S_{a2} e_{22})^2 + (S_{a3} e_{23})^2$; moreover, since $0 < \kappa - 1 = ((p - q)/q) < 1$, and based on the fact $e_{2i}^{2(\kappa-1)} \leq e_{2i}^2$ when $|e_{2i}| \geq 1$ and $e_{2i}^{2(\kappa-1)} \leq e_{2i}^{\kappa-1}$ when $|e_{2i}| < 1$, we have

$$\begin{cases} \frac{\kappa^2 c^2 \|\mathbf{S}_a^T \mathbf{e}_2^{\kappa-1}\|_2^2 \|\Phi\|_2^2 \|\zeta\|_2^2}{2a_2^2} \leq \frac{\kappa^2 c^2 \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 \|\Phi\|_2^2 \|\zeta\|_2^2}{2a_2^2} + \frac{\kappa^2 c^2 \cdot \widehat{\mathbf{S}}_a^T \cdot \widehat{\mathbf{e}}_2^{\kappa-1} \cdot \|\Phi\|_2^2 \|\zeta\|_2^2}{2a_2^2}, \\ \frac{\kappa^2 c^2 \|\mathbf{S}_a^T \mathbf{e}_2^{\kappa-1}\|_2^2 \|\mathbf{v}\|_2^2}{2a_2^2} \leq \frac{\kappa^2 c^2 \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 \|\mathbf{v}\|_2^2}{2a_2^2} + \frac{\kappa^2 c^2 \cdot \widehat{\mathbf{S}}_a^T \cdot \widehat{\mathbf{e}}_2^{\kappa-1} \|\mathbf{v}\|_2^2}{2a_2^2}, \end{cases} \quad (21)$$

where $\widehat{\mathbf{S}}_a = [S_{a1}^2, S_{a2}^2, S_{a3}^2]^T$ and $\widehat{\mathbf{e}}_2^{\kappa-1} = [e_{21}^{\kappa-1}, e_{22}^{\kappa-1}, e_{23}^{\kappa-1}]^T$. From (21), (20) can be rewritten as

$$\kappa c \mathbf{S}_a^T \mathbf{e}_2^{\kappa-1} \mathbf{D} \leq \frac{\kappa^2 c^2 \left(\|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 + \widehat{\mathbf{S}}_a^T \cdot \widehat{\mathbf{e}}_2^{\kappa-1} \right)}{2a_2^2} \cdot \boldsymbol{\eta}^T \boldsymbol{\Psi} + a_2^2, \quad (22)$$

where $\mathbf{e}_2^{\kappa-1} = \text{diag}\{e_{21}^{\kappa-1}, e_{22}^{\kappa-1}, e_{23}^{\kappa-1}\}$, $\boldsymbol{\eta} = [\|\Phi\|^2, \|\mathbf{v}\|^2]^T$, and $\boldsymbol{\Psi} = [\|\zeta\|^2, 1]^T$. Next, a virtual control law φ is designed as

$$\boldsymbol{\varphi} \triangleq [\varphi_1, \varphi_2, \varphi_3]^T = -\frac{\kappa c}{2a_2^2} (\mathbf{S}_a + \mathbf{e}_2^{3-\kappa} \mathbf{S}_a) \cdot (\widehat{\boldsymbol{\eta}}^T \boldsymbol{\Psi}) - \mathbf{e}_2^{3-\kappa} \mathbf{S}_a, \quad (23)$$

Remark 1. By choosing $1 < \kappa < 2$ and the expression form of S_{ai} ($i = 1, 2, 3$), the singularity problem for $e_{2i} = \dot{e}_{1i} = 0$ can be avoided. Moreover, function $S_{ai} = e_{1i} + c \cdot |e_{2i}|^{\kappa} \cdot \text{sign}(e_{2i})$ is continuous and differentiable although the signum operator and the absolute value are involved [31].

With the combination of (6), (13), (14), and (16), the derivative of the terminal sliding surface $\mathbf{S}_a$ is given as follows:

where $\mathbf{e}_2^{3-\kappa} = \text{diag}\{e_{21}^{3-\kappa}, e_{22}^{3-\kappa}, e_{23}^{3-\kappa}\}$, and the related adaptation law is chosen as

$$\dot{\widehat{\boldsymbol{\eta}}} = \gamma \cdot \left(\frac{\kappa^2 c^2}{2a_2^2} \cdot \left(\widehat{\mathbf{S}}_a^T \cdot \widehat{\mathbf{e}}_2^{\kappa-1} + \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 \right) \cdot \boldsymbol{\Psi} - a_2 \cdot \widehat{\boldsymbol{\eta}} \right), \quad \widehat{\boldsymbol{\eta}}(0) = 0, \gamma > 0. \quad (24)$$

In this study, $\widehat{\boldsymbol{\eta}} \triangleq [\widehat{\eta}_1, \widehat{\eta}_2]^T$ is employed to estimate $\boldsymbol{\eta}$, and the estimate error $\widetilde{\boldsymbol{\eta}}$ denotes $\widetilde{\boldsymbol{\eta}} = \boldsymbol{\eta} - \widehat{\boldsymbol{\eta}}$. It is worth noting that the time derivative of φ (in (23)) is required for the subsequent attitude controller and stability analysis. It is not difficult to find that the derivative of the virtual law φ can lead to the explosion problem of the complex term. In order to solve this problem, a first-order sliding-mode

differentiator is introduced into the closed-loop system design to estimate each element of $\dot{\varphi}$:

$$\begin{cases} \dot{\mu}_{1i} = \varsigma_i = \mu_{2i} - b_1^* \cdot |\mu_1 - \varphi_i|^{1/2} \cdot \text{sign}(\mu_1 - \varphi_i), \\ \dot{\mu}_{2i} = -b_2^* \cdot \text{sign}(\mu_{2i} - \varsigma_i), \\ i = 1, 2, 3, \end{cases} \quad (25)$$

where μ_{1i} and μ_{2i} are the states of the differentiator, ς_i is the output of the differentiator, and b_1^* and b_2^* are the positive design constants. According to (25) and Lemma 1, we obtain

$$\dot{\varphi} = \varsigma + \varepsilon, \quad (26)$$

where $\varsigma = [\varsigma_1, \varsigma_2, \varsigma_3]^T$ and $\varepsilon = [\varepsilon_1, \varepsilon_2, \varepsilon_3]^T$ is the estimation error vector of the introduced differentiator. According to Lemma 1, we can get $\|\varepsilon\| \leq \delta$ with a positive constant δ .

Noting that $\mathbf{S}_a^T \mathbf{e}_2^{\kappa-1} \mathbf{S}_a = \hat{\mathbf{S}}_a^T \hat{\mathbf{e}}_2^{\kappa-1}$ and $\mathbf{S}_a^T \mathbf{e}_2^{\kappa-1} \mathbf{e}_2^{3-\kappa} \mathbf{S}_a = \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2$, we can get from (23) that

$$\begin{aligned} \kappa c \mathbf{S}_a^T \mathbf{e}_2^{\kappa-1} \cdot \varphi &\leq -\frac{\kappa^2 c^2}{2a_2^2} \cdot \hat{\mathbf{S}}_a^T \cdot \hat{\mathbf{e}}_2^{\kappa-1} \cdot \hat{\boldsymbol{\eta}}^T \boldsymbol{\Psi} \\ &\quad - \frac{\kappa^2 c^2}{2a_2^2} \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 \cdot \hat{\boldsymbol{\eta}}^T \boldsymbol{\Psi} - \kappa c \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2. \end{aligned} \quad (27a)$$

Owing to $\hat{\boldsymbol{\eta}}^T \boldsymbol{\Psi} + \tilde{\boldsymbol{\eta}}^T \boldsymbol{\Psi} = \boldsymbol{\eta}^T \boldsymbol{\Psi}$, we have

$$\begin{cases} \frac{\kappa^2 c^2}{2a_2^2} \cdot \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 \cdot \hat{\boldsymbol{\eta}}^T \boldsymbol{\Psi} - \frac{\kappa^2 c^2}{2a_2^2} \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 \cdot \tilde{\boldsymbol{\eta}}^T \boldsymbol{\Psi} = -\frac{\kappa^2 c^2}{2a_2^2} \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 \cdot \boldsymbol{\eta}^T \boldsymbol{\Psi}, \\ \frac{\kappa^2 c^2}{2a_2^2} \cdot \hat{\mathbf{S}}_a^T \hat{\mathbf{e}}_2^{\kappa-1} \cdot \hat{\boldsymbol{\eta}}^T \boldsymbol{\Psi} - \frac{\kappa^2 c^2}{2a_2^2} \cdot \hat{\mathbf{S}}_a^T \hat{\mathbf{e}}_2^{\kappa-1} \cdot \tilde{\boldsymbol{\eta}}^T \boldsymbol{\Psi} = -\frac{\kappa^2 c^2}{2a_2^2} \cdot \hat{\mathbf{S}}_a^T \hat{\mathbf{e}}_2^{\kappa-1} \cdot \boldsymbol{\eta}^T \boldsymbol{\Psi}. \end{cases} \quad (27b)$$

By using (17), (19), (22), (24), (27a), and (27b), the following relationship can be established:

$$\begin{aligned} \mathbf{S}_a^T \dot{\mathbf{S}}_a - \frac{1}{\gamma} \tilde{\boldsymbol{\eta}}^T \dot{\tilde{\boldsymbol{\eta}}} &\leq \frac{1}{2a_1^2} \cdot \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 + \frac{a_1^2}{2} - \frac{\kappa^2 c^2}{2a_2^2} \cdot \hat{\mathbf{S}}_a^T \hat{\mathbf{e}}_2^{\kappa-1} \cdot \hat{\boldsymbol{\eta}}^T \boldsymbol{\Psi} \\ &\quad - \frac{\kappa^2 c^2}{2a_2^2} \cdot \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 \cdot \hat{\boldsymbol{\eta}}^T \boldsymbol{\Psi} - \kappa c \cdot \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 + a_2^2 \\ &\quad + \frac{\kappa^2 c^2}{2a_2^2} \left(\|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 + \hat{\mathbf{S}}_a^T \hat{\mathbf{e}}_2^{\kappa-1} \right) \cdot \boldsymbol{\eta}^T \boldsymbol{\Psi} \\ &\quad - \frac{\kappa^2 c^2}{2a_2^2} \left(\hat{\mathbf{S}}_a^T \hat{\mathbf{e}}_2^{\kappa-1} + \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 \right) \cdot \tilde{\boldsymbol{\eta}}^T \boldsymbol{\Psi} + a_2 \tilde{\boldsymbol{\eta}}^T \hat{\boldsymbol{\eta}} \\ &\leq -\left(\kappa c - (2a_1^2)^{-1} \right) \cdot \|\mathbf{S}_a^T \mathbf{e}_2\|_2^2 + 0.5a_1^2 + a_2^2 + a_2 \tilde{\boldsymbol{\eta}}^T \hat{\boldsymbol{\eta}}. \end{aligned} \quad (28)$$

To proceed, the dynamics of an auxiliary variable $\mathbf{z}$ in (16) can be calculated as

$$\dot{\mathbf{z}} = \frac{\partial \Gamma_{\mathbf{u}}}{\partial \mathbf{u}} \cdot \dot{\mathbf{u}} - \frac{\partial \Gamma_{\boldsymbol{\varphi}}}{\partial \boldsymbol{\varphi}} \cdot \dot{\boldsymbol{\varphi}} = \Gamma_{\mathbf{u}} \cdot \dot{\mathbf{u}} - \Gamma_{\boldsymbol{\varphi}} \cdot (\varsigma + \varepsilon), \quad (29)$$

where $\dot{\mathbf{u}} = [\dot{u}_1, \dot{u}_2, \dot{u}_3]^T$,

$$\Gamma_{\mathbf{u}} = \text{diag}\{\Xi_{u1}, \Xi_{u2}, \Xi_{u3}\} = \text{diag}\left\{\frac{\partial \Gamma_{\mathbf{u}}(u_1)}{\partial u_1}, \frac{\partial \Gamma_{\mathbf{u}}(u_2)}{\partial u_2}, \frac{\partial \Gamma_{\mathbf{u}}(u_3)}{\partial u_3}\right\} = \frac{\partial \Gamma_{\mathbf{u}}}{\partial \mathbf{u}}$$

$$= \begin{bmatrix} \frac{4}{(e^{u_1/u_M} + e^{-u_1/u_M})^2} & 0 & 0 \\ 0 & \frac{4}{(e^{u_2/u_M} + e^{-u_2/u_M})^2} & 0 \\ 0 & 0 & \frac{4}{(e^{u_3/u_M} + e^{-u_3/u_M})^2} \end{bmatrix},$$

$$\Gamma_{\boldsymbol{\varphi}} = \text{diag}\{\Xi_{\varphi1}, \Xi_{\varphi2}, \Xi_{\varphi3}\} = \text{diag}\left\{\frac{\partial \Gamma_{\boldsymbol{\varphi}}(\varphi_1)}{\partial \varphi_1}, \frac{\partial \Gamma_{\boldsymbol{\varphi}}(\varphi_2)}{\partial \varphi_2}, \frac{\partial \Gamma_{\boldsymbol{\varphi}}(\varphi_3)}{\partial \varphi_3}\right\} = \frac{\partial \Gamma_{\boldsymbol{\varphi}}}{\partial \boldsymbol{\varphi}}$$

$$= \begin{bmatrix} \frac{4}{(e^{\varphi_1/\varphi_M} + e^{-\varphi_1/\varphi_M})^2} & 0 & 0 \\ 0 & \frac{4}{(e^{\varphi_2/\varphi_M} + e^{-\varphi_2/\varphi_M})^2} & 0 \\ 0 & 0 & \frac{4}{(e^{\varphi_3/\varphi_M} + e^{-\varphi_3/\varphi_M})^2} \end{bmatrix}$$

$$= \text{diag} \left\{ \frac{4}{\left(e^{\varphi_1/(u_M \cdot \tanh(1)-b)} + e^{-\left(\varphi_1/(u_M \cdot \tanh(1)-b) \right)} \right)^2}, \frac{4}{\left(e^{\varphi_2/(u_M \cdot \tanh(1)-b)} + e^{-\left(\varphi_2/(u_M \cdot \tanh(1)-b) \right)} \right)^2}, \frac{4}{\left(e^{\varphi_3/(u_M \cdot \tanh(1)-b)} + e^{-\left(\varphi_3/(u_M \cdot \tanh(1)-b) \right)} \right)^2} \right\}. \quad (30)$$

Now, the input update law $\dot{u}_i$ ($i = 1, 2, 3$) is designed as

$$\dot{u}_i = \begin{cases} \Xi_{ui}^{-1} \cdot \left(\sigma_i - \left(\frac{u_i^2}{2u_M^2} + b \right) \cdot (1 + \sigma_i^2) \right), & \text{if } u_i \geq u_M; \\ \Xi_{ui}^{-1} \cdot \sigma_i, & \text{if } -u_M < u_i < u_M; \\ \Xi_{ui}^{-1} \cdot \left(\sigma_i + \left(\frac{u_i^2}{2u_M^2} + b \right) \cdot (1 + \sigma_i^2) \right), & \text{if } u_i \leq -u_M; \end{cases} \quad (31)$$

where $u_i(0) = 0$ and $\sigma \triangleq [\sigma_1, \sigma_2, \sigma_3]^T = -(\|\zeta\|_2^2 \mathbf{z}/2) - \mathbf{c} \mathbf{z}^{q/p} - \mathbf{c} \mathbf{z} - (\mathbf{c} \mathbf{z} \|\mathbf{S}_a\|^{(p+q)/p} \|\mathbf{z}\|_2^2)$, $(\|\zeta\|_2^2 \triangleq \sum_{i=1}^3 \zeta_i^2)$, $\mathbf{z}^{q/p} \triangleq [z_1^{q/p}, z_2^{q/p}, z_3^{q/p}]^T$, $\|\mathbf{S}_a\|^{(p+q)/p} \triangleq \sum_{i=1}^3 |\mathbf{S}_{ai}|^{(p+q)/p}$, and $\|\mathbf{z}\|_2^2 \triangleq \sum_{i=1}^3 z_i^2$. To facilitate the stability analysis, we focus our attention on the term $\mathbf{z}^T \dot{\mathbf{z}}$ which satisfies the following relationship:

- ① If $u_i \geq u_M$, we obtain $z_i = \Gamma_u(u_i) - \Gamma_\varphi(\varphi_i) > 0$; according to the fact $\Gamma_u(u_i) \geq \Gamma_u(u_M) = u_M \cdot \tanh(1)$ and $|\Gamma_\varphi(\varphi_i)| \leq \varphi_M = u_M \tanh(1) - b$, it yields $\sum_{i=1}^3 z_i (u_i^2 / (2u_M^2) + b) (1 + \sigma_i^2) \geq 0$.

Then, taking into account that

$$\begin{aligned} |\Xi_{\varphi_i}| &= \left| \frac{\partial \Gamma_\varphi(\varphi_i)}{\partial \varphi_i} \right| \\ &= \left| \frac{4}{\left(e^{\varphi_i/(u_M \cdot \tanh(1)-b)} + e^{-\left(\varphi_i/(u_M \cdot \tanh(1)-b) \right)} \right)^2} \right| \leq 1, \\ \mathbf{z}^T \Gamma_\varphi(\zeta + \varepsilon) &\leq 0.5 \cdot \left(\|\mathbf{z}\|_2^2 \|\zeta\|_2^2 + \|\Gamma_\varphi\|_2^2 + \|\mathbf{z}\|_2^2 + \delta \cdot \|\Gamma_\varphi\|_2^2 \right), \end{aligned} \quad (32)$$

this leads to

$$\begin{aligned} \mathbf{z}^T \dot{\mathbf{z}} &= \sum_{i=1}^3 \Xi_{ui} \cdot z_i \cdot \dot{u}_i - \mathbf{z}^T \Gamma_\varphi(\zeta + \varepsilon) \\ &= \mathbf{z}^T \boldsymbol{\sigma} - \sum_{i=1}^3 z_i \cdot \left(\frac{u_i^2}{2u_M^2} + b \right) \cdot (1 + \sigma_i^2) - \mathbf{z}^T \Gamma_\varphi(\zeta + \varepsilon) \\ &\leq \mathbf{z}^T \left(-\mathbf{c} \mathbf{z}^{q/p} - \mathbf{c} \mathbf{z} - \frac{\|\zeta\|_2^2 \mathbf{z}}{2} \right) - c \|\mathbf{S}_a\|^{(p+q)/p} + \frac{\|\mathbf{z}\|_2^2 \|\zeta\|_2^2}{2} \\ &\quad + \frac{3 \cdot (1 + \delta)}{2} + \frac{\|\mathbf{z}\|_2^2}{2} \\ &= -\mathbf{c} \mathbf{z}^{(p+q)/p} - c \|\mathbf{S}_a\|^{(p+q)/p} - \left(c - \frac{1}{2} \right) \|\mathbf{z}\|_2^2 + \frac{3}{2} (1 + \delta). \end{aligned} \quad (33)$$

- ② If $|u_i| < u_M$, we can prove that

$$\begin{aligned} \mathbf{z}^T \dot{\mathbf{z}} &= \sum_{i=1}^3 \Xi_{ui} z_i \dot{u}_i - \mathbf{z}^T \cdot \Gamma_\varphi(\zeta + \varepsilon) = \mathbf{z}^T \Gamma_u - \mathbf{z}^T \Gamma_\varphi(\zeta + \varepsilon) \\ &\leq \mathbf{z}^T \left(-\mathbf{c} \mathbf{z}^{q/p} - \mathbf{c} \mathbf{z} - \frac{\|\zeta\|_2^2 \mathbf{z}}{2} \right) - c \|\mathbf{S}_a\|^{(p+q)/p} + \frac{\|\mathbf{z}\|_2^2 \|\zeta\|_2^2}{2} \\ &\quad + \frac{3 \cdot (1 + \delta)}{2} + \frac{\|\mathbf{z}\|_2^2}{2} = -\mathbf{c} \mathbf{z}^{(p+q)/p} - c \|\mathbf{S}_a\|^{(p+q)/p} \\ &\quad - \left(c - \frac{1}{2} \right) \|\mathbf{z}\|_2^2 + \frac{3(1 + \delta)}{2}. \end{aligned} \quad (34)$$

- ③ If $u_i \leq -u_M$, it can be easily inferred that $\Gamma_u(u_i) \leq \Gamma_u(-u_M) = -u_M \cdot \tanh(1)$. Owing to $|\Gamma_\varphi(\varphi_i)| \leq \varphi_M = u_M \cdot \tanh(1) - b$, we further obtain $z_i = \Gamma_u(u_i) - \Gamma_\varphi(\varphi_i) \leq 0$, and this ensures that $\sum_{i=1}^3 z_i \cdot (u_i^2 / (2u_M^2) + b) \cdot (1 + \sigma_i^2) \leq 0$. Therefore,

$$\begin{aligned} \mathbf{z}^T \dot{\mathbf{z}} &= \sum_{i=1}^3 \Xi_{ui} z_i \dot{u}_i - \mathbf{z}^T \Gamma_\varphi(\zeta + \varepsilon) \\ &= \mathbf{z}^T \boldsymbol{\sigma} + \sum_{i=1}^3 z_i \cdot \left(\frac{u_i^2}{2u_M^2} + b \right) \cdot (1 + \sigma_i^2) - \mathbf{z}^T \cdot \Gamma_\varphi(\zeta + \varepsilon) \\ &\leq \mathbf{z}^T \left(-\mathbf{c} \mathbf{z}^{q/p} - \mathbf{c} \mathbf{z} - \frac{\|\zeta\|_2^2 \mathbf{z}}{2} \right) - c \|\mathbf{S}_a\|^{(p+q)/p} + \frac{\|\mathbf{z}\|_2^2 \|\zeta\|_2^2}{2} \\ &\quad + \frac{3 \cdot (1 + \delta)}{2} + \frac{\|\mathbf{z}\|_2^2}{2} \\ &= -\mathbf{c} \mathbf{z}^{(p+q)/p} - c \|\mathbf{S}_a\|^{(p+q)/p} - \left(c - \frac{1}{2} \right) \|\mathbf{z}\|_2^2 + \frac{3(1 + \delta)}{2}. \end{aligned} \quad (35)$$

Noting that the design parameter c satisfies $c > 0.5$ and combining with the above analysis, we thus obtain

$$\mathbf{z}^T \dot{\mathbf{z}} \leq -c \cdot \mathbf{z}^{(p+q)/p} - c \cdot \|\mathbf{S}_a\|^{(p+q)/p} + \frac{3 \cdot (1 + \delta)}{2}. \quad (36)$$

Remark 2. It is noteworthy that the input update law $\dot{u}_i$ is necessary for the stability of the closed-loop system. Moreover, it can prevent u_i from exceeding the working

range. From (31), when $u_i = u_M \cdot \sqrt{1 - 2b}$ (design parameter b satisfies $2b \ll 1$), it means $\dot{u}_i = \Xi_{ui}^{-1}(\sigma_i - (u_i^2/(2u_M^2) + b) \cdot (1 + \sigma_i^2)) \leq 0$; therefore, this prevents u_i from getting greater than $u_M \cdot \sqrt{1 - 2b}$. When $u_i = -u_M \sqrt{1 - 2b}$, then $\dot{u}_i = \Xi_{ui}^{-1}(\sigma_i + (u_i^2/(2u_M^2) + b) \cdot (1 + \sigma_i^2)) \geq 0$, and it implies that this prevents u_i from getting smaller than $u_i = -\sqrt{1 - 2b} \cdot u_M$.

The following theorem established the criteria for selecting the controller parameters to warrant the bounded stability of the attitude tracking error e_i ($i = 1, 2, 3$).

Theorem 1. Consider system (5) with Assumptions 1–2, and if the controller parameters (a_1 , a_2 , c , b , and κ) are chosen as follows:

$$\begin{cases} 0 < a_1, \\ a_2 < 1, \\ c > 0.5, \\ \kappa \cdot c > (2a_1^2)^{-1}, \\ 2b \ll 1, \end{cases} \quad (37)$$

then the control structure composed of the virtual control law φ (in (23)) with parameter update (24) and input update law $\dot{u}_i$ (in (31)) can insure that all the signals of the designed closed-loop system are bounded and e_{li} ($i = 1, 2, 3$) converges to a small neighborhood of the equilibrium point in finite time.

Proof. Consider the Lyapunov candidate function as

$$V(t) = \frac{\mathbf{S}_a^T \mathbf{S}_a}{2} + \frac{\mathbf{z}^T \mathbf{z}}{2} + \frac{\tilde{\eta}^T \tilde{\eta}}{2\gamma}. \quad (38)$$

Taking the time derivative of $V(t)$ and applying (28) and (36), the following inequality can be obtained based on $\tilde{\eta} = \eta - \hat{\eta}$:

$$\begin{aligned} \dot{V}(t) &= \mathbf{S}_a^T \dot{\mathbf{S}}_a + \mathbf{z}^T \dot{\mathbf{z}} - \gamma^{-1} \tilde{\eta}^T \dot{\tilde{\eta}} \\ &\leq -c \cdot \mathbf{z}^{(p+q)/p} - c \cdot \|\mathbf{S}_a\|^{(p+q)/p} + a_2 \tilde{\eta}^T \tilde{\eta} + \frac{a_1^2}{2} + a_2^2 \\ &\quad + \frac{3}{2} (1 + \delta). \end{aligned} \quad (39)$$

By using Young's inequality, we can obtain that

$$\begin{aligned} a_2 \tilde{\eta}^T \tilde{\eta} &= a_2 \tilde{\eta}^T (\eta - \hat{\eta}) \\ &= -a_2 \tilde{\eta}^T \tilde{\eta} + a_2 \tilde{\eta}^T \eta \leq a_2 \left(-\tilde{\eta}^T \tilde{\eta} + \frac{1}{2} \tilde{\eta}^T \tilde{\eta} + \frac{1}{2} \eta^T \eta \right) \\ &= -\frac{a_2 \gamma}{2\gamma} \tilde{\eta}^T \tilde{\eta} + \frac{a_2 \eta^T \eta}{2} - \left(\frac{a_2 \gamma}{2\gamma} \tilde{\eta}^T \tilde{\eta} \right)^{(p+q)/2p} + \left(\frac{a_2 \gamma}{2\gamma} \tilde{\eta}^T \tilde{\eta} \right)^{(p+q)/2p}. \end{aligned} \quad (40)$$

For Theorem 1 proof, the two cases are required to be addressed in the analysis of (40).

① If $a_2 \gamma \cdot \tilde{\eta}^T \tilde{\eta} / (2\gamma) > 1$, the following inequality is true:

$$-\frac{a_2 \gamma \cdot \tilde{\eta}^T \tilde{\eta}}{2\gamma} + \frac{a_2 \gamma \cdot \tilde{\eta}^T \tilde{\eta}^{(p+q)/2p}}{2\gamma} \leq 0. \quad (41)$$

② If $a_2 \gamma \cdot \tilde{\eta}^T \tilde{\eta} / (2\gamma) \leq 1$, the following relationship can be established:

$$-\frac{a_2 \gamma \cdot \tilde{\eta}^T \tilde{\eta}}{2\gamma} + \frac{a_2 \gamma \cdot \tilde{\eta}^T \tilde{\eta}^{(p+q)/2p}}{2\gamma} \leq 1. \quad (42)$$

By the combination of (41) and (42), we have

$$a_2 \tilde{\eta}^T \tilde{\eta} \leq -\left(\frac{a_2 b \cdot \tilde{\eta}^T \tilde{\eta}}{2\gamma} \right)^{(p+q)/2p} + 0.5 \cdot a_2 \cdot \|\eta\|_2^2 + 1. \quad (43)$$

Using (43) results in

$$\begin{aligned} \dot{V}(t) &\leq -c \cdot 2^{(q+p)/2p} \cdot \left(\frac{\mathbf{z}^T \mathbf{z}}{2} \right)^{(q+p)/2p} \\ &\quad - c \cdot 2^{(q+p)/2p} \cdot \left(\frac{\mathbf{S}_a^T \mathbf{S}_a}{2} \right)^{(q+p)/2p} - (a_2 \gamma)^{(q+p)/2p} \\ &\quad \cdot \left(\frac{a_2 \gamma \cdot \tilde{\eta}^T \tilde{\eta}}{2\gamma} \right)^{(q+p)/2p} + \Delta \leq -\bar{\omega} \cdot V^{(q+p)/2p} + \Delta, \end{aligned} \quad (44)$$

where $\bar{\omega} = \min\{c \cdot 2^{(q+p)/2p}, (c \cdot \gamma)^{(q+p)/2p}\}$ and $\Delta = 0.5 \cdot a_1^2 + a_2^2 + 1.5 \cdot (1 + \delta) + 0.5 a_2^2 \cdot \|\eta\|_2^2 + 1$. Using Lemma 2, the decrease in $V(t)$ can force the sliding-mode variable of the closed-loop system into $V^{(q+p)/2p} \leq \Delta / [(1 - \beta^*) \cdot \bar{\omega}]$ with $0 < \beta^* < 1$, which implies that the terminal sliding-mode variable is bounded in finite time as

$$\lim_{t \rightarrow T_{f1}} \mathbf{S}_a(t) \in \left(\|\mathbf{S}_a\| \leq \left(\frac{\Delta}{(1 - \beta^*) \bar{\omega}} \cdot 2^{p/2(p+q)} \right)^{p/(p+q)} \right). \quad (45)$$

In the expression of (45), the finite time T_{f1} satisfies that $T_{f1} \leq ([V(x_0)]^{(p-q)/p} / [\bar{\omega} \beta^* \cdot ((p-q)/p)])$ in which $V(x_0)$ is the initial value of V . To describe the boundedness of attitude tracking errors, we need to reconsider each element of the NTSS shown in (15):

$$e_{1i} + c \cdot |e_{2i}|^\kappa \cdot \text{sign}(e_{2i}) = S_{ai}, \quad i = 1, 2, 3, \quad (46)$$

where $|S_{ai}| \leq ((\Delta / (1 - \beta^*) \bar{\omega}) \cdot 2^{p/2(p+q)})^{p/(p+q)} \triangleq \xi$. Then, (46) can be rearranged as

$$e_{1i} + \bar{c} \cdot |e_{2i}|^\kappa \cdot \text{sign}(e_{2i}) = 0, \quad (47)$$

where $\bar{c} \triangleq c - S_{ai} \cdot |e_{2i}|^{-\kappa} \cdot \text{sign}(e_{2i})$. From (45) and (47), we consider the following two cases:

① If $\bar{c} \leq 0$, then it has been in the region $c|e_{2i}|^\kappa \leq \xi$, and it is easy to obtain that $|e_{1i}|$ converges to the region $|e_{1i}| = |S_{ai} - c| \cdot |e_{2i}|^\kappa \cdot \text{sign}(e_{2i})| \leq 2\xi$ in a finite time T_{f1}

- ② If $\bar{c} > 0$, (47) can be regarded as the sliding mode, and according to [31], it can be shown that the time required for e_{1i} along the nonsingular terminal sliding surface to reach 0 is $T_{f2} \leq ((0.5 \cdot e_{1i}^2(T_{f1}))^{1-\bar{\kappa}} / [\alpha^* (1-\bar{\kappa})])$ with $\bar{\kappa} = 0.5(1 + \kappa^{-1})$ and $\alpha^* = 2^{\bar{\kappa}} \cdot (1/\bar{c})^{1/\kappa}$; therefore, the convergence time of the tracking error e_{1i} is $T_f = T_{f1} + T_{f2}$.

According to the above analysis, the stability of the designed closed-loop system is ensured through the sense of Lyapunov and the finite-time reachability is obtained, and the total convergence time of the closed-loop system is $T_f \leq T_{f1} + T_{f2}$. $\square$

Remark 3. Note that the convergence accuracy of the tracking error depends on the size of parameters $\bar{\omega}$ and Δ . The smaller the desired e_{1i} ($i = 1, 2, 3$), the larger the required parameters $1/a_1$, $1/a_2$, κ , c , and γ . In practical application, such control parameters can be determined by using the trial-and-error method under condition (37) until a good performance is obtained.

Remark 4. From the point of view of improving the robust capability of the closed-loop system, the information of lumped disturbance should be integrated into the control scheme design accordingly. In the design of the closed-loop system, the fuzzy logic system is employed to obtain the information of the nonlinear mixture term $\mathbf{D}$. And, $\|\Phi\|$ is used in the fuzzy logic system estimation, as can be observed from $\eta = [\|\Phi\|^2, \|\nu\|^2]^T$. So, the quantity of design update laws (24) can be completely independent from the amount of fuzzy logic rules.

Remark 5. In order to use the fuzzy approximation, the approximated function needs to be kept in a compact set during the operation process of the closed-loop system. For the proposed control strategy, such a condition is naturally fulfilled by the physical limitation of the quadrotor, and not only system states can be guaranteed to be bounded, but also all signals of the closed-loop system are confined in a compact set during the entire control process. The result is that the fuzzy logic system and fuzzy approximation can be reasonably used in the closed-loop system and perform its functions as approximation and learning for the lumped nonlinear function.

Remark 6. Compared with the input-saturation approximation approaches [32, 33], the proposed closed-loop control system utilizes the self-adjustment mechanism to prevent the design control command u_i from exceeding u_M , which is adaptively compensated for the whole control process.

4. Simulation Studies

4.1. Parameter Setting. In order to validate the theoretical result, a simulation study is performed in this section (based

on MATLAB/Simulink environment). The inertial matrix $\mathbf{J}$ and the Coriolis term $\mathbf{N}$ are given as follows:

$$\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}, \quad (48)$$

$$\mathbf{N} = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix},$$

with $J_{11} = I_1 \sin^2(\beta) + I_2 \cos^2(\beta) \sin^2(\alpha) + I_3 \cos^2(\beta) \cos^2(\alpha)$, $J_{12} = \cos(\beta) \cos(\alpha) \sin(\alpha) (I_2 - I_3)$, $J_{13} = -I_1 \sin(\beta)$, $J_{21} = \cos(\beta) \cos(\alpha) \sin(\alpha) (I_2 - I_3)$, $J_{22} = I_2 \cos^2(\alpha) + I_3 \sin^2(\alpha)$, $J_{23} = 0$, $J_{31} = -I_1 \sin(\beta)$, $J_{32} = 0$, $J_{33} = I_1$, $N_{11} = ((I_1 \beta \sin 2\beta + I_2 (-\beta \sin 2\beta \sin^2 \sigma + \dot{\sigma} \cos^2 \beta \sin 2\sigma) - I_3 (\beta \sin 2\beta \cos^2 \sigma + \dot{\sigma} \cos^2 \beta \sin 2\sigma))/2)$, $N_{12} = (I_1 \dot{\alpha} \sin 2\beta/2) - I_2 ((\beta \sin \beta \sin 2\sigma/2) - \dot{\sigma} \cos \beta \cos 2\sigma + (\dot{\alpha} \sin 2\beta \sin^2 \sigma/2)) + I_3 (-\dot{\alpha} (\sin 2\beta \cos^2 \sigma/2) - \dot{\sigma} \cos \beta \cos 2\sigma + (\beta \sin \beta \sin 2\sigma/2))$, $N_{13} = -I_1 \beta \cos \beta + (I_2 - I_1) (\dot{\alpha} \cos^2 \beta \sin 2\sigma/2)$, $N_{21} = (-I_1 \dot{\alpha} \sin 2\beta/2) + (I_2 \dot{\alpha} \sin 2\beta \sin^2 \sigma/2) + (I_3 \dot{\alpha} \sin 2\beta \cos^2 \sigma/2)$, $N_{22} = (\dot{\sigma} \sin 2\sigma (I_2 - I_3)/2)$, $N_{23} = I_1 \dot{\alpha} \cos \beta + I_2 (-\beta \sin 2\sigma/2) + \dot{\alpha} \cos \beta \cos 2\sigma + I_3 ((\beta \sin 2\sigma/2) - \dot{\alpha} \cos \beta \cos 2\sigma)$, $N_{31} = (\dot{\alpha} \cos^2 \beta (\sin 2\sigma) (I_3 - I_2)/2)$, $N_{32} = -I_1 \dot{\alpha} \cos \beta + I_2 ((\beta \sin 2\sigma/2) - \dot{\alpha} \cos \beta \cos 2\sigma)$, and $N_{33} = 0$, where $I_1 = I_2 = 4.2 \times 10^{-3}$ (N·m/rad/s²) and $I_3 = 8.5 \times 10^{-3}$ (N·m/rad/s²).

Meanwhile, the disturbance $\mathbf{d}$ is selected as

$$\mathbf{d} = \begin{cases} 0.5 \cdot \mathbf{d}^*, & \text{for } 0 \leq t < 6 \text{ s;} \\ 2.5 \cdot \mathbf{d}^*, & \text{for } 6 \text{ s} \leq t < 12 \text{ s;} \end{cases} \quad (49)$$

where

$$\mathbf{d}^* = 0.01 \cdot \begin{bmatrix} \sin\left(\frac{\pi t}{125}\right) + \sin\left(\frac{\pi t}{250}\right) \\ \sin\left(\frac{\pi t}{125}\right) + \cos\left(\frac{\pi t}{250}\right) \\ \cos\left(\frac{\pi t}{125}\right) + \sin\left(\frac{\pi t}{250}\right) \end{bmatrix}. \quad (50)$$

The initial values of the quadrotor system are $\alpha(0) = -0.2$ rad, $\beta(0) = 0.5$ rad, $\sigma(0) = 1.2$ rad, and $\dot{\Theta} = 0.01$ rad/s. The control authority is assumed to be $u_M = 0.4$, and the attitude angle commands are set to be

$$\Theta_d = \begin{bmatrix} \alpha_d \\ \beta_d \\ \sigma_d \end{bmatrix} = \begin{bmatrix} 0.8 \sin(t) \sin(0.4t) \text{ rad} \\ 0.6 \cos(0.5t) (1 - \cos(0.7t)) \text{ rad} \\ 0.6 \cos(0.7t) (1 - \cos(0.45t)) \text{ rad} \end{bmatrix}. \quad (51)$$

The parameters for the controller are chosen as follows: $p = 5$, $q = 3$, $a_1 = a_2 = 0.05$, $c = 2$, $b = 0.001$, and $\gamma = 5$. The differentiator parameters are selected as $b_1^* = b_2^* = 5$; moreover, the fuzzy membership functions are given as

FIGURE 1: Attitude tracking $\alpha - \alpha_d$. FIGURE 3: Attitude tracking $\sigma - \sigma_d$. FIGURE 2: Attitude tracking $\beta - \beta_d$.

FIGURE 4: Attitude tracking error.

$$\begin{aligned}
 \mu_{A_i^1}(x_i) &= \exp \left[- \left(\frac{x_i + (\pi/3)}{(\pi/18)} \right)^2 \right], \\
 \mu_{A_i^2}(x_i) &= \exp \left[- \left(\frac{x_i + (\pi/6)}{(\pi/18)} \right)^2 \right], \\
 \mu_{A_i^3}(x_i) &= \exp \left[- \left(\frac{x_i}{(\pi/18)} \right)^2 \right], \\
 \mu_{A_i^4}(x_i) &= \exp \left[- \left(\frac{x_i - (\pi/6)}{(\pi/18)} \right)^2 \right], \\
 \mu_{A_i^5}(x_i) &= \exp \left[- \left(\frac{x_i - (\pi/3)}{(\pi/18)} \right)^2 \right].
 \end{aligned} \tag{52}$$

FIGURE 5: Dynamics of the terminal sliding surface.

4.2. Result Analysis. The attitude tracking response result is exhibited in Figures 1–3. Obviously, we can see that the command signals can be tracked by our control strategy with consideration of external disturbances and control input constraints. As can be seen from Figures 4 and 5, the proposed method achieves the goal of attitude control with high accuracy and fast convergence speed for attitude

tracking error trajectories e_{1i} ($i = 1, 2, 3$). This is mainly because the proposed finite-time control scheme using the self-adaptation algorithm is capable of eliminating quickly the adverse influence caused by system uncertainty and external disturbances. During the increased disturbances ($t \geq 15$ s), the control performance based on the proposed

FIGURE 6: Control input.

control method is not obviously reduced because its self-adaption algorithm exerts accommodation process such that it can restore control performance as much as possible and further avoid the unstability of the closed-loop system with consideration of the increased external disturbances and control input constraints. The control input is as illustrated in Figure 6. Based on the above result and analysis, the effectiveness of the designed method can be validated.

5. Conclusion

In this study, a new model-free compensation control strategy using the finite-time technique is developed for attitude control of the quadrotor with adverse factors. The closed-loop system and the saturation suppression method are designed based on the NTSMC structure and the input parameter updating online strategy, respectively. A rigorous stability analysis and proof of the designed closed-loop system are combined with nonsingular terminal sliding-mode theory and Lyapunov theory. The numerical simulation on the attitude control of the quadrotor is performed to validate the developed method.

Data Availability

The raw/processed data required to reproduce these findings cannot be shared at this time as the data also form a part of an ongoing study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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