

Research Article

The Dynamics Behavior of Coupled Generalized van der Pol Oscillator with Distributed Order

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In this paper, we presented different behaviors such as chaotic and hyperchaotic of the generalized van der Pol oscillator with distributed order. We introduced the parameter intervals of these behaviors by computing the Lyapunov exponents of the oscillator, which is a good test for classifying the dynamical systems' solutions. The active control approach with the Laplace transform technique was used to realize the antisynchronization and control of the proposed oscillator. Finally, numerical investigations have been carried out on the dynamics of the proposed oscillator to verify the reliability of our analytical results.

1. Introduction

In 1920, van der Pol invented the van der Pol oscillator [1]. It describes the oscillation of a triode in an electrical circuit. It is a fundamental mathematical model, where it has many numerous applications and exciting features. This oscillator is used in designing many biological models such as heartbeats [2], designing physical models such as mobile and phone oscillators [3], and modeling of electrical systems [4]. Mathematically, there are many versions of the van der Pol oscillator like

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \quad (1)$$

$$\begin{cases} \ddot{x} + x^3 + a(x^2 - 1)\dot{x} = b(\sin wt + y), \\ \ddot{y} + y^3 + a(y^2 - 1)\dot{y} = b(\sin wt + x), \end{cases} \quad (2)$$

where (1) was introduced in 1927 by Van der Pol and Van Der Mark [4], and (2) was submitted in 1991 by Kapitaniak and Steeb [5]. In this paper, we will introduce the sinusoidal forcing with amplitude b and frequency w as a fractional version of the generalized van der Pol of the form

$$\ddot{x} + x^3 + \varepsilon(x^2 - 1)\dot{x} = b \sin(wt), \quad (3)$$

where μ , ε , a , b , and w are constants.

Fractional calculus plays an essential role in modern science. It is a different and distinct method for dealing with nonlinear systems along with the integer order. Fractional order models are adequate for the description of dynamical systems rather than integer order models. We can recognize, describe, and know dynamic phenomena such as chaos, hyperchaos, synchronization, and some other aspects of fractional order models faster and more accurately than those of the integer order of nonlinear systems. At present, the application of fractional calculus in most scientific fields has attracted much attention. So, the fractional calculus on the dynamical system was essential and exciting, which had been investigated recently by many researchers [6–10]. Here, in our paper, we used distributed order as a type of fractional calculus to study the dynamic behavior of nonlinear generalized van der Pol oscillator.

Distributed order calculus has been investigated for the first time as the extinction of fractional order calculus by Caputo [11]. Caputo et al. introduced useful properties for the distributed order calculus [12–15]. Fernández-Anaya

et al. proposed the Lyapunov theorem and several features for distributed order nonlinear dynamic systems [16]. The distributed dynamic systems have various applications in engineering and physics [17–20]. Some of the solutions' properties for the viscoelastic rod derivative of the distributed order have been studied in [17]. Tavazoei [18] provided numerous conclusions about the monotonicity of responses stage describing distributed order structures in irrational transfer functions. The distributed time of order for Schrödinger-form equation implemented using the local Galerkin discontinuous method [19]. In [20, 21], the writers were safely using complex distributed order structures. On the other hand, chaos and hyperchaotic solutions for distributed order dynamical systems are essential topics. Chen et al. [20] introduced a chaotic distributed order Lorenz system. Mahmoud et al. [22] presented the chaotic complex distributed order Lü and Chen systems.

Synchronization of chaos has a critical part to play in dynamic systems. It has various applications in different fields [20, 21, 23–25] such as biology, physics, stable communication, and engineering. There are many methods of control to achieve synchronization between chaotic and hyperchaotic systems, such as linear feedback control [26], nonlinear feedback control [27], active control [28, 29], back-stepping design [30], tracking control [31], and adaptive control [24, 32]. These methods also used to hold synchronization between chaotic and hyperchaotic distributed order systems. Chen et al. [20] used the active control approach to synchronize the two chaotic Lorenz distributed order structures. Based on linear feedback control, Mahmoud et al. [22] presented the synchronization between chaotic complex distributed order Chen and Lü systems. By applying the nonlinear feedback control and direct Lyapunov procedure, the synchronization among hyperchaotic complex distributed order van der Pol oscillators was investigated in [21].

System (3) can be constructed as a system of two differential equations of the first order when $x = u_1$, $\dot{x} = u_2$; then, we have

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= -u_1^3 + \varepsilon(1 - u_1^2)u_2 + b \sin(\omega t), \end{aligned} \quad (4)$$

system (4) can be written with distributed order and complex version as

$$\begin{aligned} D^{w(\alpha)} z_1 &= z_2, \\ D^{w(\alpha)} z_2 &= -z_1^3 + \varepsilon(1 - z_1^2)z_2 + b \sin(\omega t), \end{aligned} \quad (5)$$

where $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, $\varepsilon = \varepsilon_1 + i\varepsilon_2$.

The principal aims of this paper described as follows. (1) The hyperchaotic generalized van der Pol method has been introduced with complex parameter distributed order ($\varepsilon = \varepsilon_1 + i\varepsilon_2$) of the form (4). (2) The dynamics of the system are analyzed, and we also evaluate the parameter intervals (b , $\varepsilon_1\varepsilon_2$) when the solution of this system is chaotic and hyperchaotic (3). The solution of system (3) is transformed into a periodic solution using linear feedback control. (4) A scheme to achieve antisynchronization between two

generalized frameworks by van der Pol with distributed order hyperchaotic complex is stated. (5) The numerical simulations to test this theorem are presented.

The article is set out as follows. We are displaying some critical preliminaries in Section 2. Dynamics of the van der Pol generalized system of the distributed order complex and the parameter intervals at which chaotic and hyperchaotic solutions are calculated in Section 3. Using linear feedback control in Section 4, we manage its solution from chaotic and hyperchaotic to periodic. In Section 5, the antisynchronization between two identical systems of (3) is achieved through active control and transformation of the Laplace. The paper's conclusion is presented in Section 6.

2. Preliminaries

The following section includes some definitions of the fractional order and the distributed order derivatives [16, 22, 33, 34], with useful remark and theorem that will be used later.

Definition 1. For any $l \in \mathbb{N}^+$, ($l - 1 < \alpha < l$), the Caputo fractional derivative of function $x(t)$ is given by

$${}^C D^\alpha x(t) = \frac{1}{\Gamma(l - \alpha)} \int_0^t \frac{x^{(l)}(v)}{(t - v)^{\alpha - l + 1}} dv. \quad (6)$$

The Laplace transforms a fractional derivative of Caputo ${}^C D^\alpha x(t)$ by

$$\mathcal{L}\{{}^C D^\alpha x(t)\} = s^\alpha X(s) - \sum_{j=0}^{l-1} s^{\alpha-j-1} x^{(j)}(0). \quad (7)$$

Definition 2. The distributed derivative of a continuous function $x(t)$ is

$$D^{w(\alpha)} x(t) = \int_{l-1}^l w(\alpha) {}^C D^\alpha x(t) d\alpha \approx \sum_{j=1}^m w(\alpha_j) {}^C D^{\alpha_j} x(t) \Delta\tau_j, \quad (8)$$

where $\alpha \in [l - 1, l]$, $0 = \tau_0 < \tau_1 < \dots < \tau_m = 1$, $\Delta\tau_j = \tau_j - \tau_{j-1} = (1/m)$, $\alpha_j = (\tau_j + \tau_{j-1}/2) = (2j - 1/2m)$, $j = 1, 2, \dots, m$, $m \in \mathbb{N}$.

Remark 1. The Laplace transform of the distributed derivative is given by:

$$\begin{aligned} \mathcal{L}\{D^{w(\alpha)} x(t)\} &= \int_{l-1}^l w(\alpha) \left[s^\alpha X(s) - \sum_{j=0}^{l-1} s^{\alpha-j-1} x^{(j)}(0) \right] d\alpha \\ &= W(s)X(s) - \sum_{j=0}^{l-1} \frac{W(s)}{s^{j+1}} x^{(j)}(0^+), \end{aligned} \quad (9)$$

where $W(s) = \int_{l-1}^l w(\alpha) s^\alpha d\alpha$ and $\lim_{s \rightarrow 0} W(s) = 0$.

Theorem 1. Let $F(s) = \mathcal{L}\{f(t)\}$. If all poles of $sF(s)$ are in the open left-half complex plane, then $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$.

3. Dynamics of the Complex Generalized van der Pol Oscillator with Distributed Order

The dynamics of the generalized van der Pol oscillator distributed order with complex parameters are studying in this section. We test the intervals of the parameters where there are chaotic and hyperchaotic approaches to the system.

The real form of system (5) can be written as

$$\begin{aligned} D^{w(\alpha)} x_1 &= x_3, \\ D^{w(\alpha)} x_2 &= x_4, \\ D^{w(\alpha)} x_3 &= -x_1^3 + 3x_1x_2^2 + (\varepsilon_1x_3 - \varepsilon_2x_4)(1 - x_1^2 + x_2^2) \\ &\quad + 2(\varepsilon_1x_4 + \varepsilon_2x_3)x_1x_2 + b \sin(\omega t), \\ D^{w(\alpha)} x_4 &= -3x_1^2x_2 + x_2^3 - 2(\varepsilon_1x_3 - \varepsilon_2x_4)x_1x_2 \\ &\quad + (\varepsilon_1x_4 + \varepsilon_2x_3)(1 - x_1^2 + x_2^2). \end{aligned} \tag{10}$$

System (10) is seen as a generalization of several van der Pol oscillator variants (integer and fractional order). It has no fixed points because it is not an autonomous system. It is symmetric under the transformation $(x_1, x_2, x_3, x_4) \rightarrow (-x_1, -x_2, -x_3, -x_4)$.

Therefore, if (x_1, x_2, x_3, x_4) is a solution of (10), then $(-x_1, -x_2, -x_3, -x_4)$ is also a solution of the same system. System (10) is also dissipative in the case of $\varepsilon_1(x_1^2 - x_2^2 - 1) > 2\varepsilon_2x_1x_2$.

To show the solution's behavior and to obtain the intervals for chaotic and hyperchaotic phenomena of system (10), we calculated the Lyapunov exponents, which are an excellent test to classify the solutions.

3.1. Lyapunov Exponents of System (10). System (10) has four Lyapunov exponents λ_i , $i = 1, 2, 3, 4$, which have been measured by Wolf algorithm [35]. The classification of signs of Lyapunov exponents are stated in Table 1 [36]:

Assuming that $x(0) = (1.0826, -0.0149, -0.0990, 0.1544)^T$ is an initial condition and $w(\alpha) = \Gamma(1 - \alpha)$, for the following cases, we determined Lyapunov exponents of system (10):

(i) Case 1: Fix $(\varepsilon_1, \varepsilon_2, w) = (25, 2, 2.1)$ and vary b such that $b \in [25, 32]$ and calculate Lyapunov exponents. Some results are shown in Table 2 which plotted in Figure 1.

From Table 2, the solution of system (10) is hyperchaotic.

(ii) Case 2: Fix $(b, \varepsilon_2, w) = (30, 2, 2.1)$ and vary ε_1 such that $\varepsilon_1 \in [20, 30]$, we got the results of Lyapunov exponents shown in Table 2, plotted in Figure 2.

This implies our system (10) will have a hyperchaotic solution.

(iii) Case 3: Fix $(b, \varepsilon_1, w) = (30, 25, 2.1)$ and vary ε_2 such that $\varepsilon_2 \in [0, 10]$, we got Lyapunov exponents as shown in Table 2, plotted in Figure 3. Table 2 clearly shows that system (10) gives a chaotic and hyperchaotic solution of some intervals.

4. Supervision of Unusual Approaches of (10)

We applied the linear feedback control method to transform the system's chaotic and hyperchaotic solution (10) to periodic one. System (10) after the introduction of the control functions became

$$\begin{aligned} D^{w(\alpha)} x_1 &= x_3 - k_1x_1, \\ D^{w(\alpha)} x_2 &= x_4 - k_2x_2, \\ D^{w(\alpha)} x_3 &= -x_1^3 + 3x_1x_2^2 + (\varepsilon_1x_3 - \varepsilon_2x_4)(1 - x_1^2 + x_2^2) \\ &\quad + 2(\varepsilon_1x_4 + \varepsilon_2x_3)x_1x_2 + b \sin(\omega t) - k_3x_3, \\ D^{w(\alpha)} x_4 &= -3x_1^2x_2 + x_2^3 - 2(\varepsilon_1x_3 - \varepsilon_2x_4)x_1x_2 \\ &\quad + (\varepsilon_1x_4 + \varepsilon_2x_3)(1 - x_1^2 + x_2^2) - k_4x_4, \end{aligned} \tag{11}$$

where $K = \text{diag}(k_1, k_2, k_3, k_4) = \text{diag}(30, 40, 100, 100)$. The example $b = 30$, $\varepsilon_1 = 25$, $\varepsilon_2 = 2$, and $w = 2.1$ was chosen to make control. As shown in Figures 4 and 5, the hyperchaotic solution converted to periodic solution.

5. Chaos Synchronization of System (10)

In this section, using active control and Laplace transform, we present a theorem for achieving antisynchronization among the two same systems of (10). We consider (10) the master system, and the slave system takes the form

$$\begin{aligned} D^{w(\alpha)} y_1 &= y_3 + u_1, \\ D^{w(\alpha)} y_2 &= y_4 + u_2, \\ D^{w(\alpha)} y_3 &= -y_1^3 + 3y_1y_2^2 + (\varepsilon_1y_3 - \varepsilon_2y_4)(1 - y_1^2 + y_2^2) \\ &\quad + 2(\varepsilon_1y_4 + \varepsilon_2y_3)y_1y_2 + b \sin(\omega t) + u_3, \\ D^{w(\alpha)} y_4 &= -3y_1^2y_2 + y_2^3 - 2(\varepsilon_1y_3 - \varepsilon_2y_4)y_1y_2 \\ &\quad + (\varepsilon_1y_4 + \varepsilon_2y_3)(1 - y_1^2 + y_2^2) + u_4. \end{aligned} \tag{12}$$

where $u(t) = (u_1, u_2, u_3, u_4)^T \in \mathbb{R}^4$ is a suitable control function, which will be given in the last section.

Definition 3. The drive system (10) is said to achieve anti-synchronization with the response system (12), if

$$\lim_{t \rightarrow \infty} \|e\| = \lim_{t \rightarrow \infty} \|y(t) + x(t)\| = 0, \tag{13}$$

where $e = (e_1, e_2, \dots, e_n)^T$ is vector of synchronization error, and $\|\cdot\|$ is the matrix norm.

The system of error can be written as

TABLE 1: Signs from exponents of Lyapunov and the corresponding solution form.

λ_1	λ_2	λ_3	λ_4	Dynamics
-	-	-	-	Solution goes to equilibrium point
0	-	-	-	Periodic solution (limit cycles)
0	0	-	-	Quasiperiodic solution (2-torus)
0	0	0	-	Quasiperiodic solution (3-torus)
+	-	-	-	Chaotic behavior
+	+	-	-	Hyperchaotic behavior of order 2
+	+	+	-	Hyperchaotic behavior of order 3

TABLE 2: Lyapunov exponent of system (10) for three cases.

Case 1: Fix $(\epsilon_1, \epsilon_2, w) = (25, 2, 2.1)$ and vary b					
B	λ_1	λ_2	λ_3	λ_4	Dynamics
26	6.0041	3.1382	-6.1039	-6.7879	Hyperchaotic behavior of order 2
28	9.3759	3.7670	-4.5096	-6.2100	Hyperchaotic behavior of order 2
30	7.4983	4.1911	-5.3663	-7.3997	Hyperchaotic behavior of order 2
Case 2: Fix $(b, \epsilon_2, w) = (30, 2, 2.1)$ and vary ϵ_1					
ϵ_1	λ_1	λ_2	λ_3	λ_4	Dynamics
21	8.0825	4.8029	-10.4143	-11.0868	Hyperchaotic behavior of order 2
25	7.4983	4.1911	-5.3663	-7.3997	Hyperchaotic behavior of order 2
27	8.7146	4.6502	-4.9561	-6.7303	Hyperchaotic behavior of order 2
Case 3: Fix $(b, \epsilon_1, w) = (30, 25, 2.1)$ and vary ϵ_2					
ϵ_2	λ_1	λ_2	λ_3	λ_4	Dynamics
0.7	1.9130	-0.0303	-10.6506	-11.3279	Chaotic behavior
2	7.4983	4.1911	-5.3663	-7.3997	Hyperchaotic behavior of order 2
9	7.1457	3.4489	-6.1396	-7.6788	Hyperchaotic behavior of order 2

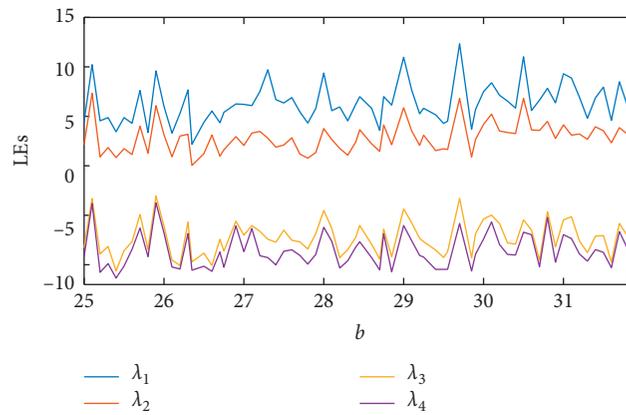


FIGURE 1: Lyapunov exponents of system (10) for $\epsilon_1 = 25, \epsilon_2 = 2, w = 2.1$ and $b \in [25, 32]$.

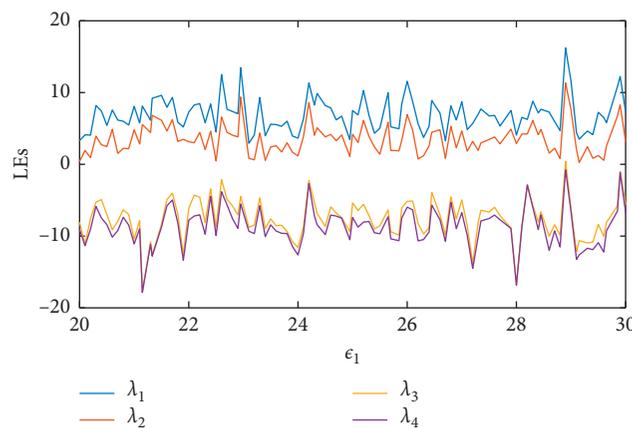


FIGURE 2: Lyapunov exponents of system (10) for $b = 30, \epsilon_2 = 2, w = 2.1$ and $\epsilon_1 \in [20, 30]$.

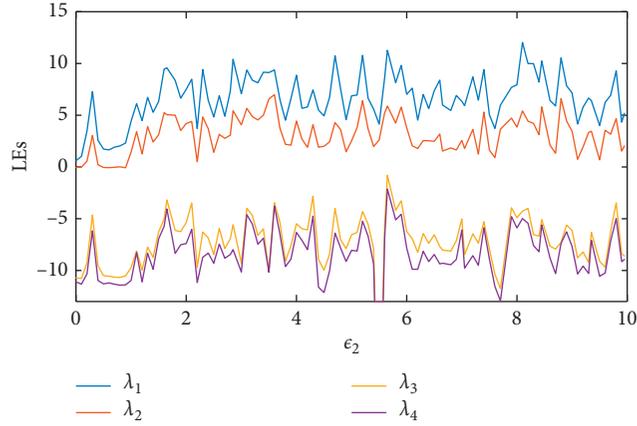


FIGURE 3: Lyapunov exponents of system (10) for $b = 30$, $\varepsilon_1 = 25$, $w = 2.1$ and $\varepsilon_2 \in [0, 10]$.

$$D^{w(\alpha)} e_1 = e_3 + u_1,$$

$$D^{w(\alpha)} e_2 = e_4 + u_2,$$

$$\begin{aligned} D^{w(\alpha)} e_3 = & \varepsilon_1 e_3 - \varepsilon_2 e_4 - x_1^3 + 3x_1 x_2^2 + (\varepsilon_1 x_3 - \varepsilon_2 x_4)(-x_1^2 + x_2^2) \\ & + 2(\varepsilon_1 x_4 + \varepsilon_2 x_3)x_1 x_2 - y_1^3 + 3y_1 y_2^2 \\ & + (\varepsilon_1 y_3 - \varepsilon_2 y_4)(-y_1^2 + y_2^2) + 2(\varepsilon_1 y_4 + \varepsilon_2 y_3)y_1 y_2 \\ & + 2b \sin(wt) + u_3, \end{aligned}$$

$$\begin{aligned} D^{w(\alpha)} e_4 = & \varepsilon_1 e_4 + \varepsilon_2 e_3 - 3x_1^2 x_2 + x_2^3 - 2(\varepsilon_1 x_3 - \varepsilon_2 x_4)x_1 x_2 \\ & + (\varepsilon_1 x_4 + \varepsilon_2 x_3)(-x_1^2 + x_2^2) - 3y_1^2 y_2 + y_2^3 \\ & - 2(\varepsilon_1 y_3 - \varepsilon_2 y_4)y_1 y_2 \\ & + (\varepsilon_1 y_4 + \varepsilon_2 y_3)(-y_1^2 + y_2^2) + u_4. \end{aligned}$$

(14)

Theorem 2. *The antisynchronization between the master system (10) and the slave system (12) will be achieved if the control functions chose as follows:*

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = & \begin{pmatrix} -k_1 e_1 \\ -k_2 e_2 \\ x_1^3 - 3x_1 x_2^2 - (\varepsilon_1 x_3 - \varepsilon_2 x_4)(-x_1^2 + x_2^2) - 2(\varepsilon_1 x_4 + \varepsilon_2 x_3)x_1 x_2 + y_1^3 \\ 3x_1^2 x_2 - x_2^3 + (\varepsilon_1 x_3 - \varepsilon_2 x_4)x_1 x_2 - (\varepsilon_1 x_4 + \varepsilon_2 x_3)(-x_1^2 + x_2^2) + 3y_1^2 y_2 \end{pmatrix} \\ & + \begin{pmatrix} 0 \\ 0 \\ -3y_1 y_2^2 - (\varepsilon_1 y_3 - \varepsilon_2 y_4)(-y_1^2 + y_2^2) - 2(\varepsilon_1 y_4 + \varepsilon_2 y_3)y_1 y_2 - 2b \sin(wt) - k_3 e_3 \\ -y_2^3 + 2(\varepsilon_1 y_3 - \varepsilon_2 y_4)y_1 y_2 - (\varepsilon_1 y_4 + \varepsilon_2 y_3)(-y_1^2 + y_2^2) - k_4 e_4 \end{pmatrix}, \end{aligned} \quad (15)$$

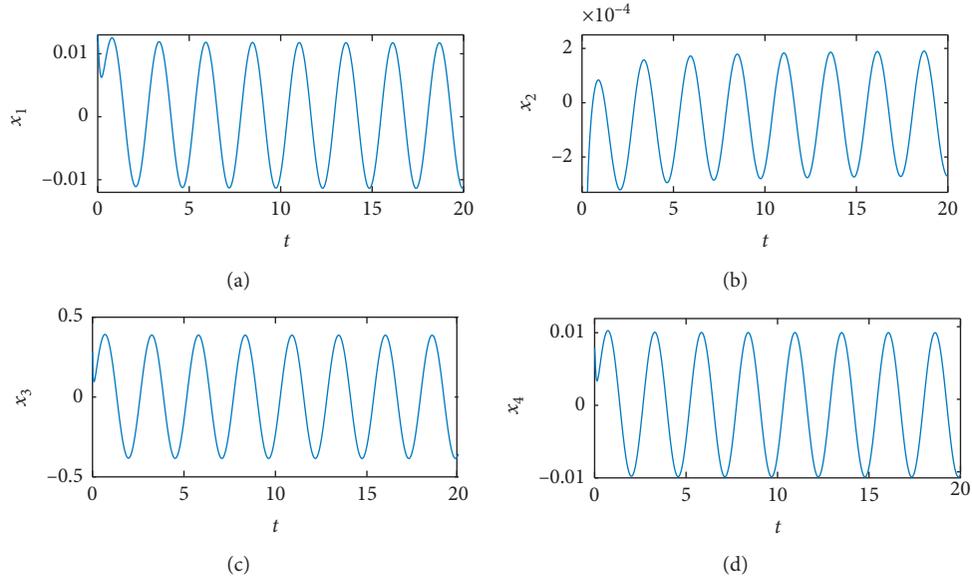


FIGURE 4: The distributed order hyperchaotic complex generalized van der Pol system with control (11): (a) (t, x_1) graph, (b) (t, x_2) graph, (c) (t, x_3) graph, and (d) (t, x_4) graph.

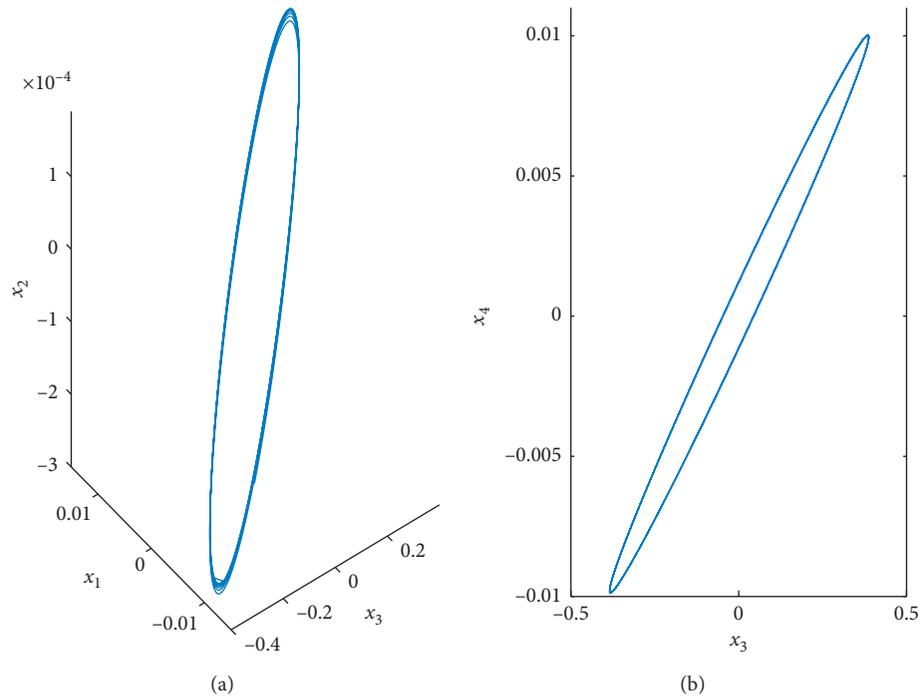


FIGURE 5: The hyperchaotic complex distributed by order generalized van der Pol system with control (11): (a) (x_3, x_1, x_2) space, and (b) (x_3, x_4) space.

where $K = \text{diag}(k_1, k_2, k_3, k_4)$ is the gain matrix.

Proof. Using the control functions (15), the error system (14) can be written as

$$\begin{aligned}
 D^{w(\alpha)}e_1 &= e_3 - k_1e_1, \\
 D^{w(\alpha)}e_2 &= e_4 - k_2e_2, \\
 D^{w(\alpha)}e_3 &= \varepsilon_1e_3 - \varepsilon_2e_4 - k_3e_3, \\
 D^{w(\alpha)}e_4 &= \varepsilon_1e_4 + \varepsilon_2e_3 - k_4e_4.
 \end{aligned} \tag{16}$$

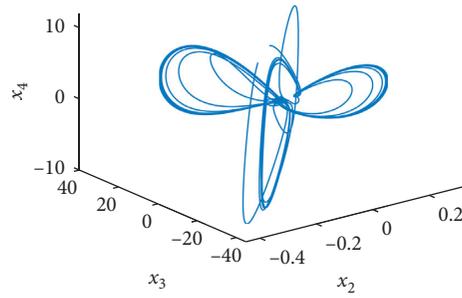


FIGURE 6: The hyperchaotic solution of system (10) in (x_2, x_3, x_4) space.

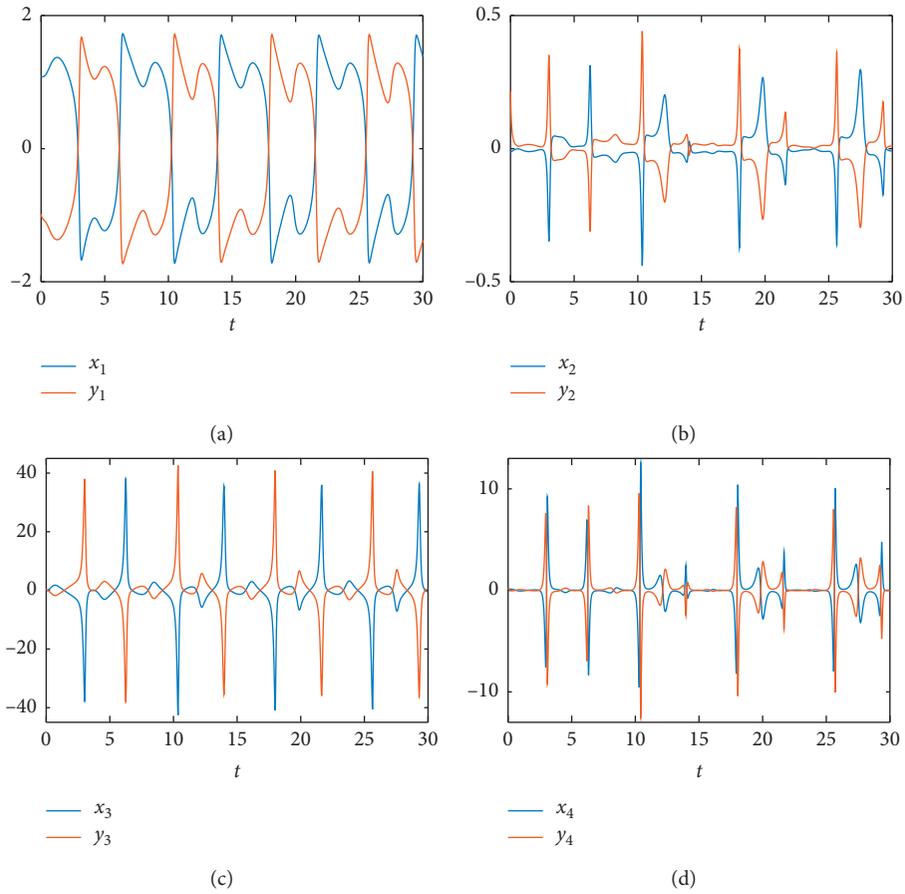


FIGURE 7: Synchronization of the state variables of the master system (10) and the slave system (12): (a) x_1 and y_1 versus t , (b) x_2 and y_2 versus t , and (c) x_3 and y_3 versus t , (d) x_4 and y_4 versus t .

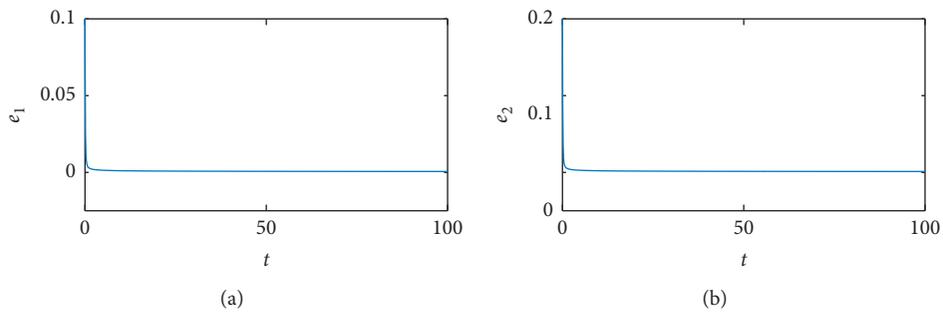


FIGURE 8: Continued.

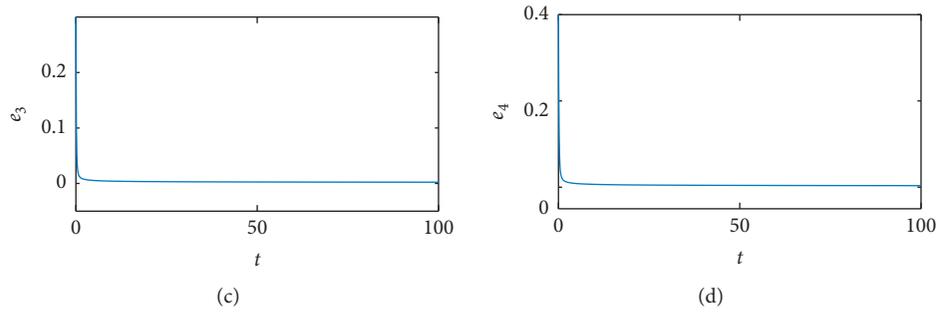


FIGURE 8: Synchronization error of the master system (10) and the slave system (12): (a) (t, e_1) graph, (b) (t, e_2) graph, (c) (t, e_3) graph, and (d) (t, e_4) graph.

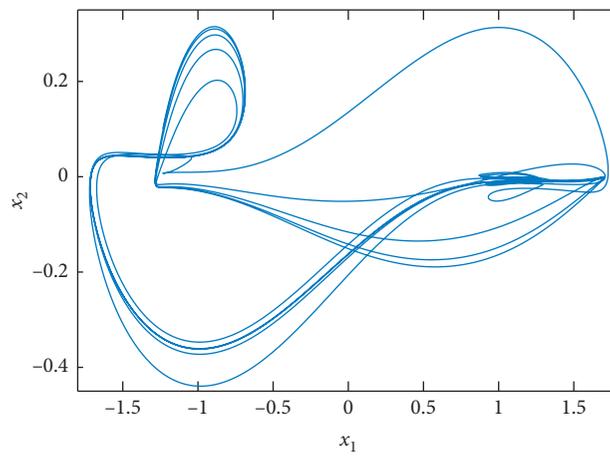


FIGURE 9: The hyperchaotic solution of system (10) in (x_1, x_2) space.

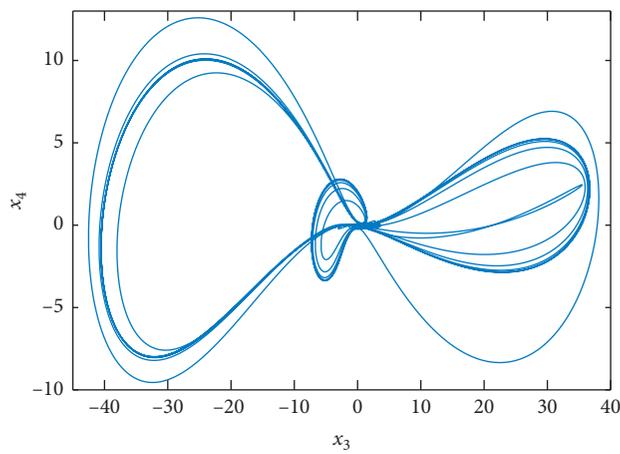


FIGURE 10: The hyperchaotic solution of system (10) in (x_3, x_4) space.

By transforming system (16) by Laplace and applying Remark 1 in $\mathcal{L}\{e_i(t)\} = E_i(s)$, $i = 1, 2, 3, 4$, then we get:

$$\begin{aligned} E_1(s) &= \frac{E_3(s)}{W(s) + k_1} + \frac{W(s)e_1(0)}{s(W(s) + k_1)}, \\ E_2(s) &= \frac{E_4(s)}{W(s) + k_2} + \frac{W(s)e_2(0)}{s(W(s) + k_2)}, \\ E_3(s) &= \frac{\varepsilon_2 E_4(s)}{W(s) - \varepsilon_1 + k_3} + \frac{W(s)e_3(0)}{s(W(s) - \varepsilon_1 + k_3)}, \\ E_4(s) &= \frac{\varepsilon_2 E_3(s)}{W(s) - \varepsilon_1 + k_4} + \frac{W(s)e_4(0)}{s(W(s) - \varepsilon_1 + k_4)}. \end{aligned} \quad (17)$$

By using Theorem 1 and Remark 1, we deduced that $\lim_{t \rightarrow \infty} e_i(t) = 0$, $i = 1, 2, 3, 4$. Anti synchronization between the master system (10) and the slave system (12) can therefore be accomplished.

Numerically, if we take $K = \text{diag}(40, 40, 50, 50)$, and the initial values of the master system (10) and the slave system (12) are $x_0 = (1.0826, -0.0149, -0.0990, 0.1544)^T$, $y_0 = (-0.94, 0.2, 0.3, 0.25)^T$ respectively, and the same parameters of Figure 6, the antisynchronization between master system (10) and slave system (12) achieved as shown in Figures 7 and 8. Figure 7 shows the antisynchronization between the state variables in the master and slave systems. The errors of the synchronization approach to zero, as shown in Figure 8. It clear that there exists an agreement between numerical simulations and Theorem 2. \square

6. Conclusions

In this work, we have investigated a new generalized van der Pol oscillator distributed order with a complex parameter (10). The literature is to be a generalization of several variants of van der Pol oscillator. We calculated Lyapunov exponents of that system for several parameter values, and we noticed that the system contains chaotic and hyperchaotic phenomena depicted in Figures 6, 9, and 10. We applied the linear feedback control method of system (10), and we could convert the chaotic and hyperchaotic solution to periodic one, as shown in Figures 4 and 5. We introduced Theorem 2 to fulfill the antisynchronization between two identical distributed order generalized van der Pol oscillators by using active control and Laplace transform method; the numerical simulations were consistent with the analytical study. The results are shown in Figures 7 and 8.

Data Availability

The authors declare that all data sources are original.

Conflicts of Interest

The authors declare no conflicts of interest.

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