Hindawi
Mathematical Problems in Engineering
Volume 2020, Article ID 5682587, 26 pages
https://doi.org/10.1155/2020/5682587

*Research Article*

# A Control Chart for Exponentially Distributed Characteristics Using Modified Multiple Dependent State Sampling

**Ahmad O. Albazli, Muhammad Aslam, and Saeed A. Dobbah**

*Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah 21551, Saudi Arabia*

Correspondence should be addressed to Muhammad Aslam; aslam_ravian@hotmail.com

Received 13 April 2020; Revised 12 November 2020; Accepted 5 December 2020; Published 28 December 2020

Academic Editor: Emilio Insfran Pelozo



In this paper, a *t*-control chart based on modified multiple dependent state sampling is proposed for monitoring processes that assume time between events following exponential distribution. The chart has double control limits and employs information from a previous sample and the current sample. The control chart coefficient "constants" are estimated by considering different values of the in-control average run lengths. The detection ability of the proposed control chart is found to be better than that of control charts based on multiple dependent state sampling in terms of average run lengths and the standard deviation of run lengths and better than generalized multiple dependent state sampling in terms of average run lengths. Case studies with real data are included as illustrative examples for the implementation of the proposed chart.

## 1. Introduction

In this era of business globalization, organizations tend to adjust their strategies and tendencies to allow them to succeed and remain ahead of rivals and to expand into new markets. Thus, ensuring the effectiveness and efficiency of the products or services they provide should boost customer satisfaction and build loyalty. In addition, implementing policies to monitor and control business processes and procedures, as a function of management, should help organizations augment their profits and remain economically sustainable [1]. Accordingly, monitoring and controlling processes are among the primary tasks that industries need to embark on [2]. Furthermore, the importance of the control function is not limited to the development and improvement of quality; rather, it also provides important information that informs decision-making processes and aids in the application of sufficient corrective plans [3]. Statistical process control (SPC) techniques and tools have, for decades, been applied to processes in a variety of fields to facilitate the creation of a decision-making road map [4]. In particular, employing control charts, which were first developed and introduced in 1924 by Dr. Walter A. Shewhart, is a common practice for monitoring quality and detecting assignable causes of process variation that may cause out-of-control conditions [5].

Shewhart's control charts are efficient at detecting large shifts in a process [6], but they are not sufficiently sensitive to detecting small process shifts. Accordingly, the probability of declaring a process out-of-control while it is actually in-control (type I error or false alarm) using Shewhart's control charts is high [7]. In addition, a Shewhart control chart based on a single sampling scheme assumes a large average run length (ARL) and involves a delay in indicating when a process will be out-of-control. Therefore, it is possible to enhance the performance of Shewhart's control charts by using different and well-structured sampling schemes and implementing supplementary run rules. For instance, Al-Marshadi et al. [8] proposed a control chart using a repetitive sampling scheme for a service performance index based on the proportion of customer complaints and proved that the proposed control chart is more efficient than single sampling. Abujiya and Lee [9] evaluated the performance of Shewhart's $\overline{X}$ chart using ranked set sampling (RSS) in comparison with traditional simple random sampling and concluded that the control chart based on RSS is more robust. The performance of a combined Shewhart exponential weighted moving average control chart based on double median RSS is analyzed by Abujiya et al. [10]. The results of the study show that the detection performance is enhanced when a sampling method on combined schemes is used. Tran [11, 12] showed that applying run rules control charts could notably improve the performance of Shewhart's control charts.

Usually, the assumption when designing any control chart for process monitoring is that real data values are at least approximately normally distributed [13]. However, in many real-life situations, this assumption is invalid and the process output is skewed. In this case, using a Shewhart control chart based on an assumption of normality can lead to fallacious results. Therefore, when the assumption of normality is not met in the case of skewed data, additional steps may be necessary to adjust the data so that this expectation is justified. For instance, Santiago and Smith [14] introduced a control chart to monitor the time interval between events, known as the time-between-events (TBE) chart or $t$-control chart, to be used when data follows the exponential distribution. They used the variable transformation provided by Nelson [15] to transform the data to be approximately normal.

The multiple dependent state (MDS) sampling scheme is proposed by Wortham and Baker [16]. In the inspection procedure of the MDS plan, decision criterion is formed to accept the sample drawn from a lot, reject it, or accept or reject the lot depending on the sample itself in combination with the states of the preceding lots. According to Balamurali and Jun [17], it is recommended that between two and five preceding subgroups be used. The number of these subgroups (say, $m$) is set in advance by the industrial engineers/quality control inspectors. The smaller values of $m$ are set for the strict inspection of the process.

Various studies were carried out on control charts under MDS. Based on this sampling plan, Aslam et al. [18, 19] proposed a control chart for the exponential distribution and gamma distribution. Additionally, Yan et al. [20] developed an MDS sampling scheme based on a coefficient of variation of normally distributed quality characteristics and showed the superiority of the proposed plan over the single sampling plan. The form of MDS is then generalized by Aslam et al. [21, 22] to generalize multiple dependent state sampling (GMDS) for the application of variable and attribute quality characteristics. It is proven that introducing an additional parameter to MDS to form GMDS increases the monitoring sensitivity significantly.

As pointed out earlier, MDS and GMDS are conditional sampling schemes that use information from the current and previous subgroups to make decisions about the state of the process. Similarly, the modified multiple dependent state (MMDS) sampling scheme uses information from the current and previous subgroups to make decisions about the state of the process in a more flexible manner, as illustrated in the next section. MMDS is developed to enhance the detection ability of process shifts. The difference between MMDS and MDS is that, in MDS, the process is declared in-control if all specified subgroups are in a controlled state, while, in MMDS, the process is declared in-control state even though one subgroup from the specified subgroups is in an in-decision area. Therefore, the MMDS is more flexible than the MDS in making decisions about the state of the process.

By exploring the literature and to the best of our knowledge, there is no work on designing $t$-charts using MMDS for skewed process output. In this paper, MMDS is used to develop a new $t$-chart to monitor skewed variables, assuming that the TBE follows an exponential distribution. In terms of ARL and the standard deviation of run length (SDRL), a performance comparison between the control charts using MMDS and MDS is conducted. Additionally, a performance comparison between the control charts using MMDS and GMDS in terms of ARL is conducted. It is expected that the control chart using MMDS would be the most efficient in terms of ARL and SDRL among the charts included in this study.

The rest of the paper is organized as follows: the design of the proposed control chart is presented in Section 2. Section 3 discusses the efficiency of the proposed control chart when the data are skewed and follow an exponential distribution. The performance of the proposed chart is evaluated compared to other existing $t$-charts in Section 4. Real examples from industry are presented in Section 5, and some concluding remarks are given in the last section.

## 2. The Proposed $t$-Chart Using the MMDS Design

Suppose that $T$ is a random variable with an exponential distribution that represents a quality characteristic of interest and has the following probability density function:

$$f(t) = \frac{1}{\theta} e^{-t/\theta}, \quad t > 0, \tag{1}$$

where $\theta$ is the scale and shape parameter of the exponential distribution, since it is equal to the mean and standard deviation. Let $\theta_0$ be the mean when the process is in-control.

According to Johnson and Kotz [23], if $T$ has an exponential distribution with a scale parameter $\theta$, the transformed random variable $T^* = T^{1/\beta}$ follows the Weibull distribution with shape parameter $\beta$ and scale parameter $\theta^{1/\beta}$. Nelson [15] provided transformation techniques to convert exponential data to Weibull distribution with a shape parameter based on this fact. Nelson [15] suggested that $\beta = 3.6$ makes an approximately normal curve; therefore, symmetric control limits can be implemented when proposing a control chart for nonconforming items. Thus, it may be more convenient to monitor the process using this transformed variable.

In this section, based on the statistic $T^*$, we present the following t-chart using the MMDS for individual measurements (sample size = 1) considering two pairs of control limits: a pair of outer control limits, $UCL_1$ and $LCL_1$, and a pair of inner control limits, $UCL_2$ and $LCL_2$.

(i) Step 1:

Measure the quality characteristic $T$ of a randomly selected sample of the production process. Then, calculate $T^*$:

$$T^* = T^{1/3.6}. \quad (2)$$

(ii) Step 2:

Declare the process to be in-control if $\mathrm{LCL}_2 \leq T^* \leq \mathrm{UCL}_2$.

Declare the process to be out-of-control if $T^* > \mathrm{UCL}_1$ or $T^* < \mathrm{LCL}_1$.

Otherwise, go to Step 3.

(iii) Step 3:

Declare the process to be in-control if there are at least $(m-1)$ out of the $m$ proceeding subgroups declared as in-control, and one $m$ may be in an indecision area (i.e., $\mathrm{UCL}_2 < T^* \leq \mathrm{UCL}_1$, or $\mathrm{LCL}_1 < T^* \leq \mathrm{LCL}_2$

When the process is in-control, we have the mean $E(T^*)$ and the second moment $E[(T^*)^2]$, for statistic $T^*$ as follows:

$$\mu_{T^*} = E(T^*) = \theta_0^* \Gamma\left(1 + \frac{1}{\beta}\right),$$
$$E\left[(T^*)^2\right] = [\theta_0^*]^2 \Gamma\left(1 + \frac{2}{\beta}\right), \quad (3)$$

where $\theta_0^* = \theta_0^{1/3.6}$ and $\Gamma(.)$ is a gamma function. Based on the above equations, we have the standard deviation of the statistic $T^*$ as follows:

$$\sigma_{T^*} = \theta_0^* \sqrt{\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma\left(1 + \frac{1}{\beta}\right)^2}. \quad (4)$$

If $T$ has an exponential distribution with the scale parameter $\theta$, then $T^*$ follows a normal distribution with $\theta_0^* = \theta_0^{1/3.6}$.

The outer control limits for the proposed chart are given by

$$\begin{aligned}
\mathrm{UCL}_1 &= \mu_{T^*} + k_1 \sigma_{T^*} \\
&= \theta_0^* \Gamma\left(1 + \frac{1}{\beta}\right) + k_1 \theta_0^* \sqrt{\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma\left(1 + \frac{1}{\beta}\right)^2} \\
&= \theta_0^* \left[\Gamma\left(1 + \frac{1}{\beta}\right) + k_1 \sqrt{\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma\left(1 + \frac{1}{\beta}\right)^2}\right] \\
&= \theta_0^* C_{U1}, \\
\mathrm{LCL}_1 &= \mu_{T^*} - k_1 \sigma_{T^*} \\
&= \theta_0^* \Gamma\left(1 + \frac{1}{\beta}\right) - k_1 \theta_0^* \sqrt{\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma\left(1 + \frac{1}{\beta}\right)^2} \\
&= \theta_0^* \left[\Gamma\left(1 + \frac{1}{\beta}\right) - k_1 \sqrt{\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma\left(1 + \frac{1}{\beta}\right)^2}\right] \\
&= \theta_0^* C_{L1}.
\end{aligned} \quad (5)$$

The inner control limits for the proposed chart are given by

$$
\begin{aligned}
\mathrm{UCL}_2 &= \mu_{T^*} + k_2\sigma_{T^*}, \\
&= \theta_0^*\Gamma\left(1+\frac{1}{\beta}\right) + k_2\theta_0^*\sqrt{\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)^2}, \qquad (6)\\
&= \theta_0^*\left[\Gamma\left(1+\frac{1}{\beta}\right) + k_2\sqrt{\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)^2}\,\right], \qquad (7)\\
&= \theta_0^* C_{U2}, \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{LCL}_2 &= \mu_{T^*} - k_2\sigma_{T^*} \\
&= \theta_0^*\Gamma\left(1+\frac{1}{\beta}\right) - k_2\theta_0^*\sqrt{\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)^2}, \qquad (9)\\
&= \theta_0^*\left[\Gamma\left(1+\frac{1}{\beta}\right) - k_2\sqrt{\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)^2}\,\right], \qquad (10)\\
&= \theta_0^* C_{L2}. \qquad (11)
\end{aligned}
$$

In the above, $k_1$ and $k_2$ $(k_1 > k_2)$ are control coefficients to be determined by considering the target in-control ARL, say $r_0$.

The probability of being declared in-control for the proposed control chart, say $P_{\mathrm{in}}^0$, when the process is actually in an in-control state $(\theta = \theta_0)$, is given as follows:

$$
\begin{aligned}
P_{\mathrm{in}}^0 &= P\left(\mathrm{LCL}_2 \le T^* \le \mathrm{UCL}_2\right) + \left\{P\left(\mathrm{UCL}_2 \le T^* \le \mathrm{UCL}_1\right) + P\left(\mathrm{LCL}_1 \le T^* \le \mathrm{LCL}_2\right)\right\} \\
&\quad \cdot \left[\begin{array}{c} \left\{P\left(\mathrm{LCL}_2 \le T^* \le \mathrm{UCL}_2\right)\right\}^m + m\left\{P\left(\mathrm{UCL}_2 \le T^* \le \mathrm{UCL}_1\right) + P\left(\mathrm{LCL}_1 \le T^* \le \mathrm{LCL}_2\right)\right\} \\ P\left(\mathrm{LCL}_2 \le T^* \le \mathrm{UCL}_2\right)^{m-1} \end{array}\right],
\end{aligned} \tag{12}
$$

where

$$
\begin{aligned}
P\left(\mathrm{LCL}_2 \le T^* \le \mathrm{UCL}_2\right) &= \exp\left\{-C_{L2}^{\beta}\right\} - \exp\left\{-C_{U2}^{\beta}\right\}, \\
P\left(\mathrm{UCL}_2 \le T^* \le \mathrm{UCL}_1\right) &= \exp\left\{-C_{U2}^{\beta}\right\} = \exp\left\{-C_{U1}^{\beta}\right\}, \\
P\left(\mathrm{LCL}_1 \le T* \le \mathrm{LCL}_2\right) &= \exp\left\{-C_{L1}^{\beta}\right\} - \exp\left\{-C_{L2}^{\beta}\right\}.
\end{aligned} \tag{13}
$$

In this case, (8) can be written as follows:

$$
\begin{aligned}
P_{\mathrm{in}}^0 &= \exp\left\{-C_{L2}^{\beta}\right\} - \exp\left\{-C_{U2}^{\beta}\right\} + \left\{\exp\left\{-C_{U2}^{\beta}\right\} - \exp\left\{-C_{U1}^{\beta}\right\} + \exp\left\{-C_{L1}^{\beta}\right\} - \exp\left\{-C_{L2}^{\beta}\right\}\right\} \\
&\quad \cdot \left[\left\{\exp\left\{-C_{L2}^{\beta}\right\} - \exp\left\{-C_{U2}^{\beta}\right\}\right\}^m + m\left\{\exp\left\{-C_{U2}^{\beta}\right\} - \exp\left\{-C_{U1}^{\beta}\right\} + \exp\left\{-C_{L1}^{\beta}\right\} - \exp\left\{-C_{L2}^{\beta}\right\}\right\}\left(\exp\left\{-C_{L2}^{\beta}\right\} - \exp\left\{-C_{U2}^{\beta}\right\}\right)^{m-1}\right].
\end{aligned} \tag{14}
$$

The in-control ARL and SDRL are defined as

$$\mathrm{ARL}_0 = \frac{1}{1 - P_{\mathrm{in}}^0},$$
$$\mathrm{SDRL}_0 = \sqrt{\frac{P_{\mathrm{in}}^0}{\left[1 - P_{\mathrm{in}}^0\right]^2}}. \tag{15}$$

Now, suppose that the process mean is shifted "equal to $\theta_1$." Then, the probability of declaring the process as out-of-control for the proposed control chart, say $P_{in}^1$, is given as follows:

$$\begin{aligned} P_{\mathrm{in}}^1 &= P\left(\mathrm{LCL}_2 \le T^* \le \mathrm{UCL}_2 \mid \theta = \theta_1\right) + \left\{P\left(\mathrm{UCL}_2 \le T^* \le \mathrm{UCL}_1 \mid \theta = \theta_1\right) + P\left(\mathrm{LCL}_1 \le T^* \le \mathrm{LCL}_2 \mid \theta = \theta_1\right)\right\} \\ &\cdot \left[\left\{P\left(\mathrm{LCL}_2 \le T^* \le \mathrm{UCL}_2 \mid \theta = \theta_1\right)\right\}^m + m\left\{P\left(\mathrm{UCL}_2 \le T^* \le \mathrm{UCL}_1 \mid \theta = \theta_1\right) + P\left(\mathrm{LCL}_1 \le T^* \right.\right.\right. \\ &\left.\left.\left.\le \mathrm{LCL}_2 \mid \theta = \theta_1\right)\right\} P\left(\mathrm{LCL}_2 \le T^* \le \mathrm{UCL}_2 \mid \theta = \theta_1\right)^{m-1}\right]. \end{aligned} \tag{16}$$

The ARL and SDRL for the shifted process are defined as

$$\mathrm{ARL}_1 = \frac{1}{1 - P_{\mathrm{in}}^1},$$
$$\mathrm{SDRL}_1 = \sqrt{\frac{P_{\mathrm{in}}^1}{\left[1 - P_{\mathrm{in}}^1\right]^2}}. \tag{17}$$

## 3. Performance Evaluation of the Proposed Chart

To measure the performance of any control chart, ARL is used as a sole measure or is combined with other measures [24]. The ARL is the number of in-control observations, on average, before a change in process level or an out-of-control observation is indicated [25].

Usually, along with the ARL, the SDRL is also computed. When the process behaves consistently over time, greater values of ARL and SDRL are expected, and the process is considered to be statistically in-control. By contrast, smaller values of ARL and SDRL are desired when the process shifts or is declared out-of-control.

Using the R programming language and software for statistical computation and simulation [26], the values of $k_1$ and $k_2$ and the values of the ARL and SDRL for the in-control and out-of-control processes are determined and presented in Tables 1–4 for $r_0$ = 200, 300, 370, and 500, respectively. It is noted that the values of the ARL and SDRL remain as targeted when the process is in-control (the ratio $\theta_0/\theta_1$ is equal to 1); however, the values of the out-of-control ARL and SDRL decrease as the shift in the process mean increases (the ratio $\theta_0/\theta_1$ decreases).

## 4. Performance Comparison

In this section, the efficiency of the proposed t-chart under MMDS sampling is evaluated compared to the $t$-chart under the MDS and GMDS sampling schemes. In many studies, a simulation approach is extensively employed to demonstrate the performance and effectiveness of control charts [27–31]. Using simulated data is convenient and is usually done when exact statistics of the proposed process are not available. Thus, this approach is performed in this research to compare the detection sensitivity of the aforementioned $t$-charts.

*4.1. Comparison of the Proposed Chart with the MDS Chart.* A simulation approach is adopted in this study to compare the proposed chart and the $t$-chart under MDS sampling (available here). First, an exponential distribution of 1000 observations is generated. Then, the limit coefficients under MMDS and MDS are computed. Choosing $r_0$ = 200, 300, 370, and 500 and $m$ = 2 to 6, the ARL and SDRL values of the $t$-chart under both schemes are compared for different shift ratios of $\theta_0/\theta_1$, as shown in Tables 5–8. From Figures 1–8, it is observed that the proposed chart detects process shifts more quickly; for example, when $r_0$ = 370 and $m$ = 6, it took the chart almost 221 samples to detect the out-of-control signal after the occurrence of the shift of "$\theta_0/\theta_1 = 0.9$" in the process mean, while, under the MDS scheme, it required 230 samples to detect the shift. Similarly, the SDRL of the proposed chart required 220.43 samples, compared to the 230.79 samples for the $t$-chart under the MDS scheme. In the same manner, when the shift in the process mean increases, the number of samples needed to detect the shift in the proposed chart becomes smaller than in the MDS sampling scheme. To illustrate, the control charts under the MMDS and MDS are constructed and shown in Figures 9 and 10. It is evident that observation 36 falls in the in-decision area with the $t$-chart under the MDS, whereas it is considered out-of-control under the MMDS.

*4.2. Comparison of the Proposed Chart with the GMDS Chart.* GMDS sampling is a generalized form of the MDS sampling scheme that is more efficient at making decisions about the state of the process. Like MDS, GMDS relies on information about current and previous observations. The performance evaluation of GMDS is more efficient than that of MDS. When compared with the GMDS results presented by Aslam et al. [21], the performance of the proposed chart when using simulated data is investigated in this section when $r_0$ = 200, 300, and 370, as shown in Tables 9–11. The decreasing pattern of ARL demonstrates that the performance of the

Table 1: The ARL and SDRL values of the proposed chart at $r_0 = 200$ with the $m$ value varying from 2 to 10.

| $m$ | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 | | 10 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k_1$ | 2.65298 | | 2.83972 | | 3.01942 | | 2.98755 | | 3.11139 | | 3.12916 | | 2.76820 | | 3.16861 | | 2.71363 | |
| $k_2$ | 1.80389 | | 1.64421 | | 1.68708 | | 1.76101 | | 1.79793 | | 1.83897 | | 1.97868 | | 1.90199 | | 2.09203 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 199.99 | 199.49 | 200.00 | 199.50 | 200.01 | 199.51 | 200.00 | 199.50 | 200.00 | 199.50 | 200.01 | 199.51 | 200.01 | 199.51 | 200.02 | 199.52 | 200.02 | 199.51 |
| 0.90 | 133.16 | 132.66 | 130.57 | 130.07 | 132.51 | 132.01 | 129.58 | 129.08 | 130.73 | 130.23 | 129.59 | 129.09 | 124.71 | 124.21 | 127.92 | 127.42 | 125.92 | 125.42 |
| 0.80 | 81.26 | 80.76 | 77.72 | 77.22 | 79.24 | 78.74 | 75.79 | 75.29 | 76.56 | 76.06 | 75.08 | 74.58 | 70.76 | 70.26 | 72.90 | 72.40 | 72.10 | 71.60 |
| 0.70 | 46.69 | 46.19 | 43.26 | 42.76 | 43.76 | 43.26 | 41.09 | 40.59 | 41.24 | 40.74 | 40.01 | 39.51 | 37.75 | 37.25 | 38.23 | 37.73 | 38.80 | 38.29 |
| 0.60 | 25.84 | 25.34 | 23.10 | 22.59 | 22.96 | 22.45 | 21.31 | 20.80 | 21.14 | 20.64 | 20.36 | 19.85 | 19.59 | 19.08 | 19.25 | 18.75 | 20.30 | 19.79 |
| 0.50 | 14.01 | 13.50 | 12.11 | 11.60 | 11.79 | 11.28 | 10.93 | 10.41 | 10.72 | 10.21 | 10.32 | 9.80 | 10.24 | 9.73 | 9.77 | 9.26 | 10.71 | 10.20 |
| 0.40 | 7.56 | 7.05 | 6.40 | 5.88 | 6.13 | 5.61 | 5.74 | 5.22 | 5.62 | 5.09 | 5.45 | 4.92 | 5.60 | 5.08 | 5.25 | 4.73 | 5.92 | 5.40 |
| 0.30 | 4.14 | 3.60 | 3.51 | 2.96 | 3.36 | 2.81 | 3.22 | 2.68 | 3.17 | 2.63 | 3.14 | 2.59 | 3.32 | 2.77 | 3.12 | 2.57 | 3.54 | 3.00 |
| 0.20 | 2.36 | 1.79 | 2.07 | 1.49 | 2.02 | 1.43 | 2.00 | 1.42 | 2.00 | 1.42 | 2.02 | 1.43 | 2.15 | 1.57 | 2.06 | 1.47 | 2.27 | 1.70 |
| 0.10 | 1.45 | 0.81 | 1.37 | 0.71 | 1.37 | 0.71 | 1.38 | 0.73 | 1.39 | 0.74 | 1.41 | 0.76 | 1.46 | 0.82 | 1.43 | 0.79 | 1.51 | 0.87 |

Table 2: The ARL and SDRL values of the proposed chart at $r_0 = 300$ with the $m$ value varying from 2 to 10.

| $m$ | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 | | 10 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k_1$ | 2.80952 | | 2.71631 | | 2.99340 | | 2.89292 | | 2.95491 | | 3.08240832 | | 2.95759 | | 2.86055 | | 2.94165 | |
| $k_2$ | 1.64840 | | 2.18966 | | 1.76962 | | 1.87672 | | 1.89838 | | 1.90957829 | | 1.97186 | | 2.04903 | | 2.03151 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 300.00 | 299.50 | 300.00 | 299.50 | 299.98 | 299.48 | 300.02 | 299.51 | 300.01 | 299.51 | 300.03 | 299.53 | 300.02 | 299.52 | 300.01 | 299.51 | 300.00 | 299.50 |
| 0.90 | 187.23 | 186.73 | 191.28 | 190.78 | 186.86 | 186.36 | 181.02 | 180.51 | 180.56 | 180.06 | 183.17 | 182.67 | 177.47 | 176.96 | 175.92 | 175.42 | 174.86 | 174.36 |
| 0.80 | 107.38 | 106.88 | 112.26 | 111.76 | 105.23 | 104.73 | 99.49 | 98.99 | 98.42 | 97.92 | 100.01 | 99.51 | 94.97 | 94.47 | 93.99 | 93.48 | 92.32 | 91.81 |
| 0.70 | 58.18 | 57.68 | 62.35 | 61.85 | 55.09 | 54.59 | 51.36 | 50.85 | 50.16 | 49.66 | 50.49 | 49.99 | 47.61 | 47.11 | 47.33 | 46.83 | 45.80 | 45.30 |
| 0.60 | 30.45 | 29.94 | 33.45 | 32.94 | 27.60 | 27.10 | 25.62 | 25.11 | 24.69 | 24.19 | 24.48 | 23.97 | 23.18 | 22.68 | 23.25 | 22.75 | 22.22 | 21.71 |
| 0.50 | 15.66 | 15.15 | 17.57 | 17.07 | 13.62 | 13.11 | 12.71 | 12.20 | 12.14 | 11.63 | 11.88 | 11.37 | 11.41 | 10.90 | 11.57 | 11.06 | 11.00 | 10.49 |
| 0.40 | 8.05 | 7.53 | 9.17 | 8.65 | 6.83 | 6.31 | 6.48 | 5.96 | 6.20 | 5.68 | 6.03 | 5.51 | 5.92 | 5.40 | 6.08 | 5.56 | 5.82 | 5.30 |
| 0.30 | 4.22 | 3.68 | 4.84 | 4.31 | 3.62 | 3.08 | 3.53 | 2.99 | 3.42 | 2.88 | 3.35 | 2.81 | 3.38 | 2.83 | 3.51 | 2.97 | 3.41 | 2.87 |
| 0.20 | 2.33 | 1.76 | 2.66 | 2.10 | 2.11 | 1.53 | 2.13 | 1.55 | 2.11 | 1.53 | 2.10 | 1.52 | 2.15 | 1.57 | 2.23 | 1.65 | 2.20 | 1.63 |
| 0.10 | 1.42 | 0.77 | 1.57 | 0.95 | 1.40 | 0.74 | 1.43 | 0.78 | 1.43 | 0.79 | 1.43 | 0.79 | 1.46 | 0.82 | 1.49 | 0.85 | 1.48 | 0.85 |

TABLE 3: The ARL and SDRL values of the proposed chart at $r_0 = 370$ with the $m$ value varying from 2 to 10.

| $m$ | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 | | 10 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k_1$ | 2.78334 | | 2.86733 | | 3.01155 | | 2.9984146 | | 3.06503 | | 3.10255 | | 3.13290 | | 2.87685 | | 2.86028 | |
| $k_2$ | 1.86932 | | 1.80537 | | 1.80629 | | 1.8756383 | | 1.90764 | | 1.94119 | | 1.97057 | | 2.09426 | | 2.13496 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 370.10 | 369.52 | 370.06 | 369.56 | 369.99 | 369.49 | 369.99 | 369.49 | 370.00 | 369.50 | 370.00 | 369.50 | 370.02 | 369.52 | 370.04 | 369.54 | 370.00 | 369.50 |
| 0.90 | 225.34 | 224.84 | 221.27 | 220.77 | 224.24 | 223.74 | 220.17 | 219.67 | 220.93 | 220.43 | 220.41 | 219.91 | 212.52 | 212.02 | 211.30 | 210.79 | 211.13 | 210.63 |
| 0.80 | 126.83 | 126.33 | 121.59 | 121.09 | 122.95 | 122.45 | 118.60 | 118.10 | 118.42 | 117.91 | 117.28 | 116.78 | 110.77 | 110.27 | 110.31 | 109.81 | 110.22 | 109.71 |
| 0.70 | 67.87 | 67.37 | 63.16 | 62.66 | 62.79 | 62.29 | 59.66 | 59.15 | 58.87 | 58.37 | 57.70 | 57.20 | 54.24 | 53.73 | 54.40 | 53.90 | 54.38 | 53.87 |
| 0.60 | 35.22 | 34.72 | 31.73 | 31.23 | 30.74 | 30.23 | 28.88 | 28.37 | 28.10 | 27.60 | 27.27 | 26.77 | 25.82 | 25.32 | 26.16 | 25.66 | 26.19 | 25.68 |
| 0.50 | 17.97 | 17.46 | 15.72 | 15.21 | 14.83 | 14.32 | 13.88 | 13.37 | 13.36 | 12.85 | 12.90 | 12.39 | 12.42 | 11.91 | 12.72 | 12.21 | 12.78 | 12.27 |
| 0.40 | 9.13 | 8.62 | 7.85 | 7.33 | 7.28 | 6.76 | 6.86 | 6.34 | 6.59 | 6.07 | 6.39 | 5.87 | 6.30 | 5.78 | 6.52 | 6.00 | 6.59 | 6.07 |
| 0.30 | 4.71 | 4.18 | 4.07 | 3.53 | 3.77 | 3.24 | 3.63 | 3.09 | 3.53 | 2.99 | 3.48 | 2.94 | 3.52 | 2.98 | 3.67 | 3.13 | 3.74 | 3.20 |
| 0.20 | 2.54 | 1.97 | 2.27 | 1.70 | 2.16 | 1.58 | 2.14 | 1.57 | 2.13 | 1.55 | 2.14 | 1.56 | 2.20 | 1.62 | 2.29 | 1.72 | 2.34 | 1.77 |
| 0.10 | 1.49 | 0.86 | 1.43 | 0.78 | 1.41 | 0.76 | 1.43 | 0.78 | 1.43 | 0.79 | 1.45 | 0.80 | 1.47 | 0.84 | 1.51 | 0.88 | 1.53 | 0.90 |

Table 4: The ARL and SDRL values of the proposed chart at $r_0 = 500$ with the $m$ value varying from 2 to 10.

| $m$ | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 | | 10 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k_1$ | 2.92776 | | 2.88286 | | 2.84840 | | 2.86525 | | 2.93012 | | 2.90614 | | 3.02547 | | 2.87998 | | 2.88962 | |
| $k_2$ | 1.69567 | | 1.90976 | | 2.09168 | | 2.09376 | | 2.03635 | | 2.10125 | | 2.04767 | | 2.20013 | | 2.20653 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 500.04 | 499.54 | 500.00 | 499.50 | 499.99 | 499.49 | 499.99 | 499.49 | 500.06 | 499.59 | 500.00 | 499.50 | 500.00 | 499.50 | 499.99 | 499.49 | 500.01 | 499.51 |
| 0.90 | 292.08 | 291.58 | 285.90 | 285.40 | 286.47 | 285.97 | 283.00 | 282.50 | 278.35 | 277.85 | 277.08 | 276.58 | 276.95 | 276.45 | 276.92 | 276.42 | 274.92 | 274.42 |
| 0.80 | 157.62 | 157.12 | 151.64 | 151.14 | 152.41 | 151.91 | 148.41 | 147.91 | 142.59 | 142.09 | 141.58 | 141.07 | 139.19 | 138.68 | 141.54 | 141.04 | 139.34 | 138.84 |
| 0.70 | 80.74 | 80.24 | 76.62 | 76.12 | 77.27 | 76.77 | 74.03 | 73.53 | 69.19 | 68.69 | 68.61 | 68.10 | 65.68 | 65.18 | 68.63 | 68.13 | 66.98 | 66.48 |
| 0.60 | 40.05 | 39.54 | 37.59 | 37.08 | 38.00 | 37.50 | 35.82 | 35.32 | 32.60 | 32.10 | 32.32 | 31.82 | 30.11 | 29.61 | 32.37 | 31.86 | 31.38 | 30.88 |
| 0.50 | 19.53 | 19.03 | 18.19 | 17.68 | 18.42 | 17.91 | 17.15 | 16.64 | 15.33 | 14.82 | 15.23 | 14.72 | 13.94 | 13.43 | 15.31 | 14.80 | 14.82 | 14.31 |
| 0.40 | 9.52 | 9.00 | 8.85 | 8.34 | 8.98 | 8.47 | 8.34 | 7.82 | 7.45 | 6.93 | 7.44 | 6.93 | 6.81 | 6.29 | 7.56 | 7.04 | 7.37 | 6.85 |
| 0.30 | 4.73 | 4.20 | 4.45 | 3.92 | 4.54 | 4.01 | 4.28 | 3.74 | 3.90 | 3.36 | 3.94 | 3.41 | 3.68 | 3.14 | 4.08 | 3.54 | 4.03 | 3.49 |
| 0.20 | 2.48 | 1.92 | 2.41 | 1.84 | 2.49 | 1.92 | 2.41 | 1.84 | 2.29 | 1.71 | 2.34 | 1.77 | 2.25 | 1.68 | 2.44 | 1.88 | 2.44 | 1.88 |
| 0.10 | 1.45 | 0.81 | 1.47 | 0.83 | 1.52 | 0.89 | 1.51 | 0.88 | 1.49 | 0.85 | 1.51 | 0.88 | 1.49 | 0.85 | 1.56 | 0.93 | 1.56 | 0.93 |

Table 5: The ARL and SDRL comparison of the proposed chart with the MDS chart at $r_0 = 200$ with the $m$ value varying from 2 to 6.

| | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | 2 | | | | 3 | | | | 4 | | | | 5 | | | | 6 | | | |
| $k_1$ | 2.65298 | | 3.05325 | | 2.83972 | | 2.93820 | | 3.01942 | | 3.17867 | | 2.98755 | | 2.81403 | | 3.11139 | | 2.64592 | |
| $k_2$ | 1.80389 | | 1.95328 | | 1.64421 | | 2.04096 | | 1.68708 | | 2.06290 | | 1.76101 | | 2.16003 | | 1.79793 | | 2.36430 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 199.99 | 199.49 | 199.99 | 199.49 | 200.00 | 199.50 | 200.00 | 199.50 | 200.01 | 199.51 | 200.00 | 199.50 | 200.00 | 199.50 | 200.01 | 199.51 | 200.00 | 199.50 | 200.00 | 199.50 |
| 0.90 | 133.16 | 132.66 | 139.12 | 138.62 | 130.57 | 130.07 | 134.54 | 134.04 | 132.51 | 132.01 | 137.09 | 136.58 | 129.58 | 129.08 | 130.97 | 130.47 | 130.73 | 130.23 | 134.26 | 133.76 |
| 0.80 | 81.26 | 80.76 | 87.56 | 87.06 | 77.72 | 77.22 | 82.12 | 81.61 | 79.24 | 78.74 | 84.39 | 83.89 | 75.79 | 75.29 | 78.16 | 77.65 | 76.56 | 76.06 | 82.58 | 82.08 |
| 0.70 | 46.69 | 46.19 | 50.92 | 50.42 | 43.26 | 42.76 | 46.68 | 46.17 | 43.76 | 43.26 | 47.79 | 47.29 | 41.09 | 40.59 | 43.79 | 43.29 | 41.24 | 40.74 | 47.84 | 47.34 |
| 0.60 | 25.84 | 25.34 | 28.06 | 27.56 | 23.10 | 22.59 | 25.41 | 24.90 | 22.96 | 22.45 | 25.68 | 25.17 | 21.31 | 20.80 | 23.73 | 23.22 | 21.14 | 20.64 | 26.75 | 26.25 |
| 0.50 | 14.01 | 13.50 | 15.03 | 14.52 | 12.11 | 11.60 | 13.59 | 13.08 | 11.79 | 11.28 | 13.51 | 13.00 | 10.93 | 10.41 | 12.76 | 12.25 | 10.72 | 10.21 | 14.73 | 14.22 |
| 0.40 | 7.56 | 7.05 | 8.00 | 7.49 | 6.40 | 5.88 | 7.32 | 6.80 | 6.13 | 5.61 | 7.18 | 6.66 | 5.74 | 5.22 | 6.99 | 6.47 | 5.62 | 5.09 | 8.15 | 7.63 |
| 0.30 | 4.14 | 3.60 | 4.34 | 3.80 | 3.51 | 2.96 | 4.06 | 3.52 | 3.36 | 2.81 | 3.97 | 3.44 | 3.22 | 2.68 | 3.99 | 3.45 | 3.17 | 2.63 | 4.61 | 4.08 |
| 0.20 | 2.36 | 1.79 | 2.45 | 1.89 | 2.07 | 1.49 | 2.38 | 1.81 | 2.02 | 1.43 | 2.35 | 1.78 | 2.00 | 1.42 | 2.41 | 1.85 | 2.00 | 1.42 | 2.71 | 2.15 |
| 0.10 | 1.45 | 0.81 | 1.50 | 0.86 | 1.37 | 0.71 | 1.50 | 0.87 | 1.37 | 0.71 | 1.50 | 0.87 | 1.38 | 0.73 | 1.54 | 0.91 | 1.39 | 0.74 | 1.64 | 1.02 |

Table 6: The ARL and SDRL comparison of the proposed chart with the MDS chart at $r_0 = 300$ with the $m$ value varying from 2 to 6.

| | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | 2 | | | | 3 | | | | 4 | | | | 5 | | | | 6 | | | |
| $k_1$ | 2.80952 | | 3.03098 | | 2.71631 | | 2.71297 | | 2.99340 | | 3.10876 | | 2.89292 | | 3.10105 | | 2.95491 | | 3.11713 | |
| $k_2$ | 1.64840 | | 2.04333 | | 2.18966 | | 2.45933 | | 1.76962 | | 2.14471 | | 1.87672 | | 2.17995 | | 1.89838 | | 2.20569 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 300.00 | 299.50 | 300.01 | 299.51 | 300.00 | 299.50 | 300.01 | 299.51 | 299.98 | 299.48 | 300.02 | 299.52 | 300.02 | 299.51 | 300.04 | 299.54 | 300.01 | 299.51 | 300.00 | 299.50 |
| 0.90 | 187.23 | 186.73 | 197.31 | 196.81 | 191.28 | 190.78 | 193.15 | 192.65 | 186.86 | 186.36 | 194.22 | 193.72 | 181.02 | 180.51 | 191.94 | 191.44 | 180.56 | 180.06 | 190.73 | 190.23 |
| 0.80 | 107.38 | 106.88 | 117.23 | 116.73 | 112.26 | 111.76 | 114.71 | 114.21 | 105.23 | 104.73 | 112.75 | 112.25 | 99.49 | 98.99 | 110.12 | 109.62 | 98.42 | 97.92 | 108.59 | 108.09 |
| 0.70 | 58.18 | 57.68 | 64.64 | 64.14 | 62.35 | 61.85 | 64.66 | 64.15 | 55.09 | 54.59 | 60.52 | 60.01 | 51.36 | 50.85 | 58.51 | 58.01 | 50.16 | 49.66 | 57.27 | 56.77 |
| 0.60 | 30.45 | 29.94 | 34.01 | 33.50 | 33.45 | 32.94 | 35.33 | 34.82 | 27.60 | 27.10 | 31.03 | 30.53 | 25.62 | 25.11 | 29.81 | 29.31 | 24.69 | 24.19 | 29.02 | 28.52 |
| 0.50 | 15.66 | 15.15 | 17.49 | 16.99 | 17.57 | 17.07 | 18.98 | 18.47 | 13.62 | 13.11 | 15.68 | 15.17 | 12.71 | 12.20 | 15.05 | 14.54 | 12.14 | 11.63 | 14.64 | 14.13 |
| 0.40 | 8.05 | 7.53 | 8.99 | 8.48 | 9.17 | 8.65 | 10.13 | 9.62 | 6.83 | 6.31 | 8.04 | 7.52 | 6.48 | 5.96 | 7.76 | 7.25 | 6.20 | 5.68 | 7.59 | 7.07 |
| 0.30 | 4.22 | 3.68 | 4.72 | 4.19 | 4.84 | 4.31 | 5.44 | 4.91 | 3.62 | 3.08 | 4.30 | 3.77 | 3.53 | 2.99 | 4.22 | 3.68 | 3.42 | 2.88 | 4.17 | 3.64 |
| 0.20 | 2.33 | 1.76 | 2.59 | 2.03 | 2.66 | 2.10 | 2.98 | 2.43 | 2.11 | 1.53 | 2.47 | 1.91 | 2.13 | 1.55 | 2.47 | 1.90 | 2.11 | 1.53 | 2.47 | 1.91 |
| 0.10 | 1.42 | 0.77 | 1.53 | 0.91 | 1.57 | 0.95 | 1.70 | 1.09 | 1.40 | 0.74 | 1.54 | 0.91 | 1.43 | 0.78 | 1.55 | 0.92 | 1.43 | 0.79 | 1.56 | 0.93 |

Table 7: The ARL and SDRL comparison of the proposed chart with the MDS chart at $r_0 = 370$ with the $m$ value varying from 2 to 6.

| | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | 2 | | | | 3 | | | | 4 | | | | 5 | | | | 6 | | | |
| $k_1$ | 2.78334 | | 2.91731 | | 2.86733 | | 2.98925 | | 3.01155 | | 3.06655 | | 2.99841 | | 3.19171 | | 3.06503 | | 3.18454 | |
| $k_2$ | 1.86932 | | 2.13330 | | 1.80537 | | 2.16569 | | 1.80629 | | 2.19134 | | 1.87564 | | 2.20979 | | 1.90764 | | 2.23733 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 370.10 | 369.52 | 370.05 | 369.55 | 370.06 | 369.56 | 370.04 | 369.54 | 369.99 | 369.49 | 370.05 | 369.55 | 369.99 | 369.49 | 370.01 | 369.51 | 370.00 | 369.50 | 370.01 | 369.51 |
| 0.90 | 225.34 | 224.84 | 230.10 | 229.60 | 221.27 | 220.77 | 230.11 | 229.61 | 224.24 | 223.74 | 230.93 | 23.43 | 220.17 | 219.67 | 233.81 | 233.31 | 220.93 | 220.43 | 231.29 | 230.79 |
| 0.80 | 126.83 | 126.33 | 130.81 | 130.31 | 121.59 | 121.09 | 129.83 | 129.32 | 122.95 | 122.45 | 129.73 | 129.23 | 118.60 | 118.10 | 131.60 | 131.10 | 118.42 | 117.91 | 128.81 | 128.31 |
| 0.70 | 67.87 | 67.37 | 70.07 | 69.57 | 63.16 | 62.66 | 68.58 | 68.08 | 62.79 | 62.29 | 67.79 | 67.29 | 59.66 | 59.15 | 68.24 | 67.74 | 58.87 | 58.37 | 66.22 | 65.72 |
| 0.60 | 35.22 | 34.72 | 36.26 | 35.76 | 31.73 | 31.23 | 34.91 | 34.40 | 30.74 | 30.23 | 34.04 | 33.53 | 28.88 | 28.37 | 33.82 | 33.32 | 28.10 | 27.60 | 32.66 | 32.16 |
| 0.50 | 17.97 | 17.46 | 18.50 | 17.99 | 15.72 | 15.21 | 17.54 | 17.03 | 14.83 | 14.32 | 16.90 | 16.39 | 13.88 | 13.37 | 16.59 | 16.08 | 13.36 | 12.85 | 16.02 | 15.51 |
| 0.40 | 9.13 | 8.62 | 9.46 | 8.94 | 7.85 | 7.33 | 8.89 | 8.38 | 7.28 | 6.76 | 8.53 | 8.01 | 6.86 | 6.34 | 8.32 | 7.80 | 6.59 | 6.07 | 8.09 | 7.57 |
| 0.30 | 4.71 | 4.18 | 4.93 | 4.41 | 4.07 | 3.53 | 4.66 | 4.13 | 3.77 | 3.24 | 4.49 | 3.96 | 3.63 | 3.09 | 4.40 | 3.87 | 3.53 | 2.99 | 4.34 | 3.81 |
| 0.20 | 2.54 | 1.97 | 2.69 | 2.13 | 2.27 | 1.70 | 2.59 | 2.03 | 2.16 | 1.58 | 2.54 | 1.98 | 2.14 | 1.57 | 2.52 | 1.96 | 2.13 | 1.55 | 2.53 | 1.97 |
| 0.10 | 1.49 | 0.86 | 1.57 | 0.95 | 1.43 | 0.78 | 1.56 | 0.93 | 1.41 | 0.76 | 1.56 | 0.93 | 1.43 | 0.78 | 1.56 | 0.94 | 1.43 | 0.79 | 1.57 | 0.95 |

Table 8: The ARL and SDRL comparison of the proposed chart with the MDS chart at $r_0 = 500$ with the $m$ value varying from 2 to 6.

| | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | 2 | | | | 3 | | | | 4 | | | | 5 | | | | 6 | | | |
| $k_1$ | 2.92776 | | 3.09800 | | 2.88286 | | 3.05617 | | 2.84840 | | 2.93402 | | 2.86525 | | 3.15044 | | 2.93012 | | 3.15310 | |
| $k_2$ | 1.69567 | | 2.13929 | | 1.90976 | | 2.21263 | | 2.09168 | | 2.31140 | | 2.09376 | | 2.26863 | | 2.03635 | | 2.29419 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 500.04 | 499.54 | 500.00 | 499.39 | 500.00 | 499.50 | 500.00 | 499.50 | 499.99 | 499.49 | 500.01 | 499.51 | 499.99 | 499.49 | 500.01 | 499.50 | 500.06 | 499.59 | 500.04 | 499.54 |
| 0.90 | 292.08 | 291.58 | 309.62 | 309.12 | 285.90 | 285.40 | 300.82 | 300.32 | 286.47 | 285.97 | 290.69 | 290.19 | 283.00 | 282.50 | 300.60 | 300.10 | 278.35 | 277.85 | 297.97 | 297.47 |
| 0.80 | 157.62 | 157.12 | 172.79 | 172.29 | 151.64 | 151.14 | 163.89 | 163.39 | 152.41 | 151.91 | 155.61 | 155.11 | 148.41 | 147.91 | 161.70 | 161.20 | 142.59 | 142.09 | 158.83 | 158.33 |
| 0.70 | 80.74 | 80.24 | 89.70 | 89.20 | 76.62 | 76.12 | 83.60 | 83.10 | 77.27 | 76.77 | 79.12 | 78.62 | 74.03 | 73.53 | 80.75 | 80.25 | 69.19 | 68.69 | 78.69 | 78.19 |
| 0.60 | 40.05 | 39.54 | 44.59 | 44.09 | 37.59 | 37.08 | 41.11 | 40.61 | 38.00 | 37.50 | 39.15 | 38.64 | 35.82 | 35.32 | 38.80 | 38.29 | 32.60 | 32.10 | 37.60 | 37.10 |
| 0.50 | 19.53 | 19.03 | 21.75 | 21.24 | 18.19 | 17.68 | 19.97 | 19.46 | 18.42 | 17.91 | 19.24 | 18.73 | 17.15 | 16.64 | 18.52 | 18.01 | 15.33 | 14.82 | 17.93 | 17.42 |
| 0.40 | 9.52 | 9.00 | 10.62 | 10.11 | 8.85 | 8.34 | 9.80 | 9.28 | 8.98 | 8.47 | 9.58 | 9.07 | 8.34 | 7.82 | 9.05 | 8.54 | 7.45 | 6.93 | 8.81 | 8.30 |
| 0.30 | 4.73 | 4.20 | 5.31 | 4.78 | 4.45 | 3.92 | 4.97 | 4.45 | 4.54 | 4.01 | 4.96 | 4.43 | 4.28 | 3.74 | 4.68 | 4.15 | 3.90 | 3.36 | 4.61 | 4.08 |
| 0.20 | 2.48 | 1.92 | 2.78 | 2.23 | 2.41 | 1.84 | 2.69 | 2.13 | 2.49 | 1.92 | 2.73 | 2.17 | 2.41 | 1.84 | 2.62 | 2.06 | 2.29 | 1.71 | 2.62 | 2.06 |
| 0.10 | 1.45 | 0.81 | 1.58 | 0.96 | 1.47 | 0.83 | 1.58 | 0.96 | 1.52 | 0.89 | 1.62 | 1.00 | 1.51 | 0.88 | 1.59 | 0.97 | 1.49 | 0.85 | 1.60 | 0.98 |

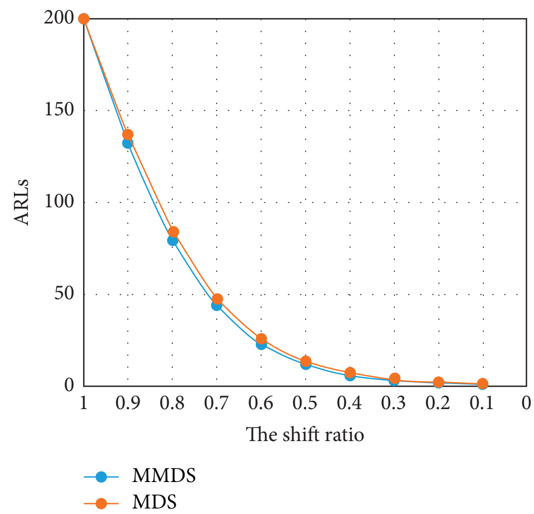


FIGURE 1: The ARL values for the MMDS and MDS at $m = 4$ and $r_0 = 200$.

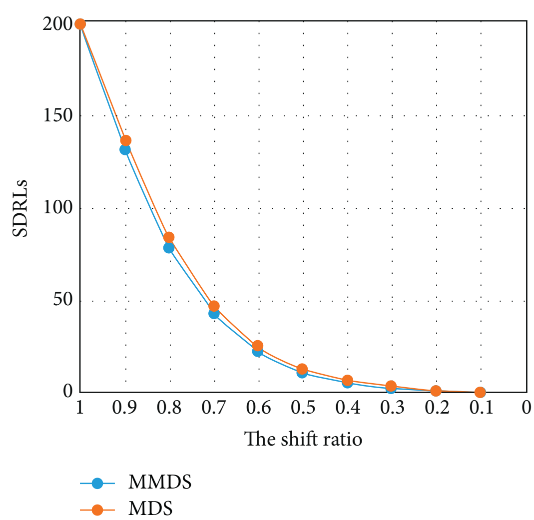


FIGURE 2: The SDRL values for the MMDS and MDS at $m = 4$ and $r_0 = 200$.

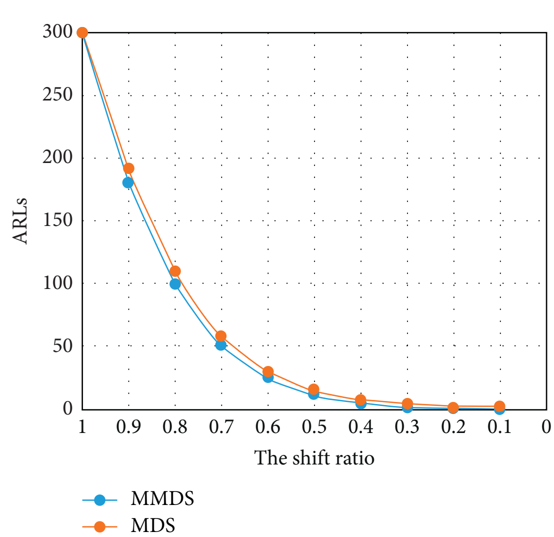


FIGURE 3: The ARL values for the MMDS and MDS at $m = 5$ and $r_0 = 300$.

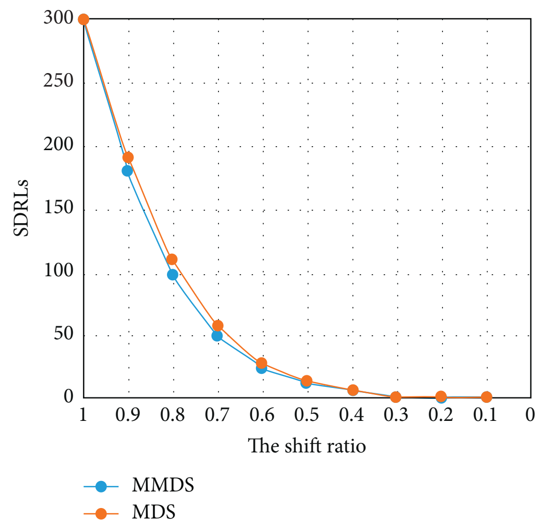


FIGURE 4: The SDRL values for the MMDS and MDS at $m = 5$ and $r_0 = 300$.

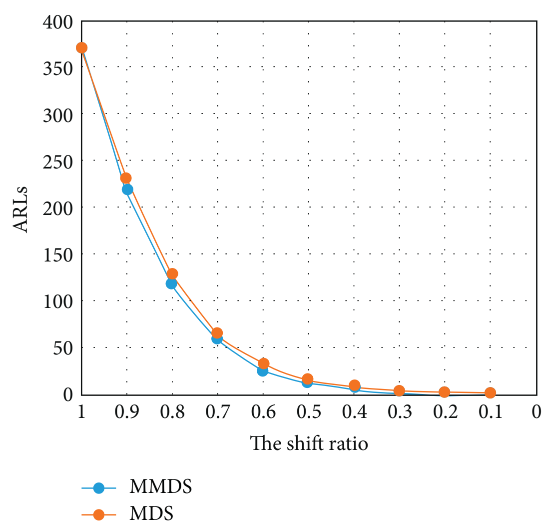


FIGURE 5: The ARL values for the MMDS and MDS at $m = 6$ and $r_0 = 370$.

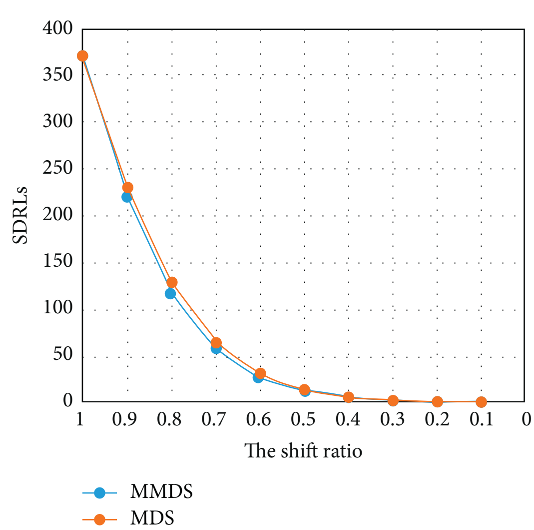


FIGURE 6: The SDRL values for the MMDS and MDS at $m = 6$ and $r_0 = 370$.

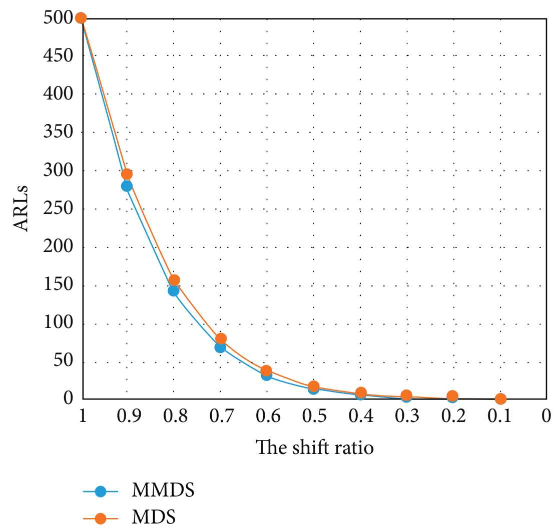


Figure 7: The ARL values for the MMDS and MDS at $m = 6$ and $r_0 = 500$.

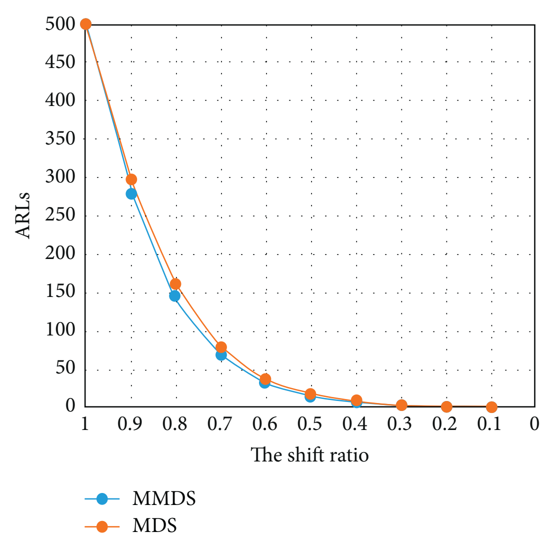


Figure 8: The SDRL values for the MMDS and MDS at $m = 6$ and $r_0 = 500$.

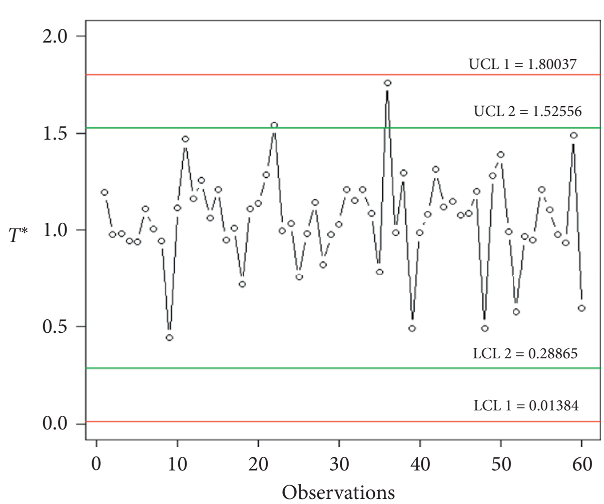


Figure 9: The MDS control chart for the simulated data at $m = 5$.

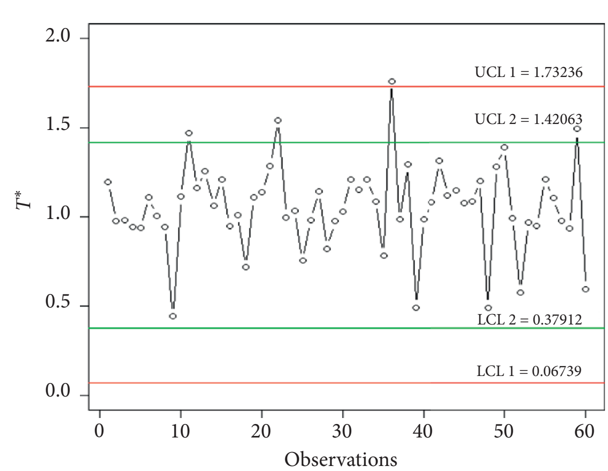


Figure 10: The MMDS control chart for the simulated data at $m = 5$.

Table 9: The ARL comparison of the proposed chart with GMDS at $r_0 = 200$.

| | MMDS | GMDS | MMDS | GMDS | MMDS | GMDS | MMDS | GMDS |
|---|---|---|---|---|---|---|---|---|
| $m$ | 4 | | 5 | | 6 | | 7 | |
| $K$ | — | 2 | — | 3 | — | 4 | — | 4 |
| $k_1$ | 3.01942 | 2.7187 | 2.98755 | 2.7197 | 3.11139 | 2.8679 | 3.12916 | 2.7106 |
| $k_2$ | 1.68708 | 1.4458 | 1.76101 | 1.5486 | 1.79793 | 1.5502 | 1.83897 | 1.4557 |
| $\theta_0/\theta_1$ | ARL | | ARL | | ARL | | ARL | |
| 1 | 200.01 | 199.99 | 200.00 | 199.99 | 200.00 | 200.00 | 200.01 | 200.00 |
| 0.90 | 132.51 | 133.58 | 129.58 | 131.89 | 130.73 | 130.12 | 129.59 | 130.62 |
| 0.80 | 79.24 | 81.86 | 75.79 | 79.60 | 76.56 | 77.04 | 75.08 | 77.93 |
| 0.70 | 43.76 | 47.12 | 41.09 | 45.01 | 41.24 | 42.45 | 40.01 | 43.45 |
| 0.60 | 22.96 | 25.99 | 21.31 | 24.34 | 21.14 | 22.29 | 20.36 | 23.09 |
| 0.50 | 11.79 | 13.96 | 10.93 | 12.83 | 10.72 | 11.45 | 10.32 | 11.92 |
| 0.40 | 6.13 | 7.41 | 5.74 | 6.72 | 5.62 | 5.93 | 5.45 | 6.13 |
| 0.30 | 3.36 | 3.97 | 3.22 | 3.60 | 3.17 | 3.22 | 3.14 | 3.27 |

Table 10: The ARL comparison of the proposed chart with GMDS at $r_0 = 300$.

| | MMDS | GMDS | MMDS | GMDS | MMDS | GMDS | MMDS | GMDS |
|---|---|---|---|---|---|---|---|---|
| $m$ | 4 | | 5 | | 6 | | 7 | |
| $K$ | — | 2 | — | 3 | — | 4 | — | 4 |
| $k_1$ | 2.74652 | 2.7682 | 2.89292 | 2.8651 | 2.95491 | 2.7608 | 3.08241 | 2.9145 |
| $k_2$ | 2.03462 | 1.5442 | 1.87672 | 1.5572 | 1.89838 | 1.7252 | 1.90958 | 1.4368 |
| $\theta_0/\theta_1$ | ARL | | ARL | | ARL | | ARL | |
| 1 | 300.01 | 300.00 | 300.02 | 300.00 | 300.01 | 300.00 | 300.03 | 299.99 |
| 0.90 | 186.19 | 190.63 | 181.02 | 188.21 | 180.56 | 187.83 | 183.17 | 185.92 |
| 0.80 | 106.03 | 111.50 | 99.49 | 107.99 | 98.42 | 107.76 | 100.01 | 105.00 |
| 0.70 | 57.00 | 61.59 | 51.36 | 58.05 | 50.16 | 58.07 | 50.49 | 55.27 |
| 0.60 | 29.58 | 32.74 | 25.62 | 29.85 | 24.69 | 29.94 | 24.48 | 27.71 |
| 0.50 | 15.11 | 16.98 | 12.71 | 14.96 | 12.14 | 15.00 | 11.88 | 13.52 |
| 0.40 | 7.76 | 8.69 | 6.48 | 7.48 | 6.20 | 7.46 | 6.03 | 6.61 |
| 0.30 | 4.14 | 4.47 | 3.53 | 3.84 | 3.42 | 3.83 | 3.35 | 3.39 |

Table 11: The ARL comparison of the proposed chart with GMDS at $r_0 = 370$.

| | MMDS | GMDS | MMDS | GMDS | MMDS | GMDS | MMDS | GMDS |
|---|---|---|---|---|---|---|---|---|
| $m$ | 4 | | 5 | | 6 | | 7 | |
| $K$ | — | 2 | — | 3 | — | 4 | — | 4 |
| $k_1$ | 3.01155 | 2.8074 | 2.99841 | 2.894 | 3.06503 | 2.8527 | 3.10255 | 2.907 |
| $k_2$ | 1.80629 | 1.5739 | 1.87564 | 1.5902 | 1.90764 | 1.6918 | 1.94119 | 1.478 |
| $\theta_0/\theta_1$ | ARL | | ARL | | ARL | | ARL | |
| 1 | 369.99 | 370.00 | 369.99 | 370.00 | 370.00 | 370.00 | 370.00 | 369.99 |
| 0.90 | 224.24 | 228.37 | 220.17 | 226.03 | 220.93 | 223.11 | 220.41 | 222.88 |
| 0.80 | 122.95 | 130.15 | 118.60 | 126.51 | 118.42 | 123.35 | 117.28 | 122.81 |
| 0.70 | 62.79 | 70.29 | 59.66 | 66.50 | 58.87 | 64.09 | 57.70 | 63.35 |
| 0.60 | 30.74 | 36.62 | 28.88 | 33.48 | 28.10 | 31.90 | 27.27 | 31.19 |
| 0.50 | 14.83 | 18.62 | 13.88 | 16.44 | 13.36 | 15.49 | 12.90 | 14.94 |
| 0.40 | 7.28 | 9.35 | 6.86 | 8.04 | 6.59 | 7.53 | 6.39 | 7.15 |
| 0.30 | 3.77 | 4.71 | 3.63 | 4.04 | 3.53 | 3.81 | 3.48 | 3.57 |

Table 12: Urinary tract infection (UTI) data from [15].

| Sample number | $T$ |
|---|---|
| 1 | 0.57014 |
| 2 | 0.07431 |
| 3 | 0.15278 |
| 4 | 0.14583 |
| 5 | 0.13889 |
| 6 | 0.14931 |
| 7 | 0.03333 |
| 8 | 0.08681 |
| 9 | 0.33681 |
| 10 | 0.03819 |
| 11 | 0.24653 |
| 12 | 0.29514 |
| 13 | 0.11944 |
| 14 | 0.05208 |
| 15 | 0.12500 |
| 16 | 0.25000 |
| 17 | 0.40069 |
| 18 | 0.02500 |
| 19 | 0.12014 |
| 20 | 0.11458 |
| 21 | 0.00347 |
| 22 | 0.12014 |
| 23 | 0.04861 |
| 24 | 0.02778 |
| 25 | 0.32639 |
| 26 | 0.64931 |
| 27 | 0.14931 |
| 28 | 0.01389 |
| 29 | 0.03819 |
| 30 | 0.46806 |
| 31 | 0.22222 |
| 32 | 0.29514 |
| 33 | 0.53472 |
| 34 | 0.15139 |
| 35 | 0.52569 |
| 36 | 0.07986 |
| 37 | 0.27083 |
| 38 | 0.04514 |
| 39 | 0.13542 |
| 40 | 0.08681 |
| 41 | 0.40347 |
| 42 | 0.12639 |
| 43 | 0.18403 |
| 44 | 0.70833 |
| 45 | 0.15625 |
| 46 | 0.24653 |
| 47 | 0.04514 |
| 48 | 0.01736 |
| 49 | 1.08889 |
| 50 | 0.05208 |
| 51 | 0.02778 |
| 52 | 0.03472 |
| 53 | 0.23611 |
| 54 | 0.35972 |

proposed chart exceeds that of the GMDS. For example, when $m = 5$ and $r_0 = 200$, it takes 129 samples to detect the out-of-control shift of "$\theta_0/\theta_1 = 0.9$" with the proposed t-chart, while it takes 131 samples with GMDS. Similarly, when $m = 5$ and $r_0 = 300$, the proposed chart needs seven fewer samples than GMDS to detect the shift. Likewise, when $m = 5$ and $r_0 = 370$, the proposed chart is found to be more efficient.

## 5. Applications

In this section, two sets of real data are employed to evaluate the performance of the proposed $t$-chart under MMDS sampling compared to the existing $t$-charts.

*5.1. Urinary Tract Infections.* The time difference between admission and discharge of male patients with urinary tract infections (UTIs) in a large hospital is discussed in Nelson's research [15]. The observations ($T$), shown in Table 12, are fitted to an exponential distribution with a mean time of 0.21 days. Here, the dataset is considered to compare the performance of the proposed chart under the MMDS scheme, in terms of the ARL and SDRL, with that of the existing $t$-chart when $r_0 = 370$, as shown in Tables 13–14 and plotted in Figures 11 and 12. For instance, when $m = 4$, the proposed chart takes 219 samples to detect the out-of-control signal after the shift of "$\theta_0/\theta_1 = 9$" occurs in the process mean, while it takes almost 226 samples to detect the shift with the t-chart under the MDS scheme. Similarly, the SDRL of the proposed chart takes 218.73 samples compared to the 255.91 samples for the $t$-chart under the MDS scheme. The control charts for the data are plotted in Figures 13 and 14. They show better detection ability for the proposed chart. Observation 21, for example, is considered to be in-control with the $t$-chart under MDS, when in fact it is located in the indecision area with the $t$-chart under the MMDS sampling scheme.

*5.2. Customer Complaint Resolution Time.* A Saudi company, one of the Middle East's largest utility companies by market value and among the top 15 in its field worldwide, provides communication channels for its customers to voice their complaints. In order to guarantee that the service provided by the company meets the highest international standards of quality and efficiency, the regulatory authority of Saudi Arabia requires it to successfully meet the standard for time to resolve complaints (TRC). Failure to do so leads to a payment to the customer once 10 working days have passed. Further compensation is paid for each additional 10 working days, whereby the required services are not completed. The time needed to resolve the complaints, or the time from submission to resolution, is found to follow an exponential distribution based on a goodness-of-fit test.

In this section, the proposed control chart is applied to monitor the time, in days, taken by the company to resolve 140 registered complaints in November 2019, as shown in Table 15. The control chart under MMDS is constructed and compared to the existing chart assuming a target ARL = 370 with the $m$ value varying from 2 to 9. From this, the control chart coefficients $k_1$ and $k_2$ are obtained. Tables 16 and 17 represent a comparison of the ARL and SDRL values of the $t$-charts under both schemes for different shift ratios ($\theta_0/\theta_1$).

Table 13: The ARL and SDRL comparison of the proposed chart with MDS for UTI data at $r_0 = 370$ with the $m$ value varying from 2 to 5.

| | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | 2 | | | | 3 | | | | 4 | | | | 5 | | | |
| $k_1$ | 2.85412 | | 3.09359 | | 2.77324 | | 3.18082 | | 2.83068 | | 2.95801 | | 3.04052 | | 3.03891 | |
| $k_2$ | 1.67334 | | 2.07679 | | 2.06525 | | 2.13187 | | 1.94589 | | 2.21982 | | 1.86324 | | 2.22984 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 370.01 | 369.51 | 370.01 | 369.51 | 369.99 | 369.49 | 369.99 | 369.49 | 369.99 | 369.49 | 369.99 | 369.49 | 370.02 | 369.52 | 370.02 | 369.52 |
| 0.90 | 224.50 | 224.00 | 239.62 | 239.12 | 225.51 | 225.01 | 239.59 | 239.09 | 219.23 | 218.73 | 226.41 | 225.91 | 222.49 | 221.99 | 227.47 | 226.97 |
| 0.80 | 125.48 | 124.98 | 139.58 | 139.07 | 126.88 | 126.38 | 138.50 | 138.00 | 119.50 | 119.00 | 126.06 | 125.56 | 120.51 | 120.01 | 126.18 | 125.68 |
| 0.70 | 66.42 | 65.92 | 75.23 | 74.73 | 67.75 | 67.25 | 73.62 | 73.12 | 61.66 | 61.15 | 65.97 | 65.47 | 60.66 | 60.15 | 65.36 | 64.86 |
| 0.60 | 34.01 | 33.50 | 38.63 | 38.13 | 35.02 | 34.51 | 37.17 | 36.67 | 30.80 | 30.30 | 33.39 | 32.89 | 29.28 | 28.77 | 32.67 | 32.17 |
| 0.50 | 17.12 | 16.61 | 19.38 | 18.88 | 17.77 | 17.26 | 18.37 | 17.86 | 15.22 | 14.71 | 16.77 | 16.26 | 14.00 | 13.49 | 16.24 | 15.73 |
| 0.40 | 8.61 | 8.10 | 9.72 | 9.20 | 9.00 | 8.48 | 9.13 | 8.62 | 7.62 | 7.10 | 8.55 | 8.03 | 6.88 | 6.36 | 8.25 | 7.74 |
| 0.30 | 4.42 | 3.89 | 4.98 | 4.45 | 4.65 | 4.12 | 4.70 | 4.17 | 4.00 | 3.46 | 4.54 | 4.01 | 3.62 | 3.08 | 4.42 | 3.88 |
| 0.20 | 2.39 | 1.82 | 2.67 | 2.11 | 2.54 | 1.98 | 2.57 | 2.01 | 2.28 | 1.71 | 2.57 | 2.01 | 2.14 | 1.56 | 2.54 | 1.98 |
| 0.10 | 1.43 | 0.79 | 1.55 | 0.93 | 1.52 | 0.89 | 1.54 | 0.92 | 1.46 | 0.82 | 1.57 | 0.95 | 1.42 | 0.77 | 1.57 | 0.95 |

Table 14: The ARL and SDRL comparison of the proposed chart with MDS for UTI data at $r_0 = 370$ with the $m$ value varying from 6 to 10.

| | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | 6 | | | | 7 | | | | 8 | | | | 9 | | | | 10 | | | |
| $k_1$ | 2.79727 | | 3.10230 | | 2.78925 | | 2.86690 | | 2.80485 | | 2.99383 | | 3.12831 | | 2.99195 | | 2.83832 | | 2.95881 | |
| $k_2$ | 2.13203 | | 2.24591 | | 2.19692 | | 2.34644 | | 2.17780 | | 2.30674 | | 2.00009 | | 2.32315 | | 2.16276 | | 2.34671 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 370.00 | 369.50 | 370.01 | 369.51 | 369.99 | 369.49 | 370.00 | 369.50 | 369.99 | 369.49 | 370.01 | 369.51 | 370.02 | 369.52 | 370.02 | 369.52 | 370.00 | 369.50 | 369.99 | 369.49 |
| 0.90 | 219.78 | 219.28 | 228.10 | 227.60 | 220.44 | 219.94 | 221.41 | 220.91 | 217.53 | 217.02 | 221.57 | 221.07 | 217.47 | 216.97 | 220.49 | 219.99 | 212.67 | 212.17 | 219.02 | 218.52 |
| 0.80 | 119.99 | 119.49 | 126.12 | 125.62 | 120.68 | 120.18 | 121.52 | 121.02 | 117.34 | 116.84 | 120.29 | 119.79 | 113.73 | 113.23 | 119.17 | 118.67 | 111.92 | 111.42 | 118.00 | 117.50 |
| 0.70 | 61.94 | 61.44 | 64.82 | 64.32 | 62.42 | 61.92 | 63.19 | 62.69 | 59.79 | 59.29 | 61.42 | 60.92 | 54.99 | 54.49 | 60.64 | 60.14 | 55.66 | 55.16 | 60.05 | 59.55 |
| 0.60 | 30.93 | 30.42 | 32.11 | 31.61 | 31.22 | 30.72 | 32.01 | 31.51 | 29.52 | 29.01 | 30.53 | 30.03 | 25.67 | 25.17 | 30.10 | 29.60 | 26.97 | 26.46 | 29.89 | 29.39 |
| 0.50 | 15.29 | 14.78 | 15.86 | 15.35 | 15.46 | 14.95 | 16.21 | 15.70 | 14.52 | 14.01 | 15.27 | 14.76 | 12.14 | 11.63 | 15.08 | 14.57 | 13.19 | 12.68 | 15.05 | 14.54 |
| 0.40 | 7.70 | 7.18 | 8.06 | 7.55 | 7.82 | 7.30 | 8.42 | 7.90 | 7.37 | 6.86 | 7.92 | 7.40 | 6.12 | 5.59 | 7.87 | 7.36 | 6.80 | 6.28 | 7.91 | 7.40 |
| 0.30 | 4.11 | 3.57 | 4.35 | 3.82 | 4.19 | 3.66 | 4.60 | 4.07 | 4.03 | 3.49 | 4.38 | 3.85 | 3.44 | 2.89 | 4.39 | 3.86 | 3.83 | 3.30 | 4.44 | 3.91 |
| 0.20 | 2.40 | 1.83 | 2.54 | 1.98 | 2.46 | 1.90 | 2.68 | 2.12 | 2.42 | 1.85 | 2.60 | 2.04 | 2.17 | 1.60 | 2.62 | 2.06 | 2.38 | 1.81 | 2.65 | 2.10 |
| 0.10 | 1.53 | 0.90 | 1.58 | 0.96 | 1.56 | 0.93 | 1.63 | 1.01 | 1.55 | 0.92 | 1.61 | 0.99 | 1.47 | 0.83 | 1.62 | 1.00 | 1.54 | 0.91 | 1.63 | 1.01 |

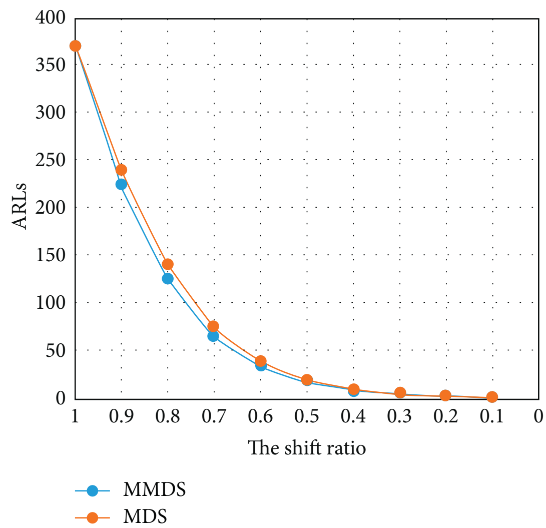


Figure 11: The ARL values for the MMDS and MDS for UTI data at $m = 2$ and $r_0 = 370$.

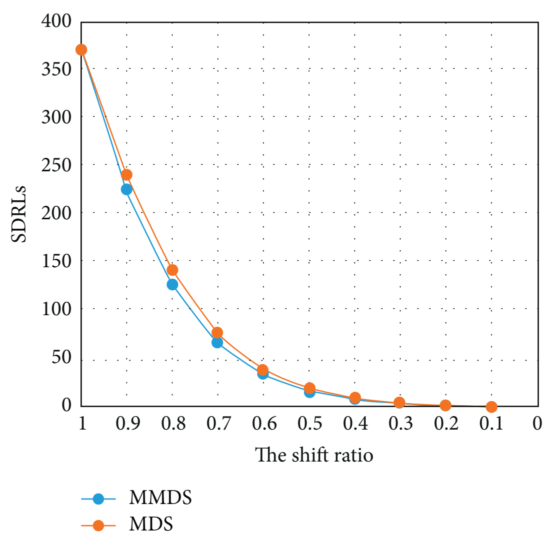


Figure 12: The SDRL values for the MMDS and MDS for UTI data at $m = 2$ and $r_0 = 370$.

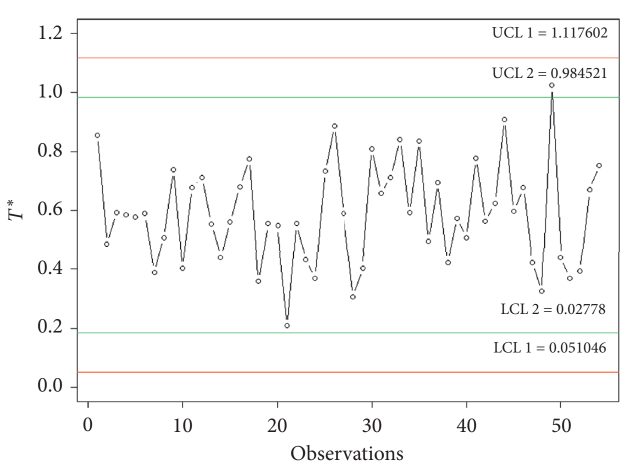


Figure 13: The MDS control chart for UTI data at $m = 4$.

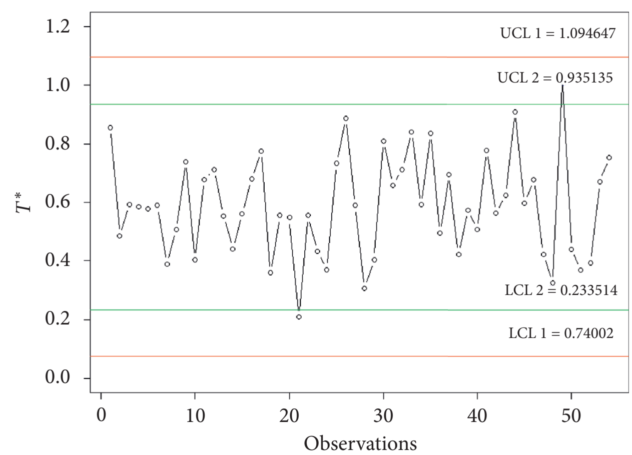


Figure 14: The MMDS control chart for UTI data at $m = 4$.

Table 15: Data for time to resolve complaints (TRC).

| Sample number | $T^*$ |
|---|---|
| 1 | 1.040 |
| 2 | 1.027 |
| 3 | 1.334 |
| 4 | 1.449 |
| 5 | 0.519 |
| 6 | 0.905 |
| 7 | 1.847 |
| 8 | 1.461 |
| 9 | 1.554 |
| 10 | 1.595 |
| 11 | 1.207 |
| 12 | 0.278 |
| 13 | 0.603 |
| 14 | 1.551 |
| 15 | 1.129 |
| 16 | 0.928 |
| 17 | 1.466 |
| 18 | 0.937 |

Table 15: Continued.

| Sample number | $T^*$ |
|---|---|
| 19 | 0.820 |
| 20 | 0.616 |
| 21 | 1.246 |
| 22 | 0.983 |
| 23 | 1.219 |
| 24 | 0.897 |
| 25 | 0.863 |
| 26 | 1.078 |
| 27 | 1.188 |
| 28 | 1.237 |
| 29 | 1.010 |
| 30 | 1.007 |
| 31 | 0.897 |
| 32 | 0.391 |
| 33 | 1.621 |
| 34 | 1.026 |
| 35 | 1.040 |
| 36 | 1.277 |
| 37 | 0.608 |
| 38 | 0.987 |
| 39 | 1.319 |
| 40 | 0.915 |
| 41 | 1.780 |
| 42 | 1.020 |
| 43 | 0.948 |
| 44 | 1.205 |
| 45 | 0.626 |
| 46 | 1.511 |
| 47 | 1.202 |
| 48 | 0.671 |
| 49 | 0.452 |
| 50 | 1.083 |
| 51 | 0.911 |
| 52 | 1.165 |
| 53 | 1.239 |
| 54 | 0.718 |
| 55 | 1.215 |
| 56 | 0.664 |
| 57 | 1.267 |
| 58 | 0.960 |
| 59 | 0.712 |
| 60 | 0.955 |
| 61 | 0.877 |
| 62 | 1.075 |
| 63 | 0.653 |
| 64 | 1.313 |
| 65 | 1.051 |
| 66 | 0.607 |
| 67 | 0.667 |
| 68 | 1.284 |
| 69 | 0.876 |
| 70 | 0.751 |
| 71 | 1.505 |
| 72 | 1.026 |
| 73 | 0.996 |
| 74 | 0.507 |
| 75 | 1.301 |
| 76 | 1.419 |
| 77 | 1.070 |
| 78 | 1.228 |
| 79 | 1.052 |

Table 15: Continued.

| Sample number | $T^*$ |
|---|---|
| 80 | 0.928 |
| 81 | 1.754 |
| 82 | 0.558 |
| 83 | 1.240 |
| 84 | 1.026 |
| 85 | 1.206 |
| 86 | 0.976 |
| 87 | 0.513 |
| 88 | 1.454 |
| 89 | 0.166 |
| 90 | 0.855 |
| 91 | 1.032 |
| 92 | 0.620 |
| 93 | 0.819 |
| 94 | 1.210 |
| 95 | 0.440 |
| 96 | 0.270 |
| 97 | 1.218 |
| 98 | 1.135 |
| 99 | 1.070 |
| 100 | 1.458 |
| 101 | 0.485 |
| 102 | 0.857 |
| 103 | 0.596 |
| 104 | 0.885 |
| 105 | 0.239 |
| 106 | 1.326 |
| 107 | 0.797 |
| 108 | 1.152 |
| 109 | 0.995 |
| 110 | 0.972 |
| 111 | 0.240 |
| 112 | 0.282 |
| 113 | 1.002 |
| 114 | 0.947 |
| 115 | 1.248 |
| 116 | 0.785 |
| 117 | 1.330 |
| 118 | 0.833 |
| 119 | 0.531 |
| 120 | 0.937 |
| 121 | 0.921 |
| 122 | 0.523 |
| 123 | 0.885 |
| 124 | 1.048 |
| 125 | 0.282 |
| 126 | 0.841 |
| 127 | 0.614 |
| 128 | 0.520 |
| 129 | 0.974 |
| 130 | 0.396 |
| 131 | 1.560 |
| 132 | 1.470 |
| 133 | 0.780 |
| 134 | 1.264 |
| 135 | 1.233 |
| 136 | 0.996 |
| 137 | 0.928 |
| 138 | 0.648 |
| 139 | 0.251 |
| 140 | 1.386 |

Table 16: The ARL and SDRL comparison of the proposed chart with MDS for TRC data at $r_0 = 370$ with $m$ value varying from 2 to 5.

| | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | 2 | | | | 3 | | | | 4 | | | | 5 | | | |
| $k_1$ | 2.88690 | | 3.10355 | | 2.78183 | | 2.89276 | | 3.17406 | | 3.18913 | | 3.00510 | | 3.11057 | |
| $k_2$ | 1.63317 | | 2.07520 | | 2.01076 | | 2.21062 | | 1.77478 | | 2.17615 | | 1.87341 | | 2.21810 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 370.00 | 369.50 | 370.00 | 369.50 | 370.00 | 369.50 | 370.01 | 369.51 | 370.01 | 369.51 | 370.00 | 369.50 | 369.99 | 369.49 | 369.99 | 369.49 |
| 0.90 | 226.02 | 225.52 | 240.19 | 239.69 | 224.25 | 223.75 | 226.60 | 226.10 | 234.95 | 234.45 | 236.45 | 235.94 | 220.52 | 220.02 | 230.45 | 229.95 |
| 0.80 | 126.81 | 126.31 | 140.14 | 139.64 | 125.42 | 124.92 | 127.11 | 126.61 | 132.87 | 132.37 | 134.69 | 134.19 | 118.88 | 118.38 | 128.66 | 128.16 |
| 0.70 | 67.16 | 66.65 | 75.59 | 75.09 | 66.56 | 66.06 | 67.36 | 66.85 | 68.68 | 68.18 | 70.60 | 70.10 | 59.80 | 59.30 | 66.63 | 66.13 |
| 0.60 | 34.31 | 33.81 | 38.81 | 38.31 | 34.19 | 33.69 | 34.58 | 34.08 | 33.48 | 32.97 | 35.26 | 34.76 | 28.93 | 28.43 | 33.15 | 32.65 |
| 0.50 | 17.21 | 16.70 | 19.46 | 18.95 | 17.27 | 16.76 | 17.56 | 17.06 | 15.90 | 15.40 | 17.33 | 16.82 | 13.89 | 13.38 | 16.37 | 15.86 |
| 0.40 | 8.62 | 8.10 | 9.74 | 9.23 | 8.72 | 8.20 | 8.99 | 8.48 | 7.63 | 7.11 | 8.64 | 8.13 | 6.86 | 6.34 | 8.26 | 7.75 |
| 0.30 | 4.41 | 3.87 | 4.98 | 4.46 | 4.51 | 3.98 | 4.74 | 4.21 | 3.86 | 3.32 | 4.51 | 3.98 | 3.63 | 3.09 | 4.40 | 3.87 |
| 0.20 | 2.38 | 1.81 | 2.67 | 2.11 | 2.47 | 1.91 | 2.64 | 2.08 | 2.16 | 1.59 | 2.53 | 1.97 | 2.14 | 1.56 | 2.53 | 1.97 |
| 0.10 | 1.42 | 0.78 | 1.55 | 0.93 | 1.50 | 0.87 | 1.58 | 0.95 | 1.40 | 0.75 | 1.55 | 0.93 | 1.42 | 0.78 | 1.57 | 0.94 |

Table 17: The ARL and SDRL comparison of the proposed chart with the MDS chart for TRC data at $r_0 = 370$ with $m$ value varying from 6 to 9.

| | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | | MMDS | | MDS | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | 6 | | | | 7 | | | | 8 | | | | 9 | | | |
| $k_1$ | 2.79685 | | 3.15202 | | 2.90606 | | 3.10523 | | 3.05044 | | 3.08986 | | 2.95615 | | 3.18391 | |
| $k_2$ | 2.13339 | | 2.24025 | | 2.01025 | | 2.26763 | | 1.98476 | | 2.28843 | | 2.04333 | | 2.29514 | |
| $\theta_0/\theta_1$ | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL | ARL | SDRL |
| 1 | 370.00 | 369.50 | 370.00 | 369.50 | 370.01 | 369.51 | 370.01 | 369.51 | 370.01 | 369.51 | 370.00 | 369.50 | 370.01 | 369.51 | 370.01 | 369.51 |
| 0.90 | 219.84 | 219.33 | 230.05 | 229.55 | 213.25 | 212.75 | 226.52 | 226.02 | 215.68 | 215.18 | 224.52 | 224.02 | 210.90 | 210.40 | 226.34 | 225.84 |
| 0.80 | 120.06 | 119.55 | 127.75 | 127.25 | 112.22 | 111.72 | 124.35 | 123.85 | 112.93 | 112.43 | 122.35 | 121.85 | 109.17 | 108.67 | 123.37 | 122.87 |
| 0.70 | 61.99 | 61.49 | 65.66 | 65.16 | 55.67 | 55.17 | 63.52 | 63.02 | 55.06 | 54.56 | 62.21 | 61.70 | 53.15 | 52.65 | 62.35 | 61.85 |
| 0.60 | 30.96 | 30.46 | 32.43 | 31.93 | 26.84 | 26.33 | 31.35 | 30.85 | 25.98 | 25.48 | 30.66 | 30.16 | 25.24 | 24.73 | 30.49 | 29.99 |
| 0.50 | 15.30 | 14.80 | 15.95 | 15.44 | 12.99 | 12.48 | 15.49 | 14.98 | 12.37 | 11.86 | 15.20 | 14.69 | 12.17 | 11.66 | 15.03 | 14.52 |
| 0.40 | 7.71 | 7.19 | 8.08 | 7.56 | 6.56 | 6.04 | 7.93 | 7.41 | 6.23 | 5.70 | 7.83 | 7.32 | 6.23 | 5.71 | 7.75 | 7.23 |
| 0.30 | 4.11 | 3.58 | 4.34 | 3.81 | 3.61 | 3.07 | 4.33 | 3.79 | 3.46 | 2.92 | 4.32 | 3.79 | 3.53 | 2.98 | 4.30 | 3.77 |
| 0.20 | 2.40 | 1.83 | 2.53 | 1.97 | 2.21 | 1.64 | 2.55 | 1.99 | 2.17 | 1.59 | 2.57 | 2.01 | 2.22 | 1.65 | 2.58 | 2.01 |
| 0.10 | 1.53 | 0.90 | 1.58 | 0.95 | 1.47 | 0.84 | 1.59 | 0.97 | 1.46 | 0.82 | 1.60 | 0.98 | 1.49 | 0.85 | 1.60 | 0.98 |

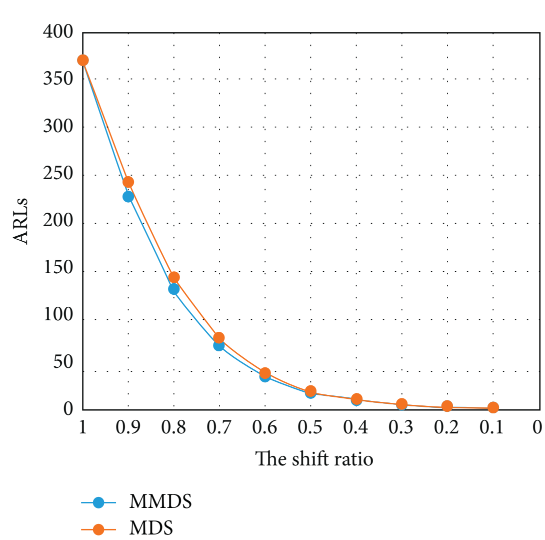


Figure 15: The ARL values for MMDS and MDS for TRC data at $m = 2$ and $r_0 = 370$.

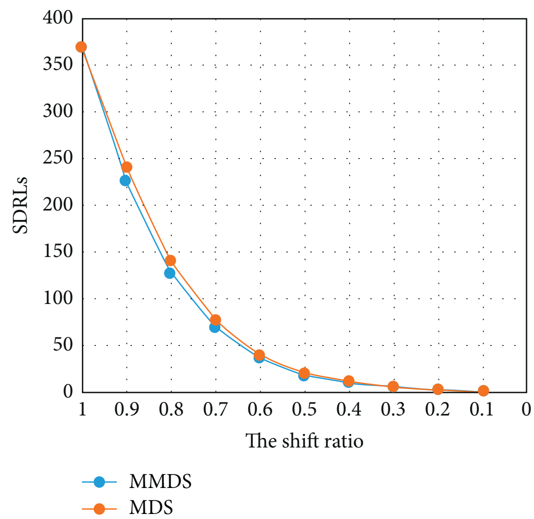


FIGURE 16: The SDRL values for MMDS and MDS for TRC data at $m = 2$ and $r_0 = 370$.

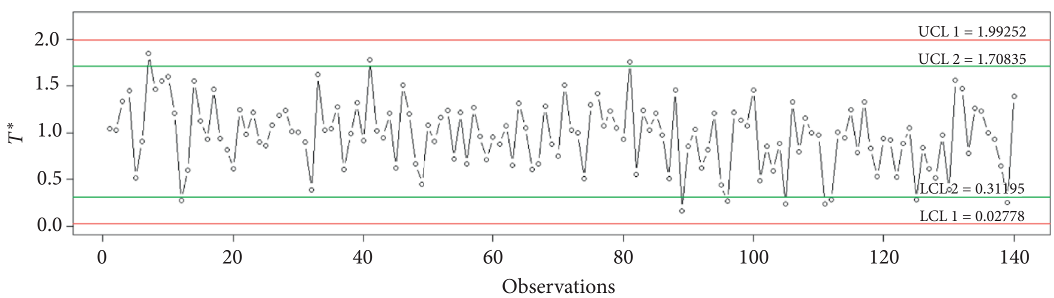


FIGURE 17: The MDS control chart for TRC data at $m = 6$.

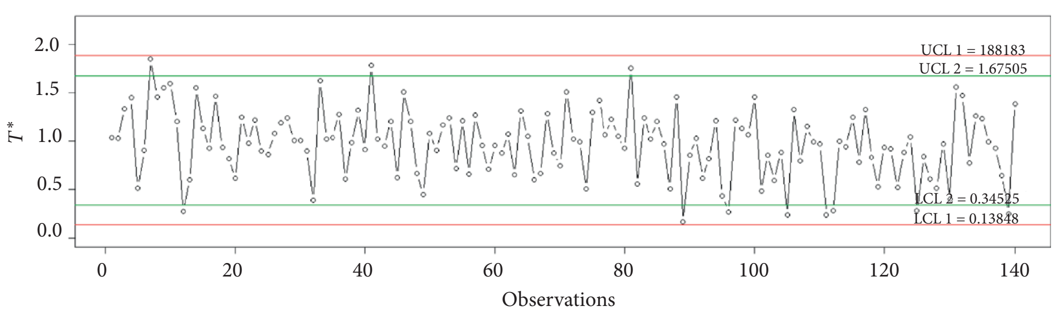


FIGURE 18: The MMDS control chart for TRC data at $m = 6$.

This example shows that the proposed chart detects process shifts more quickly than the $t$-chart under the MDS scheme, as shown in Figures 15 and 16. For example, when $m = 2$, the proposed chart takes almost 226 samples to detect the out-of-control signal after the shift of "$\theta_0/\theta_1 = 0.9$" occurs in the process mean, while the t-chart under the MDS scheme takes almost 240 samples to detect the shift. Similarly, the SDRL of the proposed chart takes 225.52 samples compared to the 234.69 samples for the $t$-chart under the MDS scheme. From Figures 17 and 18, it is apparent that the proposed chart is more sensitive to small shifts. For example, observation 7 lies near the upper outer control limit with the $t$-chart under the MMDS; therefore, it needs more attention, while the observation under MDS is obviously in the indecision area.

## 6. Conclusions

In this paper, a new $t$-control chart based on an MMDS sampling scheme is introduced to monitor processes under exponential distribution. The performance evaluation of the proposed chart is investigated using the run length characteristics. A simulation approach using the R programming language and software is used to find the control chart coefficients $k_1$ and $k_2$ for the different values of $m$ and at specified values of $r_0$. Furthermore, the ARL and SDRL values are also computed for various mean shifts to assess the shift detection ability. The superiority of the proposed chart is confirmed by comparing the values of the ARL and SDRL using the existing and proposed charts. The simulation study also shows that the proposed control chart uniformly outperforms the other $t$-charts considered in this study. Empirical examples based on the monitoring of process performance in a large hospital and a utility company are also included for the practical implementation of the proposed chart. The control chart presented in this study has been proposed for the utility company and is now in the process of implementation. The proposed chart can be extended for the testing of other distributions as well.

## Data Availability

The data are included in the article.

## Conflicts of Interest

The authors declare no conflicts of interest.

## Authors' Contributions

A.O.A., M.A., and S.A.D. contributed to the conceptualisation, data curation, formal analysis, investigation, methodology, project administration, and project validation; A.O.A. and M.A. were in charge of gathering resources, while A.O.A. contributed to the software integration, visualisation, and the writing of the first draft. M.A. and S.A.D. maintained supervisory roles, and A.O.A., M.A., and S.A.D. contributed to the writing, reviewing, and editing of the document.

## Acknowledgments

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, gratefully acknowledge the DSR technical and financial support.

## Supplementary Materials

R codes are given in the Supplementary File. (*Supplementary Materials*)

## References

[1] D. C. Montgomery and C. M. Borror, "Systems for modern quality and business improvement," *Quality Technology & Quantitative Management*, vol. 14, no. 4, pp. 343–352, 2017.

[2] J. Yu and X. Yan, "Whole process monitoring based on unstable neuron output information in hidden layers of deep belief network," *IEEE Transactions on Cybernetics*, vol. 50, no. 9, pp. 3998–4007, 2020.

[3] J. Yu and X. Yan, "Active features extracted by deep belief network for process monitoring," *ISA Transactions*, vol. 84, 2019.

[4] E. Mezzenga, V. D'Errico, A. Sarnelli et al., "Preliminary retrospective analysis of daily tomotherapy output constancy checks using statistical process control," *PLoS One*, vol. 11, no. 2, Article ID e0147936, 2016.

[5] J. C. Benneyan, R. C. Lloyd, and P. E. Plsek, "Statistical process control as a tool for research and healthcare improvement," *Quality and Safety in Health Care*, vol. 12, no. 6, pp. 458–464, 2003.

[6] S. Sukparungsee, Y. Areepong, and R. Taboran, "Exponentially weighted moving average-Moving average charts for monitoring the process mean," *PLoS One*, vol. 15, no. 2, Article ID e0228208, 2020.

[7] S. Ali, A. Pievatolo, and R. Göb, "An overview of control charts for high-quality processes," *Quality and Reliability Engineering International*, vol. 32, no. 7, pp. 2171–2189, 2016.

[8] A. H. Al-Marshadi, M. Aslam, A. H. Alharbey, N. Khan, and L. Ahmad, "Monitoring customer complaints using the repetitive sampling," *Communications in Statistics—Theory and Methods*, pp. 1–15. In press, 2020.

[9] M. Abujiya and M. H. Lee, "The three statistical control charts using ranked set sampling," in *Proceedings of the 5th International Conference on Modeling, Simulation and Applied Optimization (ICMSAO)*, pp. 1–6, Hammamet, Tunisia, April 2013.

[10] M. Abujiya, M. H. Lee, and M. Riaz, "An improved combined shewhart-EWMA chart based on double median ranked set sampling," *Journal of Computer Science & Computational Mathematics*, vol. 4, no. 2, pp. 23–28, 2014.

[11] K. P. Tran, "Designing of run rules $t$ control charts for monitoring changes in the process mean," *Chemometrics and Intelligent Laboratory Systems*, vol. 174, pp. 85–93, 2018.

[12] K. P. Tran, "Run rules median control charts for monitoring process mean in manufacturing," *Quality and Reliability Engineering*, vol. 33, no. 8, pp. 2437–2450, 2017.

[13] F. Y. Yen, K. M. B. Chong, and L. M. Ha, "Synthetic-type control charts for time-between-events monitoring," *PLoS One*, vol. 8, no. 6, Article ID e65440, 2013.

[14] E. Santiago and J. Smith, "Control charts based on the exponential distribution: adapting runs rules for the $t$ chart," *Quality Engineering*, vol. 25, no. 2, 2013.

[15] L. S. Nelson, "A control chart for parts-per-million nonconforming items," *Journal of Quality Technology*, vol. 26, no. 3, pp. 239-240, 1994.

[16] A. W. Wortham and R. C. Baker, "Multiple deferred state sampling inspection," *International Journal of Production Research*, vol. 14, no. 6, pp. 719–731, 1976.

[17] S. Balamurali and C.-H. Jun, "Multiple dependent state sampling plans for lot acceptance based on measurement data," *European Journal of Operational Research*, vol. 180, no. 3, pp. 1221–1230, 2007.

[18] M. Aslam, M. Azam, N. Khan, and C.-H. Jun, "A control chart for an exponential distribution using multiple dependent state sampling," *Quality and Quantity*, vol. 49, p. 462, 2015.

[19] M. Aslam, O.-H. Arif, and C.-H. Jun, "A control chart for gamma distribution using multiple dependent state sampling," *Industrial Engineering and Management Systems*, vol. 16, no. 1, pp. 109–117, 2017.

[20] A. Yan, S. Liu, and X. Dong, "Designing a multiple dependent state sampling plan based on the coefficient of variation," *SpringerPlus*, vol. 5, p. 1447, 2016.

[21] M. Aslam, M. A. Raza, L. Ahmad, and C. H. Jun, "Design of a t-chart using generalized multiple dependent state sampling," *Quality and Reliability Engineering International*, vol. 35, no. 6, pp. 1789–1802, 2019.

[22] M. Aslam, S. Balamurali, and C.-H. Jun, "Determination and economic design of a generalized multiple dependent state

sampling plan," *Communication in Statistics-Simulation and Computation*, pp. 1–8. In press, 2019.
[23] N. Johnson and S. Kotz, *Continuous Univariate Distributions*, Houghton Mifflin, Boston, MA, USA, 1970.
[24] W. L. Teoh, M. B. C. Khoo, and S. Y. Teh, "Optimal designs of the median run length based double sampling $\bar{X}$ chart for minimizing the average sample size," *PLoS One*, vol. 8, no. 7, Article ID e68580, 2013.
[25] J. Anhøj and A. V. Olesen, "Run charts revisited: a simulation study of run chart rules for detection of non-random variation in health care processes," *PLoS One*, vol. 9, no. 11, Article ID e113825, 2014.
[26] R Core Team, *A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria, 2016, https://www.R-project.org/.
[27] D. Santos, "Beyond six sigma—a control chart for tracking defects per billion opportunities (DPBO)," *International Journal of Industrial Engineering: Theory Applications and Practice*, vol. 16, no. 3, pp. 227–233, 2009.
[28] L. Shu, W. Huang, and W. Jiang, "A novel gradient approach for optimal design and sensitivity analysis of EWMA control charts," *Naval Research Logistics (NRL)*, vol. 61, no. 3, pp. 223–237, 2014.
[29] M. Flores, S. Naya, R. Fernández-Casal, S. Zaragoza, P. Raña, and J. Tarrío-Saavedra, "Constructing a control chart using functional data," *Mathematics*, vol. 8, no. 1, p. 58, 2020.
[30] I. A. Raji, M. H. Lee, M. Riaz, M. Abujiya, and N. Abbas, "Outliers detection models in Shewhart control charts; an application in photolithography: a semiconductor manufacturing industry," *Mathematics*, vol. 8, no. 5, p. 857, 2020.
[31] J.-H. Chen and S.-L. Lu, "An enhanced auxiliary information-based EWMA-t chart for monitoring the process mean," *Applied Sciences*, vol. 10, no. 7, p. 2252, 2020.