Research Article

Approximate Analytic Solutions of Two-Dimensional Nonlinear Klein–Gordon Equation by Using the Reduced Differential Transform Method

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In this paper, the reduced differential transform method (RDTM) is successfully implemented for solving two-dimensional nonlinear Klein–Gordon equations (NLKGEs) with quadratic and cubic nonlinearities subject to appropriate initial conditions. The proposed technique has the advantage of producing an analytical approximation in a convergent power series form with a reduced number of calculable terms. Two test examples from mathematical physics are discussed to illustrate the validity and efficiency of the method. In addition, numerical solutions of the test examples are presented graphically to show the reliability and accuracy of the method. Also, the results indicate that the introduced method is promising for solving other type systems of NLPDEs.

1. Introduction

In many applications of science and engineering such as fluid dynamics, plasma physics, hydrodynamics, solid-state physics, optical fibers, acoustics, and other disciplines, the nonlinear equations appear for modeling physical phenomenon [1]. Recently, a lot of research was carried out on real-life applications modeled by PDEs [2–12]. The Klein–Gordon equation is an important class of partial differential equations and arises in relativistic quantum mechanics and field theory, which is of great importance for the high-energy physicists [13], and is used to model many different phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles [14].

The n -dimensional nonlinear Klein–Gordon equation is given by partial differential equation [15]:

$$u_{tt}(\mathbf{x}, t) + \beta u_t(\mathbf{x}, t) = \alpha(u_{xx}(\mathbf{x}, t) + u_{yy}(\mathbf{x}, t)) - g(u(\mathbf{x}, t)) + f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, t > 0, \quad (1)$$

with initial conditions

$$\begin{aligned} u(\mathbf{x}, 0) &= \varphi_1(\mathbf{x}), \\ u_t(\mathbf{x}, 0) &= \varphi_2(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{aligned} \quad (2)$$

where Ω is a domain of R^n ($n = 1, 2, \dots$), $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ is the space variable, t is the time variable, $u(\mathbf{x}, t)$ is the unknown function, $g(u)$ is a nonlinear function of u with different types of nonlinearities, $f(\mathbf{x}, t)$ is a given function, $\varphi_1(\mathbf{x})$ and $\varphi_2(\mathbf{x})$ are prescribed initial functions, and α and β are known constants.

In the present work, we are dealing with the approximate analytical solution of the nonlinear Klein–Gordon equation (1) with a nonlinear function $g(u)$ having the form $g(u) = \gamma_1 u + \gamma_2 u^m$.

That is,

$$\begin{aligned} \frac{\partial^2 u(x, y, t)}{\partial t^2} + \beta \frac{\partial u(x, y, t)}{\partial t} &= \alpha \left(\frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} \right) \\ &\quad - (\gamma_1 u(x, y, t) + \gamma_2 u^m(x, y, t)) \\ &\quad + f(x, y, t), \end{aligned} \quad (3)$$

subject to the initial conditions

$$\begin{aligned} u(x, y, 0) &= g_1(x, y), \\ u_t(x, y, 0) &= g_2(x, y), \end{aligned} \quad (4)$$

where $t > 0$ and $(x, y) \in \mathbb{R}^2$.

Remark 1. Equation (3) is called the Klein–Gordon equation with quadratic nonlinearity if $m = 2$ and with cubic nonlinearity if $m = 3$.

In recent years, many researchers have paid attention to study the solutions of NLPDEs by various methods (see [16] and the references therein).

The reduced differential transform method (RDTM) was first proposed by Keskin and Oturanc [17] for solving partial differential equations (PDEs). It is an iterative procedure based on the use of the Taylor series solution of differential equations. RDTM has been successfully applied to solve various nonlinear partial differential equations [18–28].

The main aim of this study is to obtain the approximate analytical solutions for two-dimensional nonlinear Klein–Gordon equation (NLKGE) with quadratic and cubic nonlinearities, since most of the research focused on the numerical solutions for this problem. The reduced differential transform method is used for this purpose for several reasons. The first reason is that the method has not previously been applied to solve this problem. Secondly, this method can directly be applied to NLKGE. Thirdly, this method can reduce the size of the calculations and can provide an analytic approximation, in many cases exact solutions, in rapidly convergent power series form with elegantly computed terms (see [29] and the references therein). Moreover, the reduced differential transform method (RDTM) has an alternative approach of solving problems to overcome the demerit of discretization, linearization, or perturbations of well-known numerical and analytical methods such as Adomian decomposition, differential transform, homotopy perturbation, and variational iteration [29, 30].

The structure of the remaining parts of this paper is organized as follows. In Section 2, we begin with some basic definitions and describe the proposed method. In Section 3, we discuss the convergence analysis of the method. In Section 4, we apply the reduced differential transform method to solve two illustrative examples in order to show its ability and efficiency. Concluding remarks are given in Section 5.

2. Reduced Differential Transform Method

The basic definitions and operations of the two-dimensional reduced differential transform method [30–34] are introduced as follows.

Definition 1. If the function $u(x, y, t)$ is analytic and differentiable continuously with respect to time t and space in the domain of interest, then

$$U_k(x, y) = \frac{1}{k!} \left[\frac{\partial^k u(x, y, t)}{\partial t^k} \right]_{t=t_0}, \quad (5)$$

where the t -dimensional spectrum function $U_k(x, y)$ is the transformed function of $u(x, y, t)$.

In this study, the lower case $u(x, y, t)$ represents the original function while the upper case $U_k(x, y)$ stands for the transformed function.

Definition 2. The inverse differential transform of $U_k(x, y)$ is defined as

$$u(x, y, t) = \sum_{k=0}^{\infty} U_k(x, y) (t - t_0)^k. \quad (6)$$

Combining equations (5) and (6), we obtain

$$u(x, y, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k u(x, y, t)}{\partial t^k} \right]_{t=t_0} t^k. \quad (7)$$

Note that the function $u(x, y, t)$ can be written in a finite series as follows:

$$u_n(x, y, t) = \sum_{k=0}^n U_k(x, y) (t - t_0)^k + R_n(x, y, t), \quad (8)$$

where the tail function $R_n(x, y, t)$ is negligibly small. Therefore, the exact solution of the problem is given by

$$u(x, y, t) = \lim_{n \rightarrow \infty} u_n(x, y, t). \quad (9)$$

Lemma 1. If $w(x, y, t) = u^2(x, y, t)$, then $W_k(x, y) = \sum_{i=0}^k U_i(x, y) U_{k-i}(x, y)$.

Proof. By using Table 1, we have

$$\begin{aligned} W_k(x, y) &= \frac{1}{k!} \left[\frac{\partial^k (u(x, y, t) u(x, y, t))}{\partial t^k} \right]_{t=t_0} \\ &= \frac{1}{k!} \left[u(x, y, t) \frac{\partial^k u(x, y, t)}{\partial t^k} + k \frac{\partial u(x, y, t)}{\partial t} \frac{\partial^{k-1} u(x, y, t)}{\partial t^{k-1}} + \frac{k(k-1)}{2!} \frac{\partial^2 u(x, y, t)}{\partial t^2} \frac{\partial^{k-2} u(x, y, t)}{\partial t^{k-2}} \right. \\ &\quad \left. + \dots + k \frac{\partial^{k-1} u(x, y, t)}{\partial t^{k-1}} \frac{\partial u(x, y, t)}{\partial t} + \frac{\partial^k u(x, y, t)}{\partial t^k} u(x, y, t) \right]_{t=t_0}. \end{aligned} \quad (10)$$

TABLE 1: Basic operations of RDTM.

Original function	Transformed function
$u(x, y, t)$	$U_k(x, y) = \frac{1}{k!} [\partial^k u(x, y, t) / \partial t^k]_{t=0}$
$w(x, y, t) = \alpha u(x, y, t) + \beta v(x, y, t)$	$W_k(x, y) = \alpha U_k(x, y) + \beta V_k(x, y)$, where α and β are constants.
$w(x, y, t) = u(x, y, t)v(x, y, t)$	$W_k(x, y) = \sum_{r=0}^{\infty} U_r(x, y)V_{k-r}(x, y)$ $= \sum_{r=0}^{\infty} V_{k-r}(x, y)U_r(x, y)$
$w(x, y, t) = \partial^r / \partial t^r u(x, y, t)$	$W_k(x, y) = (k+r)!/k! U_{k+r}(x, y)$
$w(x, y, t) = \partial^2 / \partial x^2 u(x, y, t)$	$W_k(x, y) = \partial^2 / \partial x^2 U_k(x, y)$
$w(x, y, t) = \partial^2 / \partial y^2 u(x, y, t)$	$W_k(x, y) = \partial^2 / \partial y^2 U_k(x, y)$
$w(x, y, t) = \sin(\alpha x + \gamma y + \beta t)$	$W_k(x, y) = \beta^k / k! \sin(\alpha x + \gamma y + k\pi/2)$, where α, β and γ are constants.
$w(x, y, t) = \cos(\alpha x + \gamma y + \beta t)$	$W_k(x, y) = \beta^k / k! \cos(\alpha x + \gamma y + k\pi/2)$, where α, β and γ are constants.

By Leibnitz's theorem, this can be written as

$$\begin{aligned}
W_k(x, y) &= \left[\left(\frac{1}{k!} \frac{\partial^0 u(x, y, t)}{\partial t^0} \frac{\partial^k u(x, y, t)}{\partial t^k} \right) \frac{1}{0!} + \left(\frac{1}{(k-1)!} \frac{\partial u(x, y, t)}{\partial t} \frac{\partial^{k-1} u(x, y, t)}{\partial t^{k-1}} \right) \frac{1}{1!} \right. \\
&\quad + \left(\frac{1}{(k-2)!} \frac{\partial^2 u(x, y, t)}{\partial t^2} \frac{\partial^{k-2} u(x, y, t)}{\partial t^{k-2}} \right) \frac{1}{2!} + \cdots + \left(\frac{1}{(k-1)!} \frac{\partial^{k-1} u(x, y, t)}{\partial t^{k-1}} \frac{\partial u(x, y, t)}{\partial t} \right) \frac{1}{1!} \\
&\quad \left. + \left(\frac{1}{k!} \frac{\partial^k u(x, y, t)}{\partial t^k} \frac{\partial^0 u(x, y, t)}{\partial t^0} \right) \frac{1}{0!} \right]_{t=0} \\
&= \sum_{i=0}^k \frac{1}{i! (k-i)!} \left[\frac{\partial^i u(x, y, t)}{\partial t^i} \frac{\partial^{k-i} u(x, y, t)}{\partial t^{k-i}} \right]_{t=0} \\
&= \sum_{i=0}^k \frac{1}{i!} \left[\frac{\partial^i u(x, y, t)}{\partial t^i} \right]_{t=0} \frac{1}{(k-i)!} \left[\frac{\partial^{k-i} u(x, y, t)}{\partial t^{k-i}} \right]_{t=0} \\
&= \sum_{i=0}^k U_i(x, y) U_{k-i}(x, y).
\end{aligned} \tag{11}$$

Lemma 2. If $f(x, y, t) = u^3(x, y, t)$, then $W_k(x, y) = \sum_{j=0}^k \sum_{i=0}^j U_i(x, y) U_{j-i}(x, y) U_{k-j}(x, y)$.

Proof. Observe that $u^3(x, y, t) = u^2(x, y, t)u(x, y, t)$. So, by using Table 1, we have

$$\begin{aligned}
W_k(x, y) &= \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} (u^3(x, y, t)) \right]_{t=0} \\
&= \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u^2(x, y, t) u(x, y, t) \right]_{t=0} \\
&= \sum_{j=0}^k F_j(x, y) U_{k-j}(x, y),
\end{aligned} \tag{12}$$

where F_j is the reduced differential transform of $u^2(x, y, t) = \sum_{i=0}^j U_i(x, y) U_{j-i}(x, y)$ (Lemma 1).

Thus,

$$\begin{aligned}
W_k(x, y) &= \sum_{j=0}^k F_j(x, y) U_{k-j}(x, y) \\
&= \sum_{j=0}^k \sum_{i=0}^j U_i(x, y) U_{j-i}(x, y) U_{k-j}(x, y).
\end{aligned} \tag{13}$$

To demonstrate the basic concepts of RDTM, we consider the two-dimensional nonlinear Klein–Gordon equation from equation (3) subject to the initial conditions given by equation (4).

By RDTM, from equation (3), we obtain the following recurrence relation:

$$\begin{aligned}
U_{k+2}(x, y) &= \frac{1}{(k+1)(k+2)} \left[\alpha \left(\frac{\partial^2 U_k(x, y)}{\partial x^2} + \frac{\partial^2 U_k(x, y)}{\partial y^2} \right) \right. \\
&\quad \left. - \beta U_k(x, y) - N_k(x, y) + F_k(x, y) \right],
\end{aligned} \tag{14}$$

where $N_k(x, y)$ and $F_k(x, y)$ are the transformed forms of the nonlinear terms $g(u)$ and $f(x, y, t)$, respectively. Similarly, by RDTM, from the initial condition (4), we get

$$\begin{aligned} U_0(x, y) &= g_1(x, y), \\ U_1(x, y) &= g_2(x, y), \end{aligned} \quad (15)$$

respectively.

Substituting equation (15) into equation (14), we get the set of values $\{U_k(x, y)\}_{k=0}^n$. Then, the inverse reduced differential transform of these sets of values provides the n -term approximate solution as

$$\tilde{u}_n(x, y, t) = \sum_{k=0}^n U_k(x, y) t^k. \quad (16)$$

Therefore, the exact solution of the problem is given by

$$u(x, y, t) = \lim_{n \rightarrow \infty} \tilde{u}_n(x, y, t). \quad (17)$$

3. Convergence Analysis

In this section, the convergence analysis of the approximate analytical solutions which are computed from the application of RDTM [30] is presented.

Let us consider equation (3) in the following functional equation form:

$$u(x, y, t) = F(u(x, y, t)), \quad (18)$$

where F is a general nonlinear operator involving both linear and nonlinear terms.

According to RDTM, equation (3) has a solution of the form

$$u(x, y, t) = \sum_{k=0}^{\infty} U_k(x, y) t^k = \sum_{k=0}^{\infty} \beta_k. \quad (19)$$

It is noted that the solutions by RDTM are equivalent to determining the sequences

$$\begin{aligned} S_0 &= U_0(x, y) = \beta_0, \\ S_1 &= U_0(x, y) + U_1(x, y)t = \beta_0 + \beta_1, \\ S_2 &= U_0(x, y) + U_1(x, y)t + U_2(x, y)t^2 \\ &= \beta_0 + \beta_1 + \beta_2, \\ &\dots \\ S_n &= \sum_{k=0}^n U_k(x, y) t^k = \sum_{k=0}^n \beta_k, \end{aligned} \quad (20)$$

by using iterative scheme

$$S_{n+1} = F(S_n), \quad (21)$$

associated with the functional equation

$$S = F(S). \quad (22)$$

Hence, the solution obtained by RDTM, $u(x, y, t) = \sum_{k=0}^{\infty} U_k(x, y) t^k = \sum_{k=0}^{\infty} \beta_k$, is equivalent to $\{S_n\}_{n=0}^{\infty}$.

The sufficient condition for convergence of the series solution $\{S_n\}_{n=0}^{\infty}$ is given in the following theorem.

Theorem 1. Let F be an operator from a Hilbert space H into H . Then, the series solution $\{S_n\}_{n=0}^{\infty}$ converges whenever there is " α " such that $0 < \alpha < 1$, and $\|\beta_{k+1}\| \leq \alpha \|\beta_k\|$ (see for proof Ref. [30]).

Theorem 2. Let F be a nonlinear operator satisfying Lipschitz condition from Hilbert space H into H and $u(x, y, t)$ be exact solution of the given NLKGE. If the series solution $\{S_n\}_{n=0}^{\infty}$ converges, then it converges to $u(x, y, t)$ (see for proof Ref. [30]).

Definition 3. For $k \in N \cup \{0\}$, we define

$$\alpha_k = \begin{cases} \frac{\|\beta_{k+1}\|}{\|\beta_k\|} = \frac{\|U_{k+1}(x, y)t^{k+1}\|}{\|U_k(x, y)t^k\|}, & \text{if } \|\beta_k\| = \|U_k(x, y)t^k\| \neq 0, \\ 0, & \text{if } \|\beta_k\| = \|U_k(x, y)t^k\| = 0. \end{cases} \quad (23)$$

Then, we can say that the series approximate solution $\{S_n\}_{n=0}^{\infty}$ converges to the exact solution $u(x, y, t)$ when $0 \leq \alpha_k < 1$ for $k = 0, 1, 2, \dots$.

4. Test Examples

In this section, we consider two test examples that demonstrate the performance and efficiency of the RDTM for solving two-dimensional inhomogeneous nonlinear Klein–Gordon equations.

Example 1. Consider the two-dimensional NLKGE (3) with $m = 2, \alpha = 1, \beta = 0, \gamma_1 = 0, \gamma_2 = 1$.

That is,

$$\begin{aligned} u_{tt}(x, y, t) - u_{xx}(x, y, t) - u_{yy}(x, y, t) + u^2(x, y, t) \\ = -xycost + x^2y^2\cos^2t, \quad t > 0, \end{aligned} \quad (24)$$

with initial conditions

$$\begin{aligned} u(x, y, 0) &= xy, \\ u_t(x, y, 0) &= 0. \end{aligned} \quad (25)$$

Applying properties of RDTM to equation (24), we obtain the following recurrence relation:

$$\begin{aligned} U_{k+2}(x, y) &= \frac{1}{(k+1)(k+2)} \left[\frac{\partial^2 U_k(x, y)}{\partial x^2} + \frac{\partial^2 U_k(x, y)}{\partial y^2} \right. \\ &\quad \left. - N_k(x, y) + F_k(x, y) \right], \end{aligned} \quad (26)$$

where

$$N_k(x, y) = \sum_{r=0}^k U_r(x, y) U_{k-r}(x, y) \quad (27)$$

is the transformed form of $u^2(x, y, t)$ and $F_k(x, y)$ is the transformed form of the function $f(x, y, t) = -xy \cos t + x^2 y^2 \cos^2 t$.

By similar scheme from (25), we obtain

$$\begin{aligned} U_0(x, y) &= xy, \\ U_1(x, y) &= 0. \end{aligned} \quad (28)$$

Let

$$g(x, y, t) = -xy \cos t \quad (29)$$

and

$$h(x, y, t) = g^2(x, y, t) = x^2 y^2 \cos^2 t. \quad (30)$$

Applying properties of RDTM on equations (29) and (30), we get

$$G_k(x, y) = -xy \frac{1}{k!} \cos\left(k \frac{\pi}{2}\right), \quad (31)$$

and

$$H_k(x, y) = \sum_{i=0}^k G_i(x, y) G_{k-i}(x, y), \quad (32)$$

respectively.

Thus,

$$F_k(x, y) = G_k(x, y) + H_k(x, y). \quad (33)$$

We then compute the set of values of $\{u_k(x, y)\}_{k=0}^n$.

Substituting equations (27), (31), and (32) into equation (26) with $k = 0, 1, 2, \dots, n$, we obtain

$$\begin{aligned} N_0(x, y) &= \sum_{r=0}^0 U_r U_{k-r} = U_0^2(x, y) = x^2 y^2, \\ G_0(x, y) &= -xy, \\ H_0(x, y) &= \sum_{i=0}^0 G_i(x, y) G_{k-i}(x, y) = G_0^2(x, y) = x^2 y^2, \end{aligned} \quad (34)$$

respectively.

So,

$$F_0(x, y) = G_0(x, y) + H_0(x, y) = -xy + x^2 y^2. \quad (35)$$

Hence, from equation (26), we obtain

$$U_2(x, y) = \frac{1}{2} [0 + 0 - x^2 y^2 + x^2 y^2 - xy] = -\frac{xy}{2}. \quad (36)$$

Accordingly,

$$\begin{aligned} N_1(x, y) &= 2U_1(x, y)U_0(x, y) = 0, \\ G_1(x, y) &= 0, \\ H_1(x, y) &= \sum_{i=0}^1 G_i(x, y) G_{k-i}(x, y) = 0. \end{aligned} \quad (37)$$

Thus, $F_1(x, y) = G_1(x, y) + H_1(x, y) = 0$, and hence when $k = 1$, equation (26) yields

$$U_3(x, y) = \frac{1}{3!} \left[\frac{\partial^2 U_1(x, y)}{\partial x^2} + \frac{\partial^2 U_1(x, y)}{\partial y^2} - N_1(x, y) + F_1(x, y) \right] = 0. \quad (38)$$

Using similar procedure, we obtain the following results:

$$\begin{aligned} U_4(x, y) &= \frac{xy}{4!}, \\ U_5(x, y) &= 0, \\ U_6(x, y) &= -\frac{xy}{6!}, \end{aligned} \quad (39)$$

and so on.

Substituting all these values in equation (16), we obtain

$$\begin{aligned} u(x, y, t) &= \sum_{k=0}^n U_k(x, y) t^k, \\ &= xy \left(1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 - \frac{1}{6!} t^6 + \dots + \frac{(-1)^n}{(2n)!} t^{(2n)} \right), \end{aligned} \quad (40)$$

and hence by equation (17), the exact solution of Example 1 is $u(x, y, t) = xy \cos t$.

To make a decision on the convergence of the solution, we compute α_k using Definition 3 and Theorem 1 as follows.

First, let us take $x = y = t = 1$; then,

$$\begin{aligned} \alpha_0 &= \frac{\beta_1}{\beta_0} = \frac{U_1(x, y) t^1}{U_0(x, y) t^0} = \frac{0 \times 1^1}{1 \times 1} = 0 < 1 \\ \alpha_1 &= \frac{\beta_2}{\beta_1} = 0 < 1, \text{ since } \|\beta_1\| = 0, \text{ definition 3.1,} \\ \alpha_2 &= \frac{\beta_3}{\beta_2} = \frac{U_3(x, y) t^3}{U_2(x, y) t^2} = \frac{0 \times 1^3}{1/2! \times 1 \times 1 \times 1^2} = 0 < 1, \\ \alpha_3 &= \frac{\beta_4}{\beta_3} = 0 < 1, \text{ since } \beta_3 = U_3(x, y) t^3 = 0, \end{aligned}$$

Hence $x, y, t \geq 0$, $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \dots = \alpha_\infty = 0 < 1$.

(41)

Therefore, by Theorem 2, the solution of the given NLKGE by RDTM has a convergent solution.

Figures representing the numerical solution of Example 1 are presented below.

Figures 1(a) and 1(b), respectively, show the physical behavior of the approximate solution by RDTM of Example 1 at $t = 0.1$ for equal values of x and y in $[0, 3.2]$ and the associated absolute errors. From these figures, it can be observed that the numerical results by RDTM are in excellent agreement with the exact solution of the problem under consideration.

In Figure 1(c), we have plotted absolute errors by the proposed method corresponding to $t = 5.0, 5.2, 5.4, 5.6$ and $t = 5.8$. This figure represents the errors of numerical results by RDTM that could appear for the long time range.

FIGURE 1: Plots of the solution behavior of Example 1. (a) Approximate solution at $t = 0.1$. (b) Absolute errors at $t = 0.1$. (c) Absolute errors for different values of t . (d) Comparison of exact and approximate solutions for different values of t .

In Figure 1(d), we have compared the approximate solutions by RDTM and exact solutions corresponding to $t = 0.1$, 0.5 and $t = 1$, at equal values of x and y in $[0, 3.2]$. This figure asserts that the numerical results obtained by RDTM are in a good agreement with the exact results, especially for $t < 0.5$. In general, Figures 1(a)–1(d) show convergence of the approximate solution obtained by RDTM to the exact solution rapidly.

Example 2. Consider the two-dimensional NLKGE (3) with $m = 3$, $\alpha = 1$, $\beta = 0$, $\gamma_1 = 0$, $\gamma_2 = 1$, and $f(x, y, t) = \cos x \cos y \sin t + \cos^3 x \cos^3 y \sin^3 t$.

That is,

$$u_{tt}(x, y, t) - u_{xx}(x, y, t) - u_{yy}(x, y, t) + u^3(x, y, t) = \cos x \cos y \sin t + \cos^3 x \cos^3 y \sin^3 t, \quad t > 0, \quad (42)$$

with initial conditions

$$\begin{aligned} u(x, y, 0) &= 0, \\ u_t(x, y, 0) &= \cos x \cos y. \end{aligned} \quad (43)$$

By applying RDTM on equation (42), the following recursive equation is obtained:

$$U_{k+2}(x, y) = \frac{1}{(k+1)(k+2)} \left[\frac{\partial^2 U_k(x, y)}{\partial x^2} + \frac{\partial^2 U_k(x, y)}{\partial y^2} - N_k(x, y) + F_k(x, y) \right], \quad (44)$$

where

$$N_k(x, y) = \sum_{j=0}^k \sum_{i=0}^j U_i(x, y) U_{j-i}(x, y) U_{k-j}(x, y) \quad (45)$$

is the transformed form of $u^3(x, y, t)$ and $F_k(x, y)$ is the transformed form of the function $f(x, y, t) = \cos x \cos y \sin t + \cos^3 x \cos^3 y \sin^3 t$.

The reduced differential transform of equation (43) is

$$\begin{aligned} U_0(x, y) &= 0, \\ U_1(x, y) &= \cos x \cos y, \end{aligned} \quad (46)$$

The transformed function F_k can be well expressed as follows.

Let $g(x, y, t) = \cos x \cos y \sin t$

$$\begin{aligned} &= \frac{1}{4} [\sin(t+x-y) + \sin(t-x+y) \\ &\quad + \sin(t+x+y) + \sin(t-x-y)], \end{aligned} \quad (47)$$

and

$$h(x, y, t) = g^3(x, y, t) = \cos^3 x \cos^3 y \sin^3 t. \quad (48)$$

Then, the reduced differential transform of the equations (47) and (48) is

$$\begin{aligned} G_k(x, y) &= \frac{1}{4} \left[\frac{1}{k!} \sin\left(k\frac{\pi}{2} + x - y\right) + \frac{1}{k!} \sin\left(k\frac{\pi}{2} - x + y\right) \right. \\ &\quad \left. + \frac{1}{k!} \sin\left(k\frac{\pi}{2} + x + y\right) + \frac{1}{k!} \sin\left(k\frac{\pi}{2} - x - y\right) \right], \end{aligned} \quad (49)$$

and

$$H_k(x, y) = \sum_{j=0}^k \sum_{i=0}^j G_i(x, y) G_{j-i}(x, y) G_{k-j}(x, y), \quad (50)$$

respectively.

Thus,

$$F_k(x, y) = G_k(x, y) + H_k(x, y). \quad (51)$$

We now compute the set of values of $\{u_k(x, y)\}_{k=0}^n$.

When $k = 0$, from equations (45), (49), and (50), we obtain

$$\begin{aligned} N_0(x, y) &= \sum_{j=0}^0 \left(\sum_{i=0}^j U_i U_{j-i} U_{k-j} \right) = U_0^3(x, y) = 0, \\ G_0(x, y) &= \frac{1}{4} \sin(x-y) + \sin(-x+y) \\ &\quad + \sin(x+y) + \sin(-x-y) = 0, \end{aligned} \quad (52)$$

$$\begin{aligned} H_0(x, y) &= \sum_{j=0}^0 \sum_{i=0}^0 G_i(x, y) G_{j-i}(x, y) G_{k-j}(x, y) \\ &= G_0^3(x, y) = 0^3 = 0, \end{aligned}$$

respectively.

Thus, $F_0(x, y) = G_0(x, y) + H_0(x, y) = 0$, and therefore, equation (44) becomes

$$U_2(x, y) = \frac{1}{2} \left[\frac{\partial^2 U_0(x, y)}{\partial x^2} + \frac{\partial^2 U_0(x, y)}{\partial y^2} - N_0(x, y) + F_0(x, y) \right] = \frac{1}{2} [0 + 0 - 0 + 0] = 0. \quad (53)$$

When $k = 1$, we have

$$\begin{aligned}
N_1(x, y) &= \sum_{j=0}^1 \left(\sum_{i=0}^j U_i(x, y) U_{j-i}(x, y) U_{k-j}(x, y) \right) = 3U_0^2(x, y) U_1(x, y) = 0, \\
G_1(x, y) &= \frac{1}{8} \left[\sin\left(\frac{\pi}{2} + x - y\right) + \sin\left(\frac{\pi}{2} - (x - y)\right) + \sin\left(\frac{\pi}{2} + x + y\right) + \sin\left(\frac{\pi}{2} - (x + y)\right) \right], \\
&= \frac{1}{4} [2 \cos(x - y) + 2 \cos(x + y)], \\
&= \cos x \cos y \\
H_1(x, y) &= \sum_{j=0}^1 \sum_{i=0}^j G_i(x, y) G_{j-i}(x, y) G_{k-j}(x, y) = 3G_0^2(x, y) G_1(x, y) = 0.
\end{aligned} \tag{54}$$

As a result, $F_1(x, y) = G_1(x, y) + H_1(x, y) = \cos x \cos y$.
So, equation (44) implies

$$\begin{aligned}
U_3(x, y) &= \frac{1}{6} \left[\frac{\partial^2 U_1(x, y)}{\partial x^2} + \frac{\partial^2 U_1(x, y)}{\partial y^2} - N_1(x, y) + F_1(x, y) \right] \\
&= \frac{1}{6} \left[\frac{\partial^2 \cos x \cos y}{\partial x^2} + \frac{\partial^2 \cos x \cos y}{\partial y^2} - 0 + \cos x \cos y \right] \\
&= -\frac{1}{3!} \cos x \cos y.
\end{aligned} \tag{55}$$

In a similar manner, we obtain

$$\begin{aligned}
U_4(x, y) &= 0, \\
U_5(x, y) &= \frac{1}{5!} \cos x \cos y, \\
U_6(x, y) &= 0, \\
U_7(x, y) &= -\frac{1}{7!} \cos x \cos y,
\end{aligned} \tag{56}$$

and so on.

Then, the differential inverse transform of the set of values $\{U_k(x, y)\}_{k=0}^n$ is

$$\begin{aligned}
u(x, y, t) &= \sum_{k=0}^n U_k(x, y) t^k \\
&= \cos x \cos y t^1 - \frac{1}{3!} \cos x \cos y t^3 + \frac{1}{5!} \cos x \cos y t^5 - \frac{1}{7!} \cos x \cos y t^7 + \dots + \\
&\quad + \frac{(-1)^n}{(2n+1)!} \cos x \cos y t^{2n+1} \\
&= \cos x \cos y \left(t^1 - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \frac{1}{7!} t^7 + \dots + \frac{(-1)^n}{(2n+1)!} t^{(2n+1)} \right).
\end{aligned} \tag{57}$$

Therefore, by equation (17), the solution of Example 2 is $u(x, y, t) = \cos x \cos y \sin t$ which is exactly the same as the result obtained by backward preserving scheme [35].

To make a decision on the convergence of the solution, we compute α_k using Definition 3 and Theorem 1. For this purpose, let us take $x = y = t = 1$; then,

FIGURE 2: Continued.

FIGURE 2: Plots of the solution behavior of Example 2. (a) Approximate solution at $t=0.1$. (b) Absolute errors at $t=0.1$. (c) Absolute errors for different values of t . (d) Comparison of exact and approximate solutions for different values of t . (e) Absolute errors for different values of t . (f) Comparison of exact and approximate solutions for different values of t .

$$\begin{aligned}\alpha_0 &= \frac{\beta_1}{\beta_0} = 0 < 1, \quad \text{since } \|\beta_0\| = U_0(x, y)t^0 = 0, \text{ definition 3,} \\ \alpha_1 &= \frac{\beta_2}{|\beta_1|} = \frac{U_2(x, y)t^2}{U_1(x, y)t^1} = \frac{0 \times 1^2}{0.29192659} = 0 < 1, \\ \alpha_2 &= \frac{\beta_3}{\beta_2} = 0 < 1, \quad \text{since } \|\beta_2\| = U_2(x, y)t^2 = 0, \text{ definition 3,} \\ \alpha_3 &= \frac{u_4}{u_3} = \frac{U_4(x, y)t^4}{U_3(x, y)t^3} = 0 < 1,\end{aligned}\tag{58}$$

and so on

therefore for $x, y, t \geq 0, \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \dots = \alpha_\infty = 0 < 1$.

So, by Theorem 2, the given NLKGE has a convergent solution.

Figures representing the numerical solution of Example 2 are presented below.

In Figures 2(a) and 2(b), the physical behavior of the approximate solution by RDTM of Example 2 at $t = 0.1$ for equal values of x and y in $[0, 3.2]$ and the associated absolute errors are shown, respectively. From these figures, it can be seen that the numerical results by RDTM are in excellent agreement with the exact solution of the problem under consideration.

In Figure 2(c), we have plotted absolute errors corresponding to $t = 5.0, 5.2, 5.4, 5.6$ and $t = 5.8$ for equal

values of x and y in $[0, 3.2]$ of Example 2. This figure represents the errors of numerical results by RDTM that could appear for the long time range of the given NLKGE.

In Figure 2(d), we have compared the approximate solutions by RDTM corresponding to $t = 0.1, 0.5$ and $t = 1$ with the exact solution, at equal values of x and y in $[0, 3.2]$. This figure shows that the numerical results obtained by RDTM are in excellent agreement with the exact results.

Moreover, for equal values of x and y in different intervals, the shape of the parabola changes due to the wave nature of the NLKGEs as shown in Figures 2(e) and 2(f). In general, from Figures 2(a)–2(f), it is evident that the approximate solution rapidly converges to the exact solution.

5. Conclusion

The reduced differential transform method (RDTM) has been applied to find an approximate analytical solution of the two-dimensional nonlinear Klein–Gordon equations. The main advantage of the RDTM is that it provides the user an analytical approximation, in many cases, an exact solution, in a rapidly convergent power series form with elegantly computed terms. Two test examples are presented to show the validity of the method under consideration. The approximate solutions of Example 2 obtained by RDTM are in excellent agreement with the exact solution obtained by the backward group preserving scheme [35]. The result is also depicted in Figure 2.

Consequently, this technique can be applied to many NLPDEs arising from science and engineering fields without requiring linearization, discretization, or perturbation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

WGB proposed the main idea of this paper. YOM and AKG supervised her work from the first draft to revision. All authors approved the final manuscript for submission.

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