Research Article

Algebraic Connectivity and Disjoint Vertex Subsets of Graphs

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It is well known that the algebraic connectivity of a graph is the second small eigenvalue of its Laplacian matrix. In this paper, we mainly research the relationships between the algebraic connectivity and the disjoint vertex subsets of graphs, which are presented through some upper bounds on algebraic connectivity.

1. Introduction

A graph G is often used to model a complex network. The vertex set and the edge set of graph G are denoted by $\mathcal{V}$ and $\mathcal{E}$, respectively. A network is represented as an undirected graph $G = (\mathcal{V}, \mathcal{E})$ consisting of $N = |\mathcal{V}|$ nodes and $E = |\mathcal{E}|$ links, respectively.

Graph theory has provided chemists with a variety of useful tools, such as in the topological structure [1–3]. The Laplacian matrix of a graph G is denoted by L , and $L = D - A$, where D is a diagonal matrix whose diagonal entries are its degrees and A is the adjacency matrix of G . The Laplacian eigenvalues of a graph G are the eigenvalues of L , denoted by $0 = \mu_N \leq \mu_{N-1} \leq \dots \leq \mu_1$, which are all real and nonnegative. The second smallest Laplacian eigenvalue μ_{N-1} of a graph is well known as the algebraic connectivity, which was first studied by Fiedler [4]. The algebraic connectivity [5] of a graph is important for the connectivity of a graph [6], which can be used to measure the robustness of a graph. It has been emerged as an important parameter in many system problems [7–18]. Especially, the algebraic connectivity also plays an important role in the partitions of a complex network. For the literature on the algebraic connectivity of a graph [19], the reader is referred to [20, 21]. In this work, the relationships are researched between the algebraic connectivity and disjoint vertex subsets of graphs, which are presented through some upper bounds.

2. Preliminaries

Let $x \in R^n$ be a vector. Let B be an incidence matrix of G . Then, $x^T L x = \|B^T x\|_2^2 = \sum_{i,j \in E} (x_i - x_j)^2$. For any vector $x, y \in R^n$, the inner product of x and y is defined as (x, y) . Two upper bounds on the algebraic connectivity are given as follows.

Lemma 1 (see [20]). *For any vector $f \in R^n$, the Rayleigh inequality is as follows:*

$$\mu_{N-1} \leq \frac{(Lf, f)}{(f, f)}, \quad (1)$$

where $(f, c) = 0$, c is a constant, and $(Lf, f) = \sum_{v_i, v_j \in E} (f(v_i) - f(v_j))^2$, $f(v_i)$ is the vector f for the node v_i .

Lemma 2 (see [20]). *For any vector $f \in R^n$, we have*

$$\mu_{N-1} \leq \frac{\sum_{v_i, v_j \in E} (f(v_i) - f(v_j))^2}{\sum_{v_i \in \mathcal{V}} f^2(v_i)}, \quad (2)$$

$$\mu_{N-1} \leq \frac{\sum_{v_i, v_j \in E} (f(v_i) - f(v_j))^2}{\sum_{v_i \in \mathcal{V}} \sum_{v_j \in \mathcal{V}} (f(v_i) - f(v_j))^2}, \quad (3)$$

where $f(v_i)$ is the vector f for the node v_i .

Let A and B be two disjoint subsets of $\mathcal{N}$, respectively. The distance between two disjoint subsets A and B of $\mathcal{N}$ is denoted by $h(A, B)$. For convenience, h takes the place of $h(A, B)$. Let $h(u, A)$ be the distance between the node u and A , which is the shortest distance of the node $u \in \mathcal{N}$ to a node of the set A . Suppose $a = |A|/N$ and $b = |B|/N$. A result on the algebraic connectivity and two partitions of graphs is presented by Alon [22] and Milman [20] below.

Lemma 3 (see [23]). *For any two disjoint subsets A and B of $\mathcal{N}$, it holds*

$$\mu_{N-1} \leq \frac{1}{Nh^2} \left(\frac{1}{a} + \frac{1}{b} \right) (E - E_A - E_B), \quad (4)$$

where E_A and E_B are the number of links in the sets A and B , respectively.

Moreover, the next step consider the case of three disjoint vertex subsets of graphs [24].

3. Main Result

Let A , B , and C be the subsets of $\mathcal{N}$, respectively, where their numbers of nodes are, respectively, $|A|$, $|B|$, and $|C|$. Assume $a = |A|/N$, $b = |B|/N$, and $c = |C|/N$. Let $h(u, A)$, $h(u, B)$, and $h(u, C)$ be the distances from the node $u \in \mathcal{N}$ to subsets A , B , and C of $\mathcal{N}$, respectively. Suppose $h_s = \min\{h(A, B), h(A, C), h(B, C)\}$. Now, we construct a function $g(u)$ related to node u as follows, where the constructed function is referred to the book [25]:

$$g(u) = \frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \frac{1}{9} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \times \frac{\min\{h_s, h(u, A), h(u, B), h(u, C)\}}{h_s}. \quad (5)$$

Let $f = g - \bar{g} \neq 0$, where $\bar{g} = 1/N \sum_{n \in \mathcal{N}} g(n)$. It is easy to check that $(f, c) = 0$, where c is a constant function [26–30]. Meanwhile, $(g, c) \neq 0$ can be checked. Thus, the following cases need to be discussed.

Case 1. If the node u belongs to any one subset of $\{A, B, C\}$, then

$$0 = \min\{h_s, h(u, A), h(u, B), h(u, C)\}, \quad (6)$$

$$g(u) = \frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) > 0. \quad (7)$$

Case 2. If the node $u \in \mathcal{N} - \{A, B, C\}$, then we can see that

$$\frac{\min\{h_s, h(u, A), h(u, B), h(u, C)\}}{h_s} \leq 1, \quad (8)$$

$$g(u) = \frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \frac{1}{9} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \times \frac{\min\{h_s, h(u, A), h(u, B), h(u, C)\}}{h_s} > 0. \quad (9)$$

By Case 1 and Case 2, $(g, c) \neq 0$ holds. In contrast, if $(g, c) = 0$, then $g(u) = 0$ for each $u \in \mathcal{N}$ and $g - \bar{g} = 0$, which is a contradiction with $f = g - \bar{g} \neq 0$. From the definition $g(u)$, for any two adjacent nodes u and v , we have

$$|g(u) - g(v)| \leq \frac{1}{9h_s} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right). \quad (10)$$

Our main result is as follows.

Theorem 1. *Let A , B , and $C \in \mathcal{N}$ be three disjoint subsets of $\mathcal{N}$. Let E_A and E_B and E_C be the numbers of links in the sets A and B and C , respectively. Then,*

$$\begin{aligned} \mu_{N-1} &\leq \frac{1}{81N^2h_s^2} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) (E - E_A - E_B - E_C) \\ &= \frac{1}{81Nh_s^2} \left(\frac{1}{N_A} + \frac{1}{N_B} + \frac{1}{N_C} \right) (E - E_A - E_B - E_C). \end{aligned} \quad (11)$$

Proof. For subsets A , B , and C , by Lemma 2, we have

$$\sum_{u, v \in \mathcal{E}} (f(u) - f(v))^2 = \sum_{u, v \in \mathcal{E}} (g(u) - g(v))^2 = \sum_{u, v \in \mathcal{E} - (A \cup B \cup C)} (g(u) - g(v))^2 \leq \frac{1}{81h_s^2} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2 (E - E_A - E_B - E_C). \quad (12)$$

From (2) and

$$\bar{g} = \frac{1}{N} \left(\sum_{n \in \mathcal{N}} g(n) \right), \quad \sum_{n \in \mathcal{N}} f^2(n) \geq \sum_{n \in (A \cup B \cup C)} (g(n) - \bar{g})^2, \quad (13)$$

where $\bar{g} = 0$, and since the coordinates of the center of gravity of the three regions are the average of the triangle region, then the vectors $g(u, A) + g(u, B) + g(u, C) = 0$. The sum of the vectors of the center of gravity of the triangle to the vertices is equal to 0 [25]. The center of gravity is analogous to the mean or average from statistics [6, 31, 32]:

FIGURE 1: The graphs G_1 and G_2 . FIGURE 2: The graphs G_3 and G_4 .

$$\begin{aligned}
 \sum_{n \in \mathcal{N}} f^2(n) &\geq \sum_{n \in A \cup B \cup C} f^2(n) = \sum_{n \in A} (g(n) - \bar{g})^2 + \sum_{n \in B} (g(n) - \bar{g})^2 + \sum_{n \in C} (g(n) - \bar{g})^2 \\
 &= |A| \left(\frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \bar{g} \right)^2 + |B| \left(\frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \bar{g} \right)^2 + |C| \left(\frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \bar{g} \right)^2 \\
 &= |A| \left(\frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \right)^2 + |B| \left(\frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \right)^2 + |C| \left(\frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \right)^2 \\
 &= (|A| + |B| + |C|) \left(\frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \right)^2 \\
 &= \frac{1}{9} \left(|A| \frac{1}{a} + |B| \frac{1}{b} + |C| \frac{1}{c} \right)^2 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \\
 &= N^2 \frac{1}{9} (1 + 1 + 1)^2 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq N^2 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).
 \end{aligned} \tag{14}$$

By the above inequalities and Lemma 2, it arrives that

$$\begin{aligned}
 \mu_{N-1} &\leq \frac{1}{81N^2h_s^2} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) (E - E_A - E_B - E_C) \\
 &= \frac{1}{81Nh_s^2} \left(\frac{1}{N_C} + \frac{1}{N_B} + \frac{1}{N_C} \right) (E - E_A - E_B - E_C).
 \end{aligned} \tag{15}$$

Example 1. Figure 1 describes the graphs G_1 and G_2 , each with $N = 7$ nodes, $L = 10$ links, and a diameter $\rho = 4$. For G_1 subsets, $A = \{v_1, v_3, v_5, v_6, v_7\}$, $B = \{v_2\}$, and $C = \{v_4\}$. For G_2

subsets, $A = \{u_1, u_2, u_3\}$, $B = \{u_4\}$, $C = \{u_5, u_6, u_7\}$, and $h_s = 0.5$. Their algebraic connectivity [33] and their upper bounds on (11) are as follows. For the G_1 and G_2 aplacian matrixes,

$$\begin{aligned}
G_1 &= \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 4 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \\
G_2 &= \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}.
\end{aligned} \tag{16}$$

For G_1 , the algebraic connectivity is 0.6338, and the algebraic connectivity of G_2 is 0.5858. For G_1 upper bounds on $\mu_{N-1}(G_1) \leq (1/81N^2h_s^2)(1/a + 1/b + 1/c)(E - E_A - E_B - E_C) = 0.0931$.

For G_2 upper bounds on $\mu_{N-1}(G_2) \leq (1/81N^2h_s^2)(1/a + 1/b + 1/c)(E - E_A - E_B - E_C) = 0.0587$.

A graph with the second smallest Laplacian eigenvalue μ_{N-1} is thus more robust, in the sense of being better connected.

Proposition 1. Let A, B , and $C \in V$ be three disjoint subsets of V . Suppose $h_s = 1$ and $D = V - A - B - C$. Let m_A , m_B , m_C , and m_D be the number of links in the sets A, B, C , and D , respectively. Then,

$$\mu_{N-1} \leq \frac{(1/81)((1/a) + (1/b) + (1/c))^2 (m - m_A - m_B - m_C - m_D)}{(|A| + |B| + |C|)[(1/a) + (1/b) + (1/c) - \bar{g}]^2 + |D|[2/9((1/a) + (1/b) + (1/c)) - \bar{g}]^2}, \tag{17}$$

where

$$\bar{g} = \frac{1}{n} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left(|A| + |B| + |C| + \frac{2}{9}|D| \right), \tag{18}$$

in which $\bar{g}$ is the average of the A, B, C, D field.

Proof. For subsets A, B , and C , by Lemma 2, we have

$$\sum_{u,v \in E} (f(u) - f(v))^2 = \sum_{u,v \in E} (g(u) - g(v))^2 = \sum_{u,v \in E - (E_A \cup E_B \cup E_C \cup E_D)} (g(u) - g(v))^2 \leq \frac{1}{81} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2 (m - m_A - m_B - m_C - m_D), \tag{19}$$

where links of the sets in the node sets A, B, C , and D are E_A, E_B, E_C , and E_D , respectively. From (2), we obtain

$$\begin{aligned}
\sum_{n \in \mathcal{V}} f^2(n) &= \sum_{n \in \{A \cup B \cup C \cup D\}} f^2(n) = \sum_{n \in A} (g(n) - \bar{g})^2 \\
&\quad + \sum_{n \in B} (g(n) - \bar{g})^2 + \sum_{n \in C} (g(n) - \bar{g})^2 + \sum_{n \in D} (g(n) - \bar{g})^2 \\
&= |A| \left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \bar{g} \right]^2 + |B| \left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \bar{g} \right]^2 + |C| \left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \bar{g} \right]^2 \\
&\quad + |D| \left[\frac{2}{9} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \bar{g} \right]^2 \\
&= (|A| + |B| + |C|) \left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \bar{g} \right]^2 + |D| \left[\frac{2}{9} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \bar{g} \right]^2.
\end{aligned} \tag{20}$$

By direct computation, we have

$$\bar{g} = \frac{1}{n} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left(|A| + |B| + |C| + \frac{2}{9} |D| \right). \quad (21)$$

By the above equalities and Lemma 2, inequality (17) holds.

But, we note that the algebraic connectivity [34, 35], μ_{N-1} , should not be seen as a strict disconnection or a robustness metric [36].

Example 2. For example, Figure 2 describes the graphs G_3 and G_4 , with $n = 9$, $m = 12$, and diameter 6. By direct calculation, for G_3 subsets, $A = \{v_1 v_2 v_3\}$, $B = \{v_4 v_5 v_6\}$, $C = \{v_7\}$, and $D = \{v_8 v_9\}$, and for G_4 subsets, $A = \{u_1 u_2 u_3\}$, $B = \{u_4 u_5 u_6\}$, $C = \{u_7\}$, and $D = \{u_8 u_9\}$. Their algebraic connectivity G_3 is 0.4798 and G_4 is 0.4817, respectively. Their Laplacian matrices $L(G_3)$ and $L(G_4)$, for G_3 upper bounds on $\mu_{N-1}(G_3) \geq 0.431$ and for G_4 upper bounds on $\mu_{N-1}(G_4) \geq 0.357$.

Theorem 1 and Proposition 1 are two completely different situations. The theorem hypothesis is that A, B , and $C \in \mathcal{N}$ be three disjoint subsets of $\mathcal{N}$. The proposition supposes that A, B , and $C \in V$ be three disjoint subsets of V and $h_s = 1$ and $D = V - A - B - C$. In other words, the proposition has constraints. Moreover, it is not the same as the four disjoint subsets of $\mathcal{N}$.

Data Availability

All data, models, and codes generated or used during the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

- [1] R. García-Domenech, J. Gálvez, and L. Pogliani, "Some new trends in chemical graph theory," *Chemical Reviews*, vol. 108, no. 3, pp. 1127–1169, 2008.
- [2] J.-B. de Julián-Ortiz, C. Wang, S. Wang, and B. Wei, "Zagreb indices and multiplicative zagreb indices of eulerian graphs," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 1, pp. 67–78, 2019.
- [3] J.-B. Liu, J. Zhao, J. Min, and J. D. Cao, "On the Hosoya index of graphs formed by a fractal graph," *Fractals-Complex Geometry Patterns and Scaling in Nature and Society*, vol. 27, no. 8, pp. 105–135, 2019.
- [4] M. Fiedler, "Algebraic connectivity of graphs, Czech," *Journal of Mathematics*, vol. 23, pp. 298–305, 1973.
- [5] N. L. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, UK, 2nd edition, 1993.
- [6] J.-B. Liu, J. Zhao, and Z.-Q. Cai, "On the generalized adjacency, Laplacian and signless Laplacian spectra of the weighted edge corona networks," *Physica A: Statistical Mechanics and Its Applications*, vol. 540, p. 123073, 2020.
- [7] R. Baeza-Yates and B. Ribeiro-Neto, *Modern Information Retrieval*, Addison-Wesley, Boston, MA, USA, 1999.
- [8] P. Baldi, P. Frasconi, and P. Smyth, *Modeling the Internet and the Web: Probabilistic Methods and Algorithms*, John Wiley & Sons, Hoboken, NJ, USA, 2003.
- [9] S. Barnett, *Matrices: Methods and Applications*, Oxford University Press, Oxford, UK, 1992.
- [10] T. F. Chan, P. Ciarlet, and W. K. Szeto, "On the optimality of the median cut spectral bisection graph partitioning method," *SIAM Journal on Scientific Computing*, vol. 18, no. 3, pp. 943–948, 1997.
- [11] A. K. Chandra, P. Raghavan, W. L. Ruzzo, R. Smolensky, and P. Tiwari, "The electrical resistance of a graph captures its commute and cover times," in *Proceedings of the twenty-first annual ACM symposium on Theory of computing—STOC '89*, pp. 574–586, Seattle WA, USA, May 1989.
- [12] P. Chebotarev and E. Shamis, "The matrix-forest theorem and measuring relations in small social groups," *Automation and Remote Control*, vol. 58, no. 9, pp. 1505–1514, 1997.
- [13] P. Chebotarev and E. Shamis, "On a duality between metrics and s-proximities," *Automation and Remote Control*, vol. 59, no. 4, pp. 608–612, 1998.
- [14] K.-W. Cheung, K.-C. Tsui, and J. Liu, "Extended latent class models for collaborative recommendation," *IEEE Transactions on Systems, Man, and Cybernetics—Part A: Systems and Humans*, vol. 34, no. 1, pp. 143–148, 2004.
- [15] D. Harel and Y. Koren, "On clustering using random walks," *FST TCS 2001: Foundations of Software Technology and Theoretical Computer Science*, vol. 22, no. 45, pp. 18–41, 2001.
- [16] J. L. Herlocker, J. A. Konstan, L. G. Terveen, and J. T. Riedl, "Evaluating collaborative filtering recommender systems," *ACM Transactions on Information Systems (TOIS)*, vol. 22, no. 1, pp. 5–53, 2004.
- [17] N.-D. Ho and P. Van Dooren, "On the pseudo-inverse of the laplacian of a bipartite graph," *Applied Mathematics Letters*, vol. 18, no. 8, pp. 917–922, 2005.
- [18] Z. Huang, H. Chen, and D. Zeng, "Applying associative retrieval techniques to alleviate the sparsity problem in collaborative filtering," *ACM Transactions on Information Systems (TOIS)*, vol. 22, no. 1, pp. 116–142, 2004.
- [19] R. Brooks, "Spectral geometry and the Cheeger constant," in *Expanding Graphs*, J. Friedman, Ed., American Mathematical Society, Providence, RI, USA, pp. 5–19, 1993.
- [20] P. Van Mieghem, *Graph Spectra for Complex Networks*, Delft University of Technology, Delft, Netherlands, 2012.
- [21] N. M. M. de Abreu, "Old and new results on algebraic connectivity of graphs," *Linear Algebra and Its Applications*, vol. 423, no. 1, pp. 53–73, 2007.
- [22] N. Alon, "Eigenvalues and expanders," *Combinatorica*, vol. 6, no. 2, pp. 83–96, 1986.
- [23] N. Alon and V. D. Milman, " λ_1 , Isoperimetric inequalities for graphs, and superconcentrators," *Journal of Combinatorial Theory, Series B*, vol. 38, no. 1, pp. 73–88, 1985.
- [24] P. K. Chan, M. Schlag, and J. Zien, "Spectral k-way ratio cut partitioning and clustering," in *Proceedings of the Symposium on Research on integrated systems*, Warwick, UK, 1993.

- [25] A Court, Nathan College Geometry, *An Introduction to the Modern Geometry of the Triangle and the Circle*, Barnes & Noble, New York, NY, USA, 2nd edition, 1925.
- [26] F. Chung, *Spectral Graph Theory*, AMS Publications, Providence, RI, USA, 1997.
- [27] F. R. K. Chung, "Laplacians of graphs and Cheeger inequalities," in *Combinatorics, Paul Erdős is Eighty*, pp. 157–172, János Bolyai Mathematical Society, Budapest, Hungary, 1996.
- [28] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Birkhauser, Basel, Switzerland, 1992.
- [29] J. Cheeger, "A lower bound for the smallest eigenvalue of the Laplacian," in *Problems in Analysis*, R. C. Gunnig, Ed., pp. 195–199, Princeton University Press, Princeton, NJ, USA, 1970.
- [30] C. Delorme and S. Poljak, "Laplacian eigenvalues and the maximum cut problem," *Mathematical Programming*, vol. 62, no. 1-3, pp. 557–574, 1993.
- [31] J.-B. Liu, J. Zhao, H. He, and Z. Shao, "Valency-based topological descriptors and structural property of the generalized sierpinski networks," *Journal of Statistical Physics*, vol. 177, no. 6, pp. 1131–1147, 2019.
- [32] J.-B. Liu, Z.-Yu Shi, Y.-H. Pan, J. Cao, and M. Abdel-Aty, "Udai Al-juboori computing the laplacian spectrum of linear octagonal-quadrilateral networks and its applications," *Polycyclic Aromatic Compounds*, vol. 10, pp. 23–43, 2020.
- [33] D. M. Cvetkovic, M. Doob, I. Gutman et al., *Recent Results in the Theory of Graph Spectra*, *Annals of Discrete Mathematics*, Vol. 36, Elsevier, Amsterdam, Netherlands, 1988.
- [34] C. Godsil and G. Royle, "Algebraic graph theory," in *Graduate Texts in Mathematics*, p. 207, Springer-Verlag, New York, NY, USA, 2001.
- [35] S. J. Kirkland, J. J. Moliterno, M. Neumann, and B. L. Shader, "On graphs with equal algebraic and vertex connectivity," *Linear Algebra and Its Applications*, vol. 341, no. 1-3, pp. 45–56, 2002.
- [36] G. A. Edgar, D. H. Ullman, and D. B. West, "Problems and solutions," *The American Mathematical Monthly*, vol. 125, no. 1, pp. 81–89, 2018.